Reverses of the Triangle Inequality Via Selberg's and Boas-Bellman's Inequalities

FACTA UNIVERSITATIS (NIŠ)
SER. MATH. INFORM. 21 (2006), 29–39

REVERSES OF THE TRIANGLE INEQUALITY VIA SELBERG'S AND BOAS-BELLMAN'S INEQUALITIES

Sever S. Dragomir

Abstract. Reverses of the triangle inequality for vectors in inner product spaces via the Selberg and Boas-Bellman generalisations of Bessel's inequality are given. Applications for complex numbers are also provided.

1. Introduction

In 1966, J.B. Diaz and F.T. Metcalf [3] obtained the following reverse of the triangle inequality on utilising an argument based on the Bessel inequality in a real or complex inner product space $(H, \langle .,. \rangle)$.

Theorem 1.1. (Diaz-Metcalf, 1966) *Let $e_1, \dots, e_m$ be orthonormal vectors in H, i.e., $e_i \perp e_j$ for $i \neq j$ and $\|e_i\| = 1$, $i, j \in \{1, \dots, m\}$. Suppose the vectors $x_1, \dots, x_n \in H \setminus \{0\}$ satisfy*

$$0 \leq r_k \leq \frac{\operatorname{Re}\langle x_j, e_k \rangle}{\|x_j\|}, \quad j \in \{1, \dots, n\}, \; k \in \{1, \dots, m\}. \tag{1.1}$$

Then

$$\left(\sum_{k=1}^{n} r_k^2 \right)^{1/2} \sum_{j=1}^{n} \|x_j\| \leq \left\| \sum_{j=1}^{n} x_j \right\|, \tag{1.2}$$

where the equality holds if and only if

$$\sum_{j=1}^{n} x_j = \left(\sum_{j=1}^{n} \|x_j\| \right) \sum_{k=1}^{m} r_k e_k. \tag{1.3}$$

Received February 10, 2006.
2000 *Mathematics Subject Classification.* Primary 46C05, 26D15.

In an attempt to improve this result for the case of complex inner product spaces, the author obtained in 2004 the following result [4]:

Theorem 1.2. (Dragomir, 2004) *Let* $e_1, \ldots, e_m \in H$ *be an orthonormal family of vectors in the complex inner product space* H*. If the vectors* x_1, $\ldots$, $x_n \in H$ *satisfy the conditions*

$$0 \leq r_k \|x_j\| \leq \operatorname{Re}\langle x_j, e_k\rangle, \quad 0 \leq \rho_k \|x_j\| \leq \operatorname{Im}\langle x_j, e_k\rangle \tag{1.4}$$

for each $j \in \{1, \ldots, n\}$ *and* $k \in \{1, \ldots, m\}$, *then we have the following reverse of the generalised triangle inequality:*

$$\left[\sum_{k=1}^{m}\left(r_k^2 + \rho_k^2\right)\right]^{1/2} \sum_{j=1}^{n} \|x_j\| \leq \left\|\sum_{j=1}^{n} x_j\right\|. \tag{1.5}$$

The equality holds in (1.5) *if and only if*

$$\sum_{j=1}^{n} x_j = \left(\sum_{j=1}^{n} \|x_j\|\right) \sum_{k=1}^{m} (r_k + i\rho_k)\, e_k. \tag{1.6}$$

As particular cases of interest, we can notice the following results [4]:

Corollary 1.1. *Let* $e_1, \ldots, e_m \in H$ *be orthonormal vectors in the complex inner product space* $(H; \langle\cdot,\cdot\rangle)$ *and* $\rho_k, \eta_k \in (0, 1)$, $k \in \{1, \ldots, m\}$. *If* $x_1, \ldots, x_k \in H$ *are such that*

$$\|x_j - e_k\| \leq \rho_k, \quad \|x_j - ie_k\| \leq \eta_k \tag{1.7}$$

for each $j \in \{1, \ldots, n\}$ *and* $k \in \{1, \ldots, m\}$, *then we have the inequality*

$$\left[\sum_{k=1}^{m}\left(2 - \rho_k^2 - \eta_k^2\right)\right]^{1/2} \sum_{j=1}^{n} \|x_j\| \leq \left\|\sum_{j=1}^{n} x_j\right\|. \tag{1.8}$$

The case of equality holds in (1.8) *if and only if*

$$\sum_{j=1}^{n} x_j = \left(\sum_{j=1}^{n} \|x_j\|\right) \sum_{k=1}^{m} \left(\sqrt{1 - \rho_k^2} + i\sqrt{1 - \eta_k^2}\right) e_k. \tag{1.9}$$

Corollary 1.2. *Let* $e_1, \dots, e_m$ *be as in Corollary* 1.1 *and* $M_k \geq m_k > 0$, $N_k \geq n_k > 0$, $k \in \{1, \dots, m\}$. *If* $x_1, \dots, x_n \in H$ *are such that either*

$$\text{(1.10)}\quad \operatorname{Re}\langle M_k e_k - x_j, x_j - m_k e_k\rangle \geq 0, \quad \operatorname{Re}\langle N_k e_k - x_j, x_j - n_k e_k\rangle \geq 0$$

or, equivalently,

$$\text{(1.11)}\qquad \left\| x_j - \frac{M_k + m_k}{2} e_k \right\| \leq \frac{1}{2}(M_k - m_k),$$
$$\left\| x_j - \frac{N_k + n_k}{2} e_k \right\| \leq \frac{1}{2}(N_k - n_k),$$

for each $j \in \{1, \dots, n\}$ *and* $k \in \{1, \dots, m\}$, *then we have the inequality*

$$\text{(1.12)}\quad 2\left\{\sum_{k=1}^{m}\left[\frac{m_k M_k}{(M_k + m_k)^2} + \frac{n_k N_k}{(N_k + n_k)^2}\right]\right\}^{1/2} \sum_{j=1}^{n}\|x_j\| \leq \left\|\sum_{j=1}^{n} x_j\right\|.$$

The case of equality holds in (1.12) *if and only if*

$$\sum_{j=1}^{n} x_j = 2\left(\sum_{j=1}^{n}\|x_j\|\right)\sum_{k=1}^{m}\left(\frac{\sqrt{m_k M_k}}{M_k + m_k} + i\frac{\sqrt{n_k N_k}}{N_k + n_k}\right) e_k.$$

In the above results the vectors $\{e_1, \dots, e_m\}$ are assumed to be orthonormal and the principle tool in proving these results is the well known Bessel's inequality:

$$\text{(1.13)}\qquad \sum_{k=1}^{m}|\langle x, e_k\rangle|^2 \leq \|x\|^2, \quad x \in H.$$

If we use the following generalisation of Bessel's inequality, namely:

$$\text{(1.14)}\qquad \sum_{k=1}^{m}\frac{|\langle x, y_k\rangle|^2}{\sum_{j=1}^{m}|\langle y_k, y_j\rangle|} \leq \|x\|^2,$$

provided $x, y_1, \dots, y_m$ are vectors in H and $y_k \neq 0$, $k \in \{1, \dots, m\}$, which is known in the literature as the *Selberg inequality,* (see [6, p. 394] of [5, p. 134]), then we can obtain different reverses of the generalised triangle inequality, where the assumption of orthonormality is taken out.

A similar approach may be considered if the other generalisation of Bessel's inequality due to Boas [2] and Bellman [1] is used, namely the inequality

$$\sum_{k=1}^{m} |\langle x, y_k\rangle|^2 \le \|x\|^2 \left[\max_{1\le k\le m} \|y_k\|^2 + \left(\sum_{1\le k\ne j\le m} |\langle y_k, y_j\rangle|^2 \right)^{1/2} \right], \tag{1.15}$$

where $x, y_1, \dots, y_m$ are as above.

The main aim of this paper is to establish new reverses of the triangle inequality for $x_1, \dots, x_n$ vectors in an inner product space H in terms of another sequence $y_1, \dots, y_m$ of nonzero vectors that can be non-orthonormal. The main tools in obtaining such results are the Selberg inequality (1.14) and the Boas-Bellman inequality (1.15). Applications for complex numbers are also given.

2. The Results

The following result holds:

Theorem 2.1. *Let $(H; \langle\cdot,\cdot\rangle)$ be a complex inner product space, $x_1, \dots, x_n$, $y_1, \dots, y_m$ be vectors such that there exist the nonnegative real numbers ρ_j, η_j, $j \in \{1, \dots, m\}$ with*

$$\operatorname{Re}\langle x_i, y_j\rangle \ge \rho_j \|x_i\| \|y_j\|, \quad \operatorname{Im}\langle x_i, y_j\rangle \ge \eta_j \|x_i\| \|y_j\|, \tag{2.1}$$

for each $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$. Then

$$\left\| \sum_{i=1}^{n} x_i \right\| \ge \left(\sum_{j=1}^{m} \frac{\left(\rho_j^2 + \eta_j^2\right) \|y_j\|^2}{\sum\limits_{k=1}^{m} |\langle y_j, y_k\rangle|} \right)^{1/2} \sum_{i=1}^{n} \|x_i\|. \tag{2.2}$$

Proof. Utilising Selberg's inequality, we have

$$\left\| \sum_{i=1}^{n} x_i \right\|^2 \ge \sum_{j=1}^{m} \frac{|\langle \sum_{i=1}^{n} x_i, y_j\rangle|^2}{\sum_{k=1}^{m} |\langle y_j, y_k\rangle|}. \tag{2.3}$$

Since

$$\left| \left\langle \sum_{i=1}^{n} x_i, y_j \right\rangle \right|^2 = \left(\sum_{i=1}^{n} \operatorname{Re}\langle x_i, y_j\rangle \right)^2 + \left(\sum_{i=1}^{n} \operatorname{Im}\langle x_i, y_j\rangle \right)^2,$$

then, by (2.1), we obtain

$$\left|\left\langle\sum_{i=1}^{n}x_i,y_j\right\rangle\right|^2 \geq \rho_j^2\left\|y_j\right\|^2\left(\sum_{i=1}^{n}\|x_i\|\right)^2+\eta_j^2\left\|y_j\right\|^2\left(\sum_{i=1}^{n}\|x_i\|\right)^2 = \left(\sum_{i=1}^{n}\|x_i\|\right)^2\left(\rho_j^2+\eta_j^2\right)\left\|y_j\right\|^2, \tag{2.4}$$

for any $j\in\{1,\dots,m\}$. Therefore, by (2.3) we get

$$\left\|\sum_{i=1}^{n}x_i\right\|^2 \geq \left(\sum_{i=1}^{n}\|x_i\|\right)^2\sum_{j=1}^{m}\frac{\left(\rho_j^2+\eta_j^2\right)\|y_j\|^2}{\sum_{k=1}^{m}|\langle y_j,y_k\rangle|},$$

which is clearly equivalent to (2.2). □

Remark 2.1. If the space is real or complex and only the first condition of (2.1) is available, then

$$\left\|\sum_{i=1}^{n}x_i\right\| \geq \left(\sum_{j=1}^{m}\frac{\rho_j^2\|y_j\|^2}{\sum_{k=1}^{m}|\langle y_j,y_k\rangle|}\right)^{1/2}\sum_{i=1}^{n}\|x_i\|. \tag{2.5}$$

Remark 2.2. If $\{y_1,\dots,y_m\}$ are orthonormal and

$$\operatorname{Re}\langle x_i,y_j\rangle \geq \rho_j\|x_i\|, \quad \operatorname{Im}\langle x_i,y_j\rangle \geq \eta_j\|x_i\|, \tag{2.6}$$

for $i\in\{1,\dots,n\}$ and $j\in\{1,\dots,m\}$, then

$$\left\|\sum_{i=1}^{n}x_i\right\| \geq \left(\sum_{j=1}^{m}\left(\rho_j^2+\eta_j^2\right)\right)^{1/2}\sum_{i=1}^{n}\|x_i\| \tag{2.7}$$

and the inequality (2.5) is recaptured.

The following corollary may be of interest for applications:

Corollary 2.3. *Let $y_1,\dots,y_m$ be nonzero vectors in the complex inner product space $(H;\langle\cdot,\cdot\rangle)$ and $p_k,q_k\in(0,1)$ for $k\in\{1,\dots,m\}$. If $x_1,\dots,x_n\in H$ are such that:*

$$\|x_j-y_k\|\leq p_k<\|y_k\|, \quad \|x_j-iy_k\|\leq q_k<\|y_k\| \tag{2.8}$$

for each $i \in \{1, \ldots, n\}$ *and* $j \in \{1, \ldots, m\}$, *then:*

$$\left\| \sum_{j=1}^{n} x_j \right\| \geq \left(\sum_{k=1}^{m} \frac{2\|y_k\|^2 - p_k^2 - q_k^2}{\sum_{s=1}^{m} |\langle y_k, y_s \rangle|} \right)^{1/2} \sum_{j=1}^{n} \|x_j\|. \tag{2.9}$$

Proof. From the first inequality in (2.8) we deduce, by taking the square, that

$$\|x_j\|^2 + \|y_k\|^2 - p_k^2 \leq 2 \operatorname{Re} \langle x_j, y_k \rangle,$$

implying

$$\frac{\|x_j\|^2}{\sqrt{\|y_k\|^2 - p_k^2}} + \sqrt{\|y_k\|^2 - p_k^2} \leq \frac{2 \operatorname{Re} \langle x_j, y_k \rangle}{\sqrt{\|y_k\|^2 - p_k^2}} \tag{2.10}$$

since $\sqrt{\|y_k\|^2 - p_k^2} > 0$ for $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$.

On the other hand, obviously,

$$2\|x_j\| \leq \frac{\|x_j\|^2}{\sqrt{\|y_k\|^2 - p_k^2}} + \sqrt{\|y_k\|^2 - p_k^2}, \tag{2.11}$$

for $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$.

Hence, by (2.10) and (2.11) we have

$$\|x_j\| \sqrt{\|y_k\|^2 - p_k^2} \leq \operatorname{Re} \langle x_j, y_k \rangle. \tag{2.12}$$

Since $\operatorname{Re} \langle x_j, iy_k \rangle = \operatorname{Im} \langle ix_j, y_k \rangle$, $j \in \{1, \ldots, n\}$, $k \in \{1, \ldots, m\}$ then, by the second inequality in (2.8), we have

$$\|x_j\| \sqrt{\|y_k\|^2 - q_k^2} \leq \operatorname{Im} \langle x_j, y_k \rangle, \tag{2.13}$$

for $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$.

Now if we define

$$\rho_k = \frac{\sqrt{\|y_k\|^2 - p_k^2}}{\|y_k\|}, \quad \eta_k = \frac{\sqrt{\|y_k\|^2 - q_k^2}}{\|y_k\|}, \quad k \in \{1, \ldots, m\}$$

and apply Theorem 2.1, we get

$$\left\| \sum_{j=1}^{n} x_j \right\| \geq \left(\sum_{k=1}^{m} \frac{\left(\frac{\|y_k\|^2 - p_k^2}{\|y_k\|^2} + \frac{\|y_k\|^2 - q_k^2}{\|y_k\|^2} \right) \cdot \|y_k\|^2}{\sum_{s=1}^{m} |\langle y_k, y_s \rangle|} \right)^{1/2} \sum_{j=1}^{n} \|x_j\|,$$

which is exactly (2.9). □

Remark 2.3. If $\{y_1, \dots, y_m\}$ are orthonormal and $p_k, q_k \in (0,1)$, then out of (2.9) we can recapture (1.8) from the introduction.

The following corollary may be stated as well:

Corollary 2.4. *Let $y_1, \dots, y_m$ be nonzero vectors in the complex inner product space $(H; \langle\cdot,\cdot\rangle)$ and $M_k \geq m_k > 0$, $N_k \geq n_k > 0$ for $k \in \{1, \dots, m\}$. If $x_1, \dots, x_n \in H$ are such that:*

$$\text{Re}\,\langle M_k y_k - x_j, x_j - m_k y_k\rangle \geq 0, \quad \text{Re}\,\langle N_k i y_k - x_j, x_j - n_k i y_k\rangle \geq 0 \tag{2.14}$$

or, equivalently,

$$\begin{cases} \left\| x_j - \dfrac{M_k + m_k}{2} y_k \right\| & \leq \dfrac{1}{2}(M_k - m_k)\|y_k\|, \\ \left\| x_j - \dfrac{N_k + n_k}{2} i y_k \right\| & \leq \dfrac{1}{2}(N_k - n_k)\|y_k\| \end{cases} \tag{2.15}$$

for $j \in \{1, \dots, n\}$ and $k \in \{1, \dots, m\}$, then

$$\left\| \sum_{j=1}^n x_j \right\| \geq 2 \left(\sum_{k=1}^m \left[\frac{m_k M_k}{(M_k + m_k)^2} + \frac{n_k N_k}{(N_k + n_k)^2} \right] \frac{\|y_k\|^2}{\sum_{s=1}^m |\langle y_k, y_s\rangle|} \right)^{1/2} \sum_{j=1}^n \|x_j\|. \tag{2.16}$$

Proof. From the first inequality in (2.14) we get

$$\|x_j\|^2 + M_k m_k \|y_k\|^2 \leq (M_k + m_k)\,\text{Re}\,\langle x_j, y_k\rangle,$$

implying

$$\frac{\|x_j\|^2}{\sqrt{M_k m_k}} + \sqrt{M_k m_k}\,\|y_k\|^2 \leq \frac{M_k + m_k}{\sqrt{M_k m_k}}\,\text{Re}\,\langle x_j, y_k\rangle \tag{2.17}$$

for each $j \in \{1, \dots, n\}$ and $k \in \{1, \dots, m\}$.

Since, obviously

$$2\|x_j\|\,\|y_k\| \leq \frac{\|x_j\|^2}{\sqrt{M_k m_k}} + \sqrt{M_k m_k}\,\|y_k\|^2, \tag{2.18}$$

then, by (2.17) and (2.18), we get

$$\frac{2\sqrt{M_k m_k}}{M_k + m_k}\|x_j\|\|y_k\| \le \operatorname{Re}\langle x_j, y_k\rangle \tag{2.19}$$

for each $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$.

In a similar manner, on utilising the second part of (2.14) we get

$$\frac{2\sqrt{n_k N_k}}{N_k + n_k}\|x_j\|\|y_k\| \le \operatorname{Im}\langle x_j, y_k\rangle \tag{2.20}$$

for each $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$.

Applying Theorem 2.1 for

$$\rho_k = \frac{2\sqrt{M_k m_k}}{M_k + m_k}, \quad \eta_k = \frac{2\sqrt{n_k N_k}}{N_k + n_k}, \quad k \in \{1, \ldots, m\}$$

we deduce the desired result (2.16). □

Remark 2.4. The case when $\{y_1, \ldots, y_m\}$ becomes an orthonormal family of vectors will provide the known inequality (1.12) of the introduction.

On utilising the other generalisation of Bessel's inequality we can provide the following reverse of the triangle inequality as well:

Theorem 2.2. *With the assumptions of Theorem* 2.1 *for the vectors* x_1, $\ldots$, x_n, y_1, $\ldots$, y_m *and the nonnegative real numbers* ρ_j, η_j, $j \in \{1, \ldots, m\}$, *we have the inequality*

$$\left\|\sum_{i=1}^{n} x_i\right\| \ge \left(\frac{\sum_{j=1}^{m}\left(\rho_j^2 + \eta_j^2\right)\|y_j\|^2}{\max_{1\le i\le m}\|y_i\|^2 + \left(\sum_{1\le i\ne j\le m}|\langle y_i, y_j\rangle|^2\right)^{1/2}}\right)^{1/2} \sum_{i=1}^{n}\|x_i\|. \tag{2.21}$$

Proof. The argument is similar with the one incorporated in the proof of Theorem 2.1 by utilising the inequality

$$\sum_{k=1}^{m}\left|\left\langle \sum_{i=1}^{n} x_i, y_k\right\rangle\right|^2 \le \left\|\sum_{i=1}^{n} x_i\right\|^2 \left[\max_{1\le i\le m}\|y_i\|^2 + \left(\sum_{1\le i\ne j\le m}|\langle y_i, y_j\rangle|^2\right)^{1/2}\right]$$

that follows from (1.15). □

Remark 2.5. Similar results with those incorporated in Corollaries 2.3–2.4 may be stated as well. The details are omitted.

Remark 2.6. If one utilises the other generalisations of Bessel's inequalities as provided, for instance, in the monograph [5, Chapter 4], that one can state other reverses of the triangle inequality.

3. Applications for Complex Numbers

The above results may be used in establishing some interesting reverses of the generalised triangle inequality for complex numbers.

Proposition 3.1. *Let* $c_1, \dots, c_n$ *and* $d_1, \dots, d_m$ *be complex numbers with the property that there exists the nonnegative real numbers* ρ_k, η_k, $k \in \{1, \dots, m\}$ *with*

$$\operatorname{Re} c_j \cdot \operatorname{Re} d_k + \operatorname{Im} c_j \cdot \operatorname{Im} d_k \geq \rho_k |c_j| |d_k| \tag{3.1}$$

and

$$\operatorname{Re} d_k \cdot \operatorname{Im} c_j - \operatorname{Re} c_j \cdot \operatorname{Im} d_k \geq \eta_k |c_j| |d_k|, \tag{3.2}$$

for any $j \in \{1, \dots, n\}$ *and* $k \in \{1, \dots, m\}$. *Then*

$$\left| \sum_{j=1}^{n} c_j \right| \geq \left[\frac{\sum_{k=1}^{m} \left(\rho_k^2 + \eta_k^2 \right) |d_k|}{\sum_{s=1}^{m} |d_s|} \right]^{1/2} \sum_{j=1}^{n} |c_j| . \tag{3.3}$$

Proof. It follows from Theorem 2.1 applied for the complex inner product space $\mathbb{H} = \mathbb{C}$ endowed with the canonical inner product $\langle x, y \rangle := x\bar{y}$. The details are omitted. □

Possibly a more useful result which also has a clear geometrical interpretation is incorporated in the following.

Proposition 3.2. *Assume that the complex numbers* c_j, $d_j, j \in \{1, \dots, n\}$ *and the nonnegative real numbers* p_k, q_k $k \in \{1, \dots, m\}$ *are such that*

$$|c_j - d_k| \leq p_k < |d_k| ; \quad |c_j - i d_k| \leq q_k < |d_k| \tag{3.4}$$

for each $j \in \{1, \dots, n\}$ *and* $k \in \{1, \dots, m\}$. *Then*

$$\left|\sum_{j=1}^{n} c_j\right| \geq \frac{\left(\sum_{k=1}^{m} \frac{2|d_k|^2 - p_k^2 - q_k^2}{|d_k|}\right)^{1/2}}{\left(\sum_{s=1}^{m} |d_s|\right)^{1/2}} \sum_{j=1}^{n} |c_j| . \tag{3.5}$$

The proof is obvious by Corollary 2.3.

Further, on using Corollary 2.4 we can state:

Proposition 3.3. *If* $c_1, \dots, c_n$ *and* $d_1, \dots, d_m$ *are complex numbers such that there exists* $M_k \geq m_k > 0$, $N_k \geq n_k > 0$ *with*

$$\begin{aligned}(M_k \operatorname{Re} d_k - \operatorname{Re} c_j)(\operatorname{Re} c_j - m_k \operatorname{Re} d_k)& \\ + (M_k \operatorname{Im} d_k - \operatorname{Im} c_j)(\operatorname{Im} c_j - m_k \operatorname{Im} d_k) &\geq 0\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}(-N_k \operatorname{Im} d_k - \operatorname{Re} c_j)(\operatorname{Re} c_j + n_k \operatorname{Im} d_k)& \\ + (N_k \operatorname{Re} d_k - \operatorname{Im} c_j)(\operatorname{Im} c_j - n_k \operatorname{Re} d_k) &\geq 0\end{aligned} \tag{3.7}$$

for each $j \in \{1, \dots, n\}$ *and* $k \in \{1, \dots, m\}$, *then*

$$\left|\sum_{j=1}^{n} c_j\right| \geq 2\frac{\left(\sum_{k=1}^{m} \left[\frac{m_k M_k}{(M_k+m_k)^2} + \frac{n_k N_k}{(N_k+n_k)^2}\right] |d_k|\right)^{1/2}}{\left(\sum_{s=1}^{m} |d_s|\right)^{1/2}} \sum_{j=1}^{n} |c_j| .$$

Remark 3.7. A sufficient condition for (3.6) to occur is

$$m_k \operatorname{Re} d_k \leq \operatorname{Re} c_j \leq M_k \operatorname{Re} d_k$$

and

$$m_k \operatorname{Im} d_k \leq \operatorname{Im} c_j \leq M_k \operatorname{Im} d_k$$

for each $j \in \{1, \dots, n\}$ and $k \in \{1, \dots, m\}$ while for (3.7) is

$$-N_k \operatorname{Im} d_k \geq \operatorname{Re} c_j \geq -n_k \operatorname{Im} d_k$$

and

$$N_k \operatorname{Re} d_k \geq \operatorname{Im} c_j \geq n_k \operatorname{Re} d_k$$

for each $j \in \{1, \dots, n\}$ and $k \in \{1, \dots, m\}$.

Finally, on utilising Theorem 2.2 we can state the following reverse of the triangle inequality for complex numbers as well:

Proposition 3.4. *With the assumptions of Proposition* 3.3 *for the complex numbers* $c_1, \dots, c_n$*;* $d_1, \dots, d_m$ *and the nonnegative real numbers* ρ_k, η_k, $k \in \{1, \dots, m\}$, *we have*

$$\left|\sum_{j=1}^{n} c_j\right| \geq \left(\frac{\sum_{k=1}^{m} \left(\rho_k^2 + \eta_k^2\right) |d_k|^2}{\max_{1 \leq j \leq m} |d_j|^2 + \left(\sum_{1 \leq j \neq k \leq m} |d_j d_k|^2 \right)^{1/2}} \right)^{1/2} \sum_{j=1}^{n} |c_j| .$$

REFERENCES

1. R. Bellman: *Almost orthogonal series. Bull. Amer. Math. Soc.* **50**(1944), 517–519.
2. R.P. Boas: *A general moment problem. Amer. J. Math.* **63**(1941), 361–370.
3. J.B. Diaz and F.T. Metcalf: *A complementary triangle inequality in Hilbert and Banach spaces. Proc. Amer. Math. Soc.* **17**(1) (1966), 88–97.
4. S.S. Dragomir: *Some reverses of the generalised triangle inequality in complex inner product spaces. Linear Algebra Appl.* **402** (2005), 245–254.
5. S.S. Dragomir: *Advances in Inequalities of the Schwarz, Grüss and Bessel Type in Inner Product Spaces.* Nova Science Publishers, Inc., NY, 2004.
6. D.S. Mitrinović, J.E. Pečarić and A.M. Fink: *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht/Boston/London, 1993.

School of Computer Science and Mathematics
Victoria University
PO Box 14428, Melbourne City
Victoria 8001, Australia
sever.dragomir@vu.edu.au
http://rgmia.vu.edu.au/dragomir/