Refining Jensen's Integral Inequality for Divisions of Measurable Space

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Refining Jensen's Integral Inequality for Divisions of Measurable Space

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Abstract. In this paper we establish a refinement and some reverses for Jensen's inequality for the general Lebesgue integral on divisions of measurable space. Applications for discrete inequalities and weighted means of positive numbers are also given. Some examples related to Hermite-Hadamard inequality for convex functions are provided as well.

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1. Introduction

Let C be a convex subset of the linear space X and f a convex function on C. If $\mathbf{p} = (p_1, \ldots, p_n)$ is a *probability sequence,* i.e. $p_1, \ldots, p_n \geqslant 0$ with $\sum_{i=1}^{n} p_i = 1$ and $\mathbf{x} = (x_1, \ldots, x_n) \in C^n$, then

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leqslant \sum_{i=1}^{n} p_i f(x_i), \tag{1}$$

is well known in the literature as *Jensen's inequality.*
The Jensen inequality for convex functions plays a crucial role in the Theory of Inequalities due to the fact that other inequalities such as the arithmetic mean-geometric mean inequality, Hölder and Minkowski's inequalities, Ky Fan's inequality etc. can be obtained as particular cases of it.

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In 1989, J. Pečarić and the author obtained the following refinement of (1) (see [19]):

$$f\left(\sum_{i=1}^{n}p_ix_i\right)\leqslant\sum_{i_1,\dots,i_{k+1}=1}^{n}p_{i_1}\dots p_{i_{k+1}}f\left(\frac{x_{i_1}+\cdots+x_{i_{k+1}}}{k+1}\right)\tag{2}$$
$$\leqslant\sum_{i_1,\dots,i_k=1}^{n}p_{i_1}\dots p_{i_k}f\left(\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right)\leqslant\cdots\leqslant\sum_{i=1}^{n}p_if\left(x_i\right),$$

for $k\geqslant 1$ and $\mathbf{p}$, $\mathbf{x}$ as above.
If $q_1,\dots,q_k\geqslant 0$ with $\sum_{j=1}^{k}q_j=1$, then the following weighted refinement obtained in 1994 by the author also holds (see [6]):

$$f\left(\sum_{i=1}^{n}p_ix_i\right)\leqslant\sum_{i_1,\dots,i_k=1}^{n}p_{i_1}\dots p_{i_k}f\left(\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right)\tag{3}$$
$$\leqslant\sum_{i_1,\dots,i_k=1}^{n}p_{i_1}\dots p_{i_k}f\left(q_1x_{i_1}+\cdots+q_kx_{i_k}\right)\leqslant\sum_{i=1}^{n}p_if\left(x_i\right),$$

where $1\leqslant k\leqslant n$ and $\mathbf{p}$, $\mathbf{x}$ are as above.
For other refinements and applications related to Ky Fan's inequality, the arithmetic mean-geometric mean inequality, the generalized triangle inequality, the f-Divergence measure etc., see [1], [3]-[15], [16]-[18] and [20]-[21].
Motivated by the above results, we investigate in this paper the integral version of Jensen inequality and establish some refinements and reverses of interest for applications.
Let $(\Omega,\mathcal{A},\mu)$ be a measurable space consisting of a set Ω, a σ -algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$. For the μ-integrable positive μ-a.e. weight w consider the Lebesgue space

$$L_w\left(\Omega,\mu\right):=\{f:\Omega\to\mathbb{R},\ f\text{ is }\mu\text{-measurable and }\int_\Omega\left|f\left(t\right)\right|w\left(t\right)d\mu\left(t\right)<\infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega wd\mu$ instead of $\int_\Omega w\left(t\right)d\mu\left(t\right)$ etc.
We say that the family of measurable sets $F_n\left(\Omega\right)=\{\Omega_i\}_{i\in\{1,\dots,n\}}$ is a n-division for Ω if $\Omega=\bigcup_{i=1}^{n}\Omega_i$, $\Omega_i\cap\Omega_j=\emptyset$ for any i, $j\in\{1,...,n\}$ with $i\neq j$ and $\mu\left(\Omega_i\right)>0$ for any $i\in\{1,...,n\}$. In this situation, if $f\in L_w\left(\Omega,\mu\right)$ then $f\in L_w\left(\Omega_i,\mu\right)$ for any $i\in\{1,...,n\}$ and $\int_\Omega fwd\mu=\sum_{i=1}^{n}\int_{\Omega_i}fwd\mu$. Also, $\int_\Omega wd\mu=\sum_{i=1}^{n}\int_{\Omega_i}wd\mu$ with $\int_{\Omega_i}wd\mu>0$ for any $i\in\{1,...,n\}$.

For a given $n \geqslant 2$ we denote by $\mathfrak{D}_n(\Omega)$ the set of all n-divisions of Ω and consider the functional $\psi(\Phi, w, f, \cdot) : \mathfrak{D}_n(\Omega) \to \mathbb{R}$ defined by

$$\psi(\Phi, f, w, F_n(\Omega)) := \frac{1}{\int_\Omega w d\mu} \sum_{i=1}^n \Phi\left(\frac{\int_{\Omega_i} f w d\mu}{\int_{\Omega_i} w d\mu}\right) \int_{\Omega_i} w d\mu. \tag{4}$$

In the next section we establish some results concerning this functional that are related to Jensen's integral inequality. Applications for discrete inequalities and weighted means are provided in the third section. In the last section some applications related to the Hermite-Hadamard inequality for convex functions are also given.

2. The Main Results

The following result holds:

Theorem 2.1. *Let $\Phi : [m, M] \to \mathbb{R}$ be a convex function, $f : \Omega \to [m, M]$ a μ-measurable function such that f, $\Phi \circ f \in L_w(\Omega, \mu)$. Then for any $F_n(\Omega) \in \mathfrak{D}_n(\Omega)$ we have*

$$\frac{\int_\Omega (\Phi \circ f) w d\mu}{\int_\Omega w d\mu} \geqslant \psi(\Phi, f, w, F_n(\Omega)) \geqslant \Phi\left(\frac{\int_\Omega f w d\mu}{\int_\Omega w d\mu}\right), \tag{5}$$

where $n \geqslant 2$.

Proof. From Jensen's integral inequality we have

$$\int_{\Omega_i} (\Phi \circ f) w d\mu \geqslant \Phi\left(\frac{\int_{\Omega_i} f w d\mu}{\int_{\Omega_i} w d\mu}\right) \int_{\Omega_i} w d\mu \tag{6}$$

for any $i \in \{1, ..., n\}$.

If we sum the inequality (6) over i from 1 to n we get

$$\sum_{i=1}^n \int_{\Omega_i} (\Phi \circ f) w d\mu \geqslant \sum_{i=1}^n \Phi\left(\frac{\int_{\Omega_i} f w d\mu}{\int_{\Omega_i} w d\mu}\right) \int_{\Omega_i} w d\mu \tag{7}$$

and since

$$\sum_{i=1}^n \int_{\Omega_i} (\Phi \circ f) w d\mu = \int_{\bigcup_{i=1}^n \Omega_i} (\Phi \circ f) w d\mu = \int_\Omega (\Phi \circ f) w d\mu,$$

then from (7) we get the first part of (5).

Let

$$p_i = \int_{\Omega_i} w d\mu > 0, \;\; z_i = \frac{\int_{\Omega_i} f w d\mu}{\int_{\Omega_i} w d\mu} \in [m, M], \;\; i \in \{1, ..., n\}.$$

Then

$$P_n := \sum_{i=1}^{n} p_i = \int_{\bigcup_{i=1}^{n} \Omega_i} w d\mu = \int_{\Omega} w d\mu,$$

and

$$\sum_{i=1}^{n} p_i z_i = \sum_{i=1}^{n} \int_{\Omega_i} f w d\mu = \int_{\bigcup_{i=1}^{n} \Omega_i} f w d\mu = \int_{\Omega} f w d\mu.$$

From Jensen's discrete inequality

$$\frac{1}{P_n} \sum_{i=1}^{n} p_i \Phi(z_i) \geqslant \Phi\left(\frac{\sum_{i=1}^{n} p_i z_i}{P_n}\right)$$

we have

$$\frac{1}{\int_{\Omega_i} w d\mu} \sum_{i=1}^{n} \Phi\left(\frac{\int_{\Omega_i} f w d\mu}{\int_{\Omega_i} w d\mu}\right) \int_{\Omega_i} w d\mu \geqslant \Phi\left(\frac{\int_{\Omega_i} f w d\mu}{\int_{\Omega_i} w d\mu}\right)$$

and the second inequality in (5) is also proved. □

Remark 2.2. *The double inequality (5) is equivalent to*

$$\frac{\int_{\Omega} (\Phi \circ f) w d\mu}{\int_{\Omega} w d\mu} \geqslant \sup_{F_n(\Omega) \in \mathfrak{D}_n(\Omega)} \psi(\Phi, f, w, F_n(\Omega)) \tag{8}$$

and

$$\inf_{F_n(\Omega) \in \mathfrak{D}_n(\Omega)} \psi(\Phi, f, w, F_n(\Omega)) \geqslant \Phi\left(\frac{\int_{\Omega} f w d\mu}{\int_{\Omega} w d\mu}\right), \tag{9}$$

where $n \geqslant 2$.

For $n = 2$ we re-obtain the result from [13] where further applications for f-divergence measures in Information Theory are also given.

We use the following lemma [10].

Lemma 2.3. *Let* $\Phi : I \to \mathbb{R}$ *be a convex function on the interval of real numbers* I *and* $m, M \in \mathbb{R}$, $m < M$ *with* $[m, M] \subset \mathring{I}$, $\mathring{I}$ *is the interior of* I. *If* $g : \Omega \to \mathbb{R}$ *is* μ*-measurable, satisfies the bounds*

$$-\infty < m \leqslant g(x) \leqslant M < \infty \text{ for } \mu\text{-a.e. } x \in \Omega$$

and such that $g, \Phi\circ g \in L_p(\Omega,\mu)$, *where* $p \geqslant 0$ μ*-a.e. on* Ω *with* $\int_\Omega p d\mu = 1$, *then*

$$\begin{aligned}
0 &\leqslant \int_\Omega p\,(\Phi\circ f)\,d\mu - \Phi\left(\int_\Omega p f d\mu\right) \qquad (10)\\
&\leqslant \frac{\left(M - \int_\Omega p f d\mu\right)\left(\int_\Omega p f d\mu - m\right)}{M-m}\sup_{t\in(m,M)}\Psi_\Phi(t;m,M)\\
&\leqslant \left(M - \int_\Omega p f d\mu\right)\left(\int_\Omega p f d\mu - m\right)\frac{\Phi'_-(M)-\Phi'_+(m)}{M-m}\\
&\leqslant \frac{1}{4}(M-m)\left[\Phi'_-(M)-\Phi'_+(m)\right],
\end{aligned}$$

where $\Psi_\Phi(\cdot;m,M):(m,M)\to\mathbb{R}$ *is defined by*

$$\Psi_\Phi(t;m,M) = \frac{\Phi(M)-\Phi(t)}{M-t} - \frac{\Phi(t)-\Phi(m)}{t-m}.$$

We have the following reverse of the first inequality in (5).

Theorem 2.4. *Let* $\Phi:[m,M]\to\mathbb{R}$ *be a convex function,* $f:\Omega\to[m,M]$ *a* μ*-measurable function such that* f, $\Phi\circ f \in L_w(\Omega,\mu)$. *Then for any* $F_n(\Omega)\in\mathfrak{D}_n(\Omega)$, $n\geqslant 2$ *we have*

$$\begin{aligned}
0 &\leqslant \frac{\int_\Omega (\Phi\circ f)\,w d\mu}{\int_\Omega w d\mu} - \psi(\Phi, f, w, F_n(\Omega)) \qquad (11)\\
&\leqslant \frac{1}{M-m}\sup_{t\in(m,M)}\Psi_\Phi(t;m,M)\\
&\times \frac{1}{\int_\Omega w d\mu}\sum_{i=1}^n\left(\int_{\Omega_i} w d\mu\right)\left(M - \frac{\int_{\Omega_i} w f d\mu}{\int_{\Omega_i} w d\mu}\right)\left(\frac{\int_{\Omega_i} w f d\mu}{\int_{\Omega_i} w d\mu} - m\right)\\
&\leqslant \frac{1}{M-m}\sup_{t\in(m,M)}\Psi_\Phi(t;m,M)\left(M - \frac{\int_\Omega w f d\mu}{\int_\Omega w d\mu}\right)\left(\frac{\int_\Omega w f d\mu}{\int_\Omega w d\mu} - m\right).
\end{aligned}$$

Proof. From the second inequality in (10) for $g=f$ and $p=\frac{w}{\int_{\Omega_i} w d\mu}$, $i\in\{1,...,n\}$ we have

$$\begin{aligned}
0 &\leqslant \frac{1}{\int_{\Omega_i} w d\mu}\int_{\Omega_i} w(\Phi\circ f)\,d\mu - \Phi\left(\frac{1}{\int_{\Omega_i} w d\mu}\int_{\Omega_i} w f d\mu\right) \qquad (12)\\
&\leqslant \frac{1}{M-m}\left(M - \frac{\int_{\Omega_i} w f d\mu}{\int_{\Omega_i} w d\mu}\right)\left(\frac{\int_{\Omega_i} w f d\mu}{\int_{\Omega_i} w d\mu} - m\right)\sup_{t\in(m,M)}\Psi_\Phi(t;m,M)
\end{aligned}$$

for any $i \in \{1, ..., n\}$.
If we multiply by $\int_{\Omega_i} w d\mu > 0$ and sum over i from 1 to n we get

$$\begin{aligned}
0 &\leqslant \sum_{i=1}^{n} \int_{\Omega_i} w (\Phi \circ f) d\mu - \sum_{i=1}^{n} \Phi \left(\frac{1}{\int_{\Omega_i} w d\mu} \int_{\Omega_i} w f d\mu \right) \int_{\Omega_i} w d\mu \qquad (13) \\
&\leqslant \frac{1}{M-m} \sup_{t \in (m,M)} \Psi_{\Phi} (t; m, M) \\
&\times \sum_{i=1}^{n} \left(\int_{\Omega_i} w d\mu \right) \left(M - \frac{\int_{\Omega_i} w f d\mu}{\int_{\Omega_i} w d\mu} \right) \left(\frac{\int_{\Omega_i} w f d\mu}{\int_{\Omega_i} w d\mu} - m \right),
\end{aligned}$$

which proves the second inequality in (11).

Now, observe that the function $\Psi : [m, M] \to [0, \infty)$, $\Psi (t) = (M - t) (t - m)$ is concave and by Jensen's inequality for concave functions with

$$p_i = \int_{\Omega_i} w d\mu > 0, \ z_i = \frac{\int_{\Omega_i} w f d\mu}{\int_{\Omega_i} w d\mu} \in [m, M], \ i \in \{1, ..., n\}$$

we have

$$\begin{aligned}
&\frac{1}{\int_{\Omega} w d\mu} \sum_{i=1}^{n} \left(\int_{\Omega_i} w d\mu \right) \left(M - \frac{\int_{\Omega_i} w f d\mu}{\int_{\Omega_i} w d\mu} \right) \left(\frac{\int_{\Omega_i} w f d\mu}{\int_{\Omega_i} w d\mu} - m \right) \\
&\leqslant \left(M - \frac{1}{\int_{\Omega} w d\mu} \sum_{i=1}^{n} \left(\int_{\Omega_i} w d\mu \right) \frac{\int_{\Omega_i} w f d\mu}{\int_{\Omega_i} w d\mu} \right) \\
&\times \left(\frac{1}{\int_{\Omega} w d\mu} \sum_{i=1}^{n} \left(\int_{\Omega_i} w d\mu \right) \frac{\int_{\Omega_i} w f d\mu}{\int_{\Omega_i} w d\mu} - m \right) \\
&= \left(M - \frac{\int_{\Omega} f w d\mu}{\int_{\Omega} w d\mu} \right) \left(\frac{\int_{\Omega} f w d\mu}{\int_{\Omega} w d\mu} - m \right),
\end{aligned}$$

which proves the last part of (11). $\square$

Remark 2.5. *Since, as shown in [10],*

$$\sup_{t \in (m,M)} \Psi_{\Phi} (t; m, M) \leqslant \Phi'_{-} (M) - \Phi'_{+} (m)$$

then we have the following simpler inequality

$$
\begin{aligned}
0 &\leqslant \frac{\int_{\Omega} (\Phi \circ f)\, w d\mu}{\int_{\Omega} w d\mu} - \psi\left(\Phi, f, w, F_n(\Omega)\right) \\
&\leqslant \frac{\Phi'_{-}(M) - \Phi'_{+}(m)}{M - m} \\
&\times \sum_{i=1}^{n} \frac{\int_{\Omega_i} w d\mu}{\int_{\Omega} w d\mu} \left(M - \frac{\int_{\Omega_i} w f d\mu}{\int_{\Omega_i} w d\mu} \right) \left(\frac{\int_{\Omega_i} w f d\mu}{\int_{\Omega_i} w d\mu} - m \right) \\
&\leqslant \frac{\Phi'_{-}(M) - \Phi'_{+}(m)}{M - m} \left(M - \frac{\int_{\Omega} f w d\mu}{\int_{\Omega} w d\mu} \right) \left(\frac{\int_{\Omega} f w d\mu}{\int_{\Omega} w d\mu} - m \right).
\end{aligned} \tag{14}
$$

If we use Lemma 2.3 for the discrete measure, we can state the following result:

Lemma 2.6. *Let $\Phi : [a,b] \to \mathbb{R}$ be a convex function on the interval of real numbers $[a,b]$, $z_i \in [a,b]$, $p_i \geqslant 0$, $i \in \{1, ..., n\}$ and $\sum_{i=1}^{n} p_i = 1$. Then*

$$
\begin{aligned}
0 &\leqslant \sum_{i=1}^{n} p_i \Phi(z_i) - \Phi\left(\sum_{i=1}^{n} p_i z_i \right) \\
&\leqslant \frac{\left(b - \sum_{i=1}^{n} p_i z_i\right)\left(\sum_{i=1}^{n} p_i z_i - a\right)}{b - a} \sup_{t \in (a,b)} \Psi_{\Phi}(t; a, b) \\
&\leqslant \left(b - \sum_{i=1}^{n} p_i z_i \right) \left(\sum_{i=1}^{n} p_i z_i - a \right) \frac{\Phi'_{-}(b) - \Phi'_{+}(a)}{b - a} \\
&\leqslant \frac{1}{4}(b-a)\left[\Phi'_{-}(b) - \Phi'_{+}(a)\right].
\end{aligned} \tag{15}
$$

The following reverse of the second inequality in (5) holds:

Theorem 2.7. *Let $\Phi : [m,M] \to \mathbb{R}$ be a convex function, $f : \Omega \to [m,M]$ a μ-measurable function such that f, $\Phi \circ f \in L_w(\Omega, \mu)$. Then for any $F_n(\Omega) \in \mathfrak{D}_n(\Omega)$, $n \geqslant 2$ we have*

$$
\begin{aligned}
0 &\leqslant \psi\left(\Phi, f, w, F_n(\Omega)\right) - \Phi\left(\frac{\int_{\Omega} f w d\mu}{\int_{\Omega} w d\mu} \right) \\
&\leqslant \frac{\left(\Lambda(F_n(\Omega)) - \frac{\int_{\Omega} f w d\mu}{\int_{\Omega} w d\mu} \right) \left(\frac{\int_{\Omega} f w d\mu}{\int_{\Omega} w d\mu} - \lambda(F_n(\Omega)) \right)}{\Lambda(F_n(\Omega)) - \lambda(F_n(\Omega))} \\
&\times \sup_{t \in (\lambda(F_n(\Omega)), \Lambda(F_n(\Omega)))} \Psi_{\Phi}\left(t; \lambda(F_n(\Omega)), \Lambda(F_n(\Omega))\right) \\
&\leqslant \frac{1}{4}\left(\Lambda(F_n(\Omega)) - \lambda(F_n(\Omega)) \right) \\
&\times \sup_{t \in (\lambda(F_n(\Omega)), \Lambda(F_n(\Omega)))} \Psi_{\Phi}\left(t; \lambda(F_n(\Omega)), \Lambda(F_n(\Omega))\right),
\end{aligned} \tag{16}
$$

where

$$\lambda\left(F_n\left(\Omega\right)\right) := \min_{i\in\{1,...,n\}}\left\{\frac{\int_{\Omega_i} fwd\mu}{\int_{\Omega_i} wd\mu}\right\}, \tag{17}$$

$$\Lambda\left(F_n\left(\Omega\right)\right) := \max_{i\in\{1,...,n\}}\left\{\frac{\int_{\Omega_i} fwd\mu}{\int_{\Omega_i} wd\mu}\right\}.$$

Proof. If we write the first two inequalities in (15) for

$$p_i = \frac{\int_{\Omega_i} wd\mu}{\int_{\Omega} wd\mu} > 0,\ z_i = \frac{\int_{\Omega_i} fwd\mu}{\int_{\Omega_i} wd\mu},\ i \in \{1,...,n\}$$

and for $a = \lambda\left(F_n\left(\Omega\right)\right),\, b = \Lambda\left(F_n\left(\Omega\right)\right)$ as above we have

$$\begin{aligned}
0 &\leqslant \frac{1}{\int_{\Omega} wd\mu}\sum_{i=1}^{n}\left(\int_{\Omega_i} wd\mu\right)\Phi\left(\frac{\int_{\Omega_i} fwd\mu}{\int_{\Omega_i} wd\mu}\right) - \Phi\left(\frac{\sum_{i=1}^{n}\int_{\Omega_i} fwd\mu}{\int_{\Omega} wd\mu}\right) \qquad (18)\\
&\leqslant \frac{\sup_{t\in(\lambda(F_n(\Omega)),\Lambda(F_n(\Omega)))}\Psi_{\Phi}\left(t;\lambda\left(F_n\left(\Omega\right)\right),\Lambda\left(F_n\left(\Omega\right)\right)\right)}{\Lambda\left(F_n\left(\Omega\right)\right)-\lambda\left(F_n\left(\Omega\right)\right)}\\
&\times\left(\Lambda\left(F_n\left(\Omega\right)\right) - \sum_{i=1}^{n}\frac{\int_{\Omega_i} wd\mu}{\int_{\Omega} wd\mu}\cdot\frac{\int_{\Omega_i} fwd\mu}{\int_{\Omega_i} wd\mu}\right)\\
&\times\left(\sum_{i=1}^{n}\frac{\int_{\Omega_i} wd\mu}{\int_{\Omega} wd\mu}\cdot\frac{\int_{\Omega_i} fwd\mu}{\int_{\Omega_i} wd\mu} - \lambda\left(F_n\left(\Omega\right)\right)\right)\\
&= \frac{\left(\Lambda\left(F_n\left(\Omega\right)\right) - \frac{\int_{\Omega} fwd\mu}{\int_{\Omega} wd\mu}\right)\left(\frac{\int_{\Omega} fwd\mu}{\int_{\Omega} wd\mu} - \lambda\left(F_n\left(\Omega\right)\right)\right)}{\Lambda\left(F_n\left(\Omega\right)\right)-\lambda\left(F_n\left(\Omega\right)\right)}\\
&\times \sup_{t\in(\lambda(F_n(\Omega)),\Lambda(F_n(\Omega)))}\Psi_{\Phi}\left(t;\lambda\left(F_n\left(\Omega\right)\right),\Lambda\left(F_n\left(\Omega\right)\right)\right),
\end{aligned}$$

which proves the second inequality in (16).
The last part in (16) follows by the elementary inequality

$$\alpha\beta \leqslant \left(\frac{\alpha+\beta}{2}\right)^2,\ \alpha,\beta\in\mathbb{R}. \quad \square$$

Remark 2.8. *Since*

$$\Psi_{\Phi}\left(t;\lambda\left(F_n\left(\Omega\right)\right),\Lambda\left(F_n\left(\Omega\right)\right)\right) \leqslant \Phi'_-\left(\Lambda\left(F_n\left(\Omega\right)\right)\right) - \Phi'_+\left(\lambda\left(F_n\left(\Omega\right)\right)\right),$$

then from (16) we have the simpler inequalities

$$\begin{aligned}
0 &\leqslant \psi(\Phi, f, w, F_n(\Omega)) - \Phi\left(\frac{\int_\Omega fwd\mu}{\int_\Omega wd\mu}\right) \qquad (19)\\
&\leqslant \frac{\Phi'_-(\Lambda(F_n(\Omega))) - \Phi'_+(\lambda(F_n(\Omega)))}{\Lambda(F_n(\Omega)) - \lambda(F_n(\Omega))}\\
&\times \left(\Lambda(F_n(\Omega)) - \frac{\int_\Omega fwd\mu}{\int_\Omega wd\mu}\right)\left(\frac{\int_\Omega fwd\mu}{\int_\Omega wd\mu} - \lambda(F_n(\Omega))\right)\\
&\leqslant \frac{1}{4}(\Lambda(F_n(\Omega)) - \lambda(F_n(\Omega)))\left[\Phi'_-(\Lambda(F_n(\Omega))) - \Phi'_+(\lambda(F_n(\Omega)))\right].
\end{aligned}$$

The following reverse of Jensen inequality also holds [10]:

Lemma 2.9. *Let $\Phi : I \to \mathbb{R}$ be a convex function on the interval of real numbers I and $m, M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \mathring{I}$. If $g : \Omega \to \mathbb{R}$ is μ-measurable, satisfies the bounds*

$$-\infty < m \leqslant g(x) \leqslant M < \infty \text{ for } \mu\text{-a.e. } x \in \Omega$$

and such that $g, \Phi \circ g \in L_p(\Omega, \mu)$, where $p \geqslant 0$ μ-a.e. on Ω with $\int_\Omega pd\mu = 1$, then

$$\begin{aligned}
0 &\leqslant \int_\Omega p(\Phi \circ f)\, d\mu - \Phi\left(\int_\Omega pfd\mu\right) \qquad (20)\\
&\leqslant \left[\frac{\Phi(m) + \Phi(M)}{2} - \Phi\left(\frac{m+M}{2}\right)\right]\\
&\times \left(1 + \frac{2}{M-m}\left|\int_\Omega pfd\mu - \frac{m+M}{2}\right|\right)\\
&\leqslant 2\left[\frac{\Phi(m) + \Phi(M)}{2} - \Phi\left(\frac{m+M}{2}\right)\right].
\end{aligned}$$

We have the following result:

Theorem 2.10. *Let $\Phi : [m, M] \to \mathbb{R}$ be a convex function, $f : \Omega \to [m, M]$ a μ-measurable function such that f, $\Phi \circ f \in L_w(\Omega, \mu)$. Then for any $F_n(\Omega) \in \mathfrak{D}_n(\Omega)$, $n \geqslant 2$ we have*

$$
\begin{aligned}
0 &\leqslant \frac{\int_{\Omega}(\Phi\circ f)\,wd\mu}{\int_{\Omega}wd\mu}-\psi\left(\Phi,f,w,F_n(\Omega)\right) \qquad (21)\\
&\leqslant\left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right]\\
&\times\left(1+\frac{2}{M-m}\sum_{i=1}^{n}\left|\frac{1}{\int_{\Omega}wd\mu}\int_{\Omega_i}w\left(f-\frac{m+M}{2}\right)d\mu\right|\right)\\
&\leqslant\left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right]\\
&\times\left(1+\frac{2}{M-m}\frac{1}{\int_{\Omega}wd\mu}\int_{\Omega}w\left|f-\frac{m+M}{2}\right|d\mu\right).
\end{aligned}
$$

Proof. From the second inequality in (20) for $g=f$ and $p=\frac{w}{\int_{\Omega_i}wd\mu}$, $i\in\{1,...,n\}$ we have

$$
\begin{aligned}
0 &\leqslant \frac{1}{\int_{\Omega_i}wd\mu}\int_{\Omega_i}w(\Phi\circ f)\,d\mu-\Phi\left(\frac{1}{\int_{\Omega_i}wd\mu}\int_{\Omega_i}wfd\mu\right) \qquad (22)\\
&\leqslant\left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right]\\
&\times\left(1+\frac{2}{M-m}\left|\frac{1}{\int_{\Omega_i}wd\mu}\int_{\Omega_i}wfd\mu-\frac{m+M}{2}\right|\right)
\end{aligned}
$$

for any $i\in\{1,...,n\}$.

If we multiply the inequality (22) by $\int_{\Omega_i}wd\mu>0$ and sum over i from 1 to n we get

$$
\begin{aligned}
0 &\leqslant \sum_{i=1}^{n}\int_{\Omega_i}w(\Phi\circ f)\,d\mu-\sum_{i=1}^{n}\Phi\left(\frac{1}{\int_{\Omega_i}wd\mu}\int_{\Omega_i}wfd\mu\right)\int_{\Omega_i}wd\mu\\
&\leqslant\left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right]\\
&\times\left(\sum_{i=1}^{n}\int_{\Omega_i}wd\mu+\frac{2}{M-m}\sum_{i=1}^{n}\left(\int_{\Omega_i}wd\mu\right)\left|\frac{1}{\int_{\Omega_i}wd\mu}\int_{\Omega_i}wfd\mu-\frac{m+M}{2}\right|\right)\\
&=\left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right]\\
&\times\left(\int_{\Omega}wd\mu+\frac{2}{M-m}\sum_{i=1}^{n}\left|\int_{\Omega_i}w\left(f-\frac{m+M}{2}\right)d\mu\right|\right)
\end{aligned}
$$

any by dividing with $\int_\Omega w d\mu = \sum_{i=1}^n \int_{\Omega_i} w d\mu$, we get the second inequality in (21).
By the properties of modulus we have

$$\begin{aligned}\sum_{i=1}^n \left| \frac{1}{\int_\Omega w d\mu} \int_{\Omega_i} w \left(f - \frac{m+M}{2} \right) d\mu \right| &\leqslant \frac{1}{\int_\Omega w d\mu} \sum_{i=1}^n \int_{\Omega_i} w \left| f - \frac{m+M}{2} \right| d\mu \\ &= \frac{1}{\int_\Omega w d\mu} \int_\Omega w \left| f - \frac{m+M}{2} \right| d\mu\end{aligned}$$

and the last part of (21) is proved. □

If we use Lemma 2.3 for the discrete measure, we can state the following result:

Lemma 2.11. *Let $\Phi : [a,b] \to \mathbb{R}$ be a convex function on the interval of real numbers $[a,b]$, $z_i \in [a,b]$, $p_i \geqslant 0$, $i \in \{1,...,n\}$ and $\sum_{i=1}^n p_i = 1$. Then*

$$\begin{aligned}0 &\leqslant \sum_{i=1}^n p_i \Phi(z_i) - \Phi\left(\sum_{i=1}^n p_i z_i \right) \qquad (23)\\ &\leqslant \left[\frac{\Phi(a) + \Phi(b)}{2} - \Phi\left(\frac{a+b}{2} \right) \right] \left(1 + \frac{2}{b-a} \left| \sum_{i=1}^n p_i z_i - \frac{a+b}{2} \right| \right) \\ &\leqslant 2 \left[\frac{\Phi(a) + \Phi(b)}{2} - \Phi\left(\frac{a+b}{2} \right) \right].\end{aligned}$$

Using this lemma we can state and prove the following result as well:

Theorem 2.12. *Let $\Phi : [m,M] \to \mathbb{R}$ be a convex function, $f : \Omega \to [m,M]$ a μ-measurable function such that f, $\Phi \circ f \in L_w(\Omega,\mu)$. Then for any $F_n(\Omega) \in \mathfrak{D}_n(\Omega)$, $n \geqslant 2$ we have*

$$\begin{aligned}0 &\leqslant \psi(\Phi, f, w, F_n(\Omega)) - \Phi\left(\frac{\int_\Omega f w d\mu}{\int_\Omega w d\mu} \right) \\ &\leqslant \left[\frac{\Phi(\lambda(F_n(\Omega))) + \Phi(\Lambda(F_n(\Omega)))}{2} - \Phi\left(\frac{\lambda(F_n(\Omega)) + \Lambda(F_n(\Omega))}{2} \right) \right] \\ &\times \left(1 + \frac{2}{\Lambda(F_n(\Omega)) - \lambda(F_n(\Omega))} \left| \frac{1}{\int_\Omega w d\mu} \int_\Omega w f d\mu - \frac{\lambda(F_n(\Omega)) + \Lambda(F_n(\Omega))}{2} \right| \right) \\ &\leqslant 2 \left[\frac{\Phi(\lambda(F_n(\Omega))) + \Phi(\Lambda(F_n(\Omega)))}{2} - \Phi\left(\frac{\lambda(F_n(\Omega)) + \Lambda(F_n(\Omega))}{2} \right) \right], \qquad (24)\end{aligned}$$

where $\lambda(F_n(\Omega))$, $\Lambda(F_n(\Omega))$ are defined by (17).

Proof. If we write the first two inequalities in (15) for

$$p_i = \frac{\int_{\Omega_i} w d\mu}{\int_{\Omega} w d\mu} > 0, \ z_i = \frac{\int_{\Omega_i} f w d\mu}{\int_{\Omega_i} w d\mu}, \ i \in \{1, ..., n\}$$

and for $a = \lambda(F_n(\Omega))$, $b = \Lambda(F_n(\Omega))$ as above we have

$$\begin{aligned}
0 &\leqslant \frac{1}{\int_{\Omega} w d\mu} \sum_{i=1}^{n} \left(\int_{\Omega_i} w d\mu \right) \Phi\left(\frac{\int_{\Omega_i} f w d\mu}{\int_{\Omega_i} w d\mu} \right) - \Phi\left(\sum_{i=1}^{n} \frac{\int_{\Omega_i} w d\mu}{\int_{\Omega} w d\mu} \cdot \frac{\int_{\Omega_i} f w d\mu}{\int_{\Omega_i} w d\mu} \right) \\
&\leqslant \left[\frac{\Phi(\lambda(F_n(\Omega))) + \Phi(\Lambda(F_n(\Omega)))}{2} - \Phi\left(\frac{\lambda(F_n(\Omega)) + \Lambda(F_n(\Omega))}{2} \right) \right] \\
&\times \left\{ 1 + \frac{2}{\Lambda(F_n(\Omega)) - \lambda(F_n(\Omega))} \right. \\
&\times \left. \left| \sum_{i=1}^{n} \frac{\int_{\Omega_i} w d\mu}{\int_{\Omega} w d\mu} \frac{\int_{\Omega_i} f w d\mu}{\int_{\Omega_i} w d\mu} - \frac{\lambda(F_n(\Omega)) + \Lambda(F_n(\Omega))}{2} \right| \right\} \\
&= \left[\frac{\Phi(\lambda(F_n(\Omega))) + \Phi(\Lambda(F_n(\Omega)))}{2} - \Phi\left(\frac{\lambda(F_n(\Omega)) + \Lambda(F_n(\Omega))}{2} \right) \right] \\
&\times \left\{ 1 + \frac{2}{\Lambda(F_n(\Omega)) - \lambda(F_n(\Omega))} \right. \\
&\times \left. \left| \frac{1}{\int_{\Omega} w d\mu} \sum_{i=1}^{n} \int_{\Omega_i} w f d\mu - \frac{\lambda(F_n(\Omega)) + \Lambda(F_n(\Omega))}{2} \right| \right\}
\end{aligned}$$

that proves the required inequalities in (24). $\square$

3. Discrete Inequalities

For a nonempty finite family of indices J and positive weights w_j, $j \in J$ we denote $W_J := \sum_{j \in J} w_j$. If $\Phi : [m, M] \to \mathbb{R}$ is a convex function and $x_j \in [m, M]$, $j \in J$ then Jensen's inequality states that

$$\frac{1}{W_J} \sum_{j \in J} w_j \Phi(x_j) \geqslant \Phi\left(\frac{1}{W_J} \sum_{j \in J} w_j x_j \right). \tag{25}$$

Assume that, for $n \geqslant 2$, the family J of indices containing more than n elements and $F_n(J) = \{J_i\}_{i \in \{1, \dots, n\}}$ is a n-division for J, namely $J = \bigcup_{i=1}^{n} J_i$ and $J_i \cap J_j = \emptyset$ for any i, $j \in \{1, ..., n\}$ with $i \neq j$.

For a given $n \geqslant 2$ we denote by $\mathfrak{D}_n(J)$ the set of all n-divisions of J and consider the functional $\psi(\Phi, x, \cdot) : \mathfrak{D}_n(J) \to \mathbb{R}$ defined by

$$\psi(\Phi, x, w, F_n(J)) := \frac{1}{W_J} \sum_{i=1}^{n} W_{J_i} \Phi\left(\frac{1}{W_{J_i}} \sum_{j \in J_i} w_j x_j \right). \tag{26}$$

From the inequality (5) for the discrete measure we have

$$\frac{1}{W_J} \sum_{j \in J} w_j \Phi(x_j) \geqslant \frac{1}{W_J} \sum_{i=1}^{n} W_{J_i} \Phi\left(\frac{1}{W_{J_i}} \sum_{j \in J_i} w_j x_j \right) \geqslant \Phi\left(\frac{1}{W_J} \sum_{j \in J} w_j x_j \right), \tag{27}$$

for any $F_n(J) \in \mathfrak{D}_n(J)$.
From (14) we have

$$\begin{aligned} 0 &\leqslant \frac{1}{W_J} \sum_{j \in J} w_j \Phi(x_j) - \frac{1}{W_J} \sum_{i=1}^{n} W_{J_i} \Phi\left(\frac{1}{W_{J_i}} \sum_{j \in J_i} w_j x_j \right) \\ &\leqslant \frac{\Phi'_-(M) - \Phi'_+(m)}{M - m} \\ &\times \frac{1}{W_J} \sum_{i=1}^{n} W_{J_i} \left(M - \frac{1}{W_{J_i}} \sum_{j \in J_i} w_j x_j \right) \left(\frac{1}{W_{J_i}} \sum_{j \in J_i} w_j x_j - m \right) \\ &\leqslant \frac{\Phi'_-(M) - \Phi'_+(m)}{M - m} \left(M - \frac{1}{W_J} \sum_{j \in J} w_j x_j \right) \left(\frac{1}{W_J} \sum_{j \in J} w_j x_j - m \right), \end{aligned} \tag{28}$$

while from (19) we have

$$\begin{aligned} 0 &\leqslant \frac{1}{W_J} \sum_{i=1}^{n} W_{J_i} \Phi\left(\frac{1}{W_{J_i}} \sum_{j \in J_i} w_j x_j \right) - \Phi\left(\frac{1}{W_J} \sum_{j \in J} w_j x_j \right) \\ &\leqslant \frac{\Phi'_-(\Lambda(F_n(J))) - \Phi'_+(\lambda(F_n(J)))}{\Lambda(F_n(J)) - \lambda(F_n(J))} \\ &\times \left(\Lambda(F_n(J)) - \frac{1}{W_J} \sum_{j \in J} w_j x_j \right) \left(\frac{1}{W_J} \sum_{j \in J} w_j x_j - \lambda(F_n(J)) \right) \\ &\leqslant \frac{1}{4} \left(\Lambda(F_n(J)) - \lambda(F_n(J)) \right) \left[\Phi'_-(\Lambda(F_n(J))) - \Phi'_+(\lambda(F_n(J))) \right], \end{aligned} \tag{29}$$

for any $F_n(J) \in \mathfrak{D}_n(J)$, where

$$\lambda(F_n(J)) := \min_{i\in\{1,\dots,n\}} \left\{ \frac{1}{W_{J_i}} \sum_{j\in J_i} w_j x_j \right\}, \tag{30}$$
$$\Lambda(F_n(J)) := \max_{i\in\{1,\dots,n\}} \left\{ \frac{1}{W_{J_i}} \sum_{j\in J_i} w_j x_j \right\}.$$

Using (21) we have

$$\begin{aligned}
0 &\leqslant \frac{1}{W_J}\sum_{j\in J} w_j \Phi(x_j) - \frac{1}{W_J}\sum_{i=1}^{n} W_{J_i}\Phi\left(\frac{1}{W_{J_i}}\sum_{j\in J_i} w_j x_j\right) \\
&\leqslant \left[\frac{\Phi(m)+\Phi(M)}{2} - \Phi\left(\frac{m+M}{2}\right)\right] \\
&\times \left(1 + \frac{2}{M-m}\frac{1}{W_J}\sum_{i=1}^{n}\left|\sum_{j\in J_i} w_j\left(x_j - \frac{m+M}{2}\right)\right|\right) \\
&\leqslant \left[\frac{\Phi(m)+\Phi(M)}{2} - \Phi\left(\frac{m+M}{2}\right)\right] \\
&\times \left(1 + \frac{2}{M-m}\frac{1}{W_J}\sum_{j\in J} w_j\left|x_j - \frac{m+M}{2}\right|\right)
\end{aligned} \tag{31}$$

while from (24) we get $n \geqslant 2$ we have

$$\begin{aligned}
0 &\leqslant \frac{1}{W_J}\sum_{i=1}^{n} W_{J_i}\Phi\left(\frac{1}{W_{J_i}}\sum_{j\in J_i} w_j x_j\right) - \Phi\left(\frac{1}{W_J}\sum_{j\in J} w_j x_j\right) \\
&\leqslant \left[\frac{\Phi(\lambda(F_n(J))) + \Phi(\Lambda(F_n(J)))}{2} - \Phi\left(\frac{\lambda(F_n(J)) + \Lambda(F_n(J))}{2}\right)\right] \\
&\times \left(1 + \frac{2}{\Lambda(F_n(J)) - \lambda(F_n(J))}\left|\frac{1}{W_J}\sum_{j\in J} w_j x_j - \frac{\lambda(F_n(J)) + \Lambda(F_n(J))}{2}\right|\right) \\
&\leqslant 2\left[\frac{\Phi(\lambda(F_n(J))) + \Phi(\Lambda(F_n(J)))}{2} - \Phi\left(\frac{\lambda(F_n(J)) + \Lambda(F_n(J))}{2}\right)\right]
\end{aligned} \tag{32}$$

for any $F_n(J) \in \mathfrak{D}_n(J)$.

We consider the convex function $\Phi : (0,\infty) \to (0,\infty)$, $\Phi(t) = t^p$ with $p \in (-\infty, 0) \cup (1,\infty)$. Then from (27)

$$\frac{1}{W_J}\sum_{j\in J} w_j \Phi(x_j) \geqslant \frac{1}{W_J}\sum_{i=1}^{n} W_{J_i}^{1-p}\left(\sum_{j\in J_i} w_j x_j\right)^p \geqslant \Phi\left(\frac{1}{W_J}\sum_{j\in J} w_j x_j\right), \quad (33)$$

for any $x_j > 0$, $j \in J$.

If we set $a = \min\{x_j\}_{j\in J}$ and $b = \max\{x_j\}_{j\in J}$ then from (28) and (31) we have

$$\begin{aligned}
0 &\leqslant \frac{1}{W_J}\sum_{j\in J} w_j x_j^p - \frac{1}{W_J}\sum_{i=1}^{n} W_{J_i}^{1-p}\left(\sum_{j\in J_i} w_j x_j\right)^p \qquad (34)\\
&\leqslant p\frac{b^{p-1}-a^{p-1}}{M-m}\\
&\times \frac{1}{W_J}\sum_{i=1}^{n} W_{J_i}\left(b - \frac{1}{W_{J_i}}\sum_{j\in J_i} w_j x_j\right)\left(\frac{1}{W_{J_i}}\sum_{j\in J_i} w_j x_j - a\right)\\
&\leqslant p\frac{b^{p-1}-a^{p-1}}{M-m}\left(b - \frac{1}{W_J}\sum_{j\in J} w_j x_j\right)\left(\frac{1}{W_J}\sum_{j\in J} w_j x_j - a\right)
\end{aligned}$$

and

$$\begin{aligned}
0 &\leqslant \frac{1}{W_J}\sum_{j\in J} w_j x_j^p - \frac{1}{W_J}\sum_{i=1}^{n} W_{J_i}^{1-p}\left(\sum_{j\in J_i} w_j x_j\right)^p \qquad (35)\\
&\leqslant \left[\frac{a^p+b^p}{2} - \left(\frac{a+b}{2}\right)^p\right]\\
&\times \left(1 + \frac{2}{b-a}\frac{1}{W_J}\sum_{i=1}^{n}\left|\sum_{j\in J_i} w_j\left(x_j - \frac{a+b}{2}\right)\right|\right)\\
&\leqslant \left[\frac{a^p+b^p}{2} - \left(\frac{a+b}{2}\right)^p\right]\\
&\times \left(1 + \frac{2}{b-a}\frac{1}{W_J}\sum_{j\in J} w_j\left|x_j - \frac{a+b}{2}\right|\right).
\end{aligned}$$

If we use the inequality (29) and (32) we have

$$\begin{aligned} 0 &\leqslant \frac{1}{W_J}\sum_{i=1}^{n} W_{J_i}^{1-p}\left(\sum_{j\in J_i} w_j x_j\right)^p - \left(\frac{1}{W_J}\sum_{j\in J} w_j x_j\right)^p \\ &\leqslant p\frac{\Lambda^{p-1}\left(F_n\left(J\right)\right)-\lambda^{p-1}\left(F_n\left(J\right)\right)}{\Lambda\left(F_n\left(J\right)\right)-\lambda\left(F_n\left(J\right)\right)} \\ &\times\left(\Lambda\left(F_n\left(J\right)\right)-\frac{1}{W_J}\sum_{j\in J} w_j x_j\right)\left(\frac{1}{W_J}\sum_{j\in J} w_j x_j-\lambda\left(F_n\left(J\right)\right)\right) \\ &\leqslant \frac{1}{4}p\left(\Lambda\left(F_n\left(J\right)\right)-\lambda\left(F_n\left(J\right)\right)\right)\left[\Lambda^{p-1}\left(F_n\left(J\right)\right)-\lambda^{p-1}\left(F_n\left(J\right)\right)\right], \end{aligned} \tag{36}$$

and

$$\begin{aligned} 0 &\leqslant \frac{1}{W_J}\sum_{i=1}^{n} W_{J_i}^{1-p}\left(\sum_{j\in J_i} w_j x_j\right)^p - \left(\frac{1}{W_J}\sum_{j\in J} w_j x_j\right)^p \\ &\leqslant \left[\frac{\lambda^p\left(F_n\left(J\right)\right)+\Lambda^p\left(F_n\left(J\right)\right)}{2}-\left(\frac{\lambda\left(F_n\left(J\right)\right)+\Lambda\left(F_n\left(J\right)\right)}{2}\right)^p\right] \\ &\times\left(1+\frac{2}{\Lambda\left(F_n\left(J\right)\right)-\lambda\left(F_n\left(J\right)\right)}\left|\frac{1}{W_J}\sum_{j\in J} w_j x_j-\frac{\lambda\left(F_n\left(J\right)\right)+\Lambda\left(F_n\left(J\right)\right)}{2}\right|\right) \\ &\leqslant 2\left[\frac{\lambda^p\left(F_n\left(J\right)\right)+\Lambda^p\left(F_n\left(J\right)\right)}{2}-\left(\frac{\lambda\left(F_n\left(J\right)\right)+\Lambda\left(F_n\left(J\right)\right)}{2}\right)^p\right] \end{aligned} \tag{37}$$

where $\lambda\left(F_n\left(J\right)\right), \Lambda\left(F_n\left(J\right)\right)$ are defined by (30).

4. Some Inequalities Related to HH-Inequality

Let $\Phi : [m, M] \to \mathbb{R}$ be a convex function and $f : [a, b] \to [m, M]$ an integrable function. Consider the division of the interval $[a, b]$ given by

$$d_n : a = x_0 < x_{1<} ... < x_{n-1} < x_n = b,\ n \geqslant 2.$$

If we take $\Omega = [a, b]$ and $\Omega_1 = [a, x_1]$, $\Omega_i = (x_i, x_{i+1}]$ for $i \in \{1, ..., n-1\}$ then $\Omega = \bigcup_{i=1}^{n} \Omega_i$ and $\Omega_i \cap \Omega_j = \emptyset$ for any $i, j \in \{1, ..., n\}$ with $i \neq j$.
By making use of (4) we can consider the functional

$$\psi(\Phi, f, w, d_n) := \frac{1}{\int_a^b w(x)\,dx} \sum_{i=1}^{n} \Phi\left(\frac{\int_{x_{i-1}}^{x_i} w(x) f(x)\,dx}{\int_{x_{i-1}}^{x_i} w(x)\,dx}\right) \int_{x_{i-1}}^{x_i} w(x)\,dx, \tag{38}$$

where $w : [a,b] \to (0,\infty)$ is an integrable weight.

It is clear that all inequalities from Section 2 can be written for univariate functions f and the functional (38). We are, however, interested here in the particular case that is related to the celebrated *Hermite-Hadamard inequality*

$$\Phi\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b \Phi(t)\,dt \leqslant \frac{\Phi(a)+\Phi(b)}{2},$$

where $\Phi : [a,b] \to \mathbb{R}$ is a convex function on $[a,b]$.

Now, if we take in (38) $w(t) = 1$, $f(t) = t$, $t \in [m,M] = [a,b]$ then we can consider the simpler functional

$$\psi(\Phi, d_n) := \frac{1}{b-a}\sum_{i=1}^{n} \Phi\left(\frac{x_i + x_{i-1}}{2}\right)(x_i - x_{i-1}). \tag{39}$$

From (5) we then have

$$\frac{1}{b-a}\int_a^b \Phi(t)\,dt \geqslant \frac{1}{b-a}\sum_{i=1}^{n} \Phi\left(\frac{x_i + x_{i-1}}{2}\right)(x_i - x_{i-1}) \geqslant \Phi\left(\frac{a+b}{2}\right). \tag{40}$$

This inequality was obtained by the author in 1994, [2] (see also [14, p. 22]).

If we use the inequality (14), then we have

$$\begin{aligned} 0 &\leqslant \frac{1}{b-a}\int_a^b \Phi(t)\,dt - \frac{1}{b-a}\sum_{i=1}^{n} \Phi\left(\frac{x_i + x_{i-1}}{2}\right)(x_i - x_{i-1}) \qquad (41)\\ &\leqslant \frac{\Phi'_-(b) - \Phi'_+(a)}{b-a} \\ &\times \frac{1}{b-a}\sum_{i=1}^{n} (x_i - x_{i-1})\left(b - \frac{x_i + x_{i-1}}{2}\right)\left(\frac{x_i + x_{i-1}}{2} - a\right) \\ &\leqslant \frac{1}{4}(b-a)\left[\Phi'_-(b) - \Phi'_+(a)\right], \end{aligned}$$

while from (21), the inequality

$$0 \leqslant \frac{1}{b-a}\int_a^b \Phi(t)\,dt - \frac{1}{b-a}\sum_{i=1}^{n}\Phi\left(\frac{x_i+x_{i-1}}{2}\right)(x_i-x_{i-1}) \tag{42}$$
$$\leqslant \left[\frac{\Phi(a)+\Phi(b)}{2} - \Phi\left(\frac{a+b}{2}\right)\right]$$
$$\times \left(1+\frac{2}{(b-a)^2}\sum_{i=1}^{n}(x_i-x_{i-1})\left|\frac{x_i+x_{i-1}}{2}-\frac{a+b}{2}\right|\right)$$
$$\leqslant \frac{3}{2}\left[\frac{\Phi(a)+\Phi(b)}{2} - \Phi\left(\frac{a+b}{2}\right)\right].$$

Consider

$$\lambda(d_n) := \min_{i\in\{1,\dots,n\}}\left\{\frac{x_i+x_{i-1}}{2}\right\} = \frac{x_1+a}{2}, \tag{43}$$
$$\Lambda(d_n) := \max_{i\in\{1,\dots,n\}}\left\{\frac{x_i+x_{i-1}}{2}\right\} = \frac{x_{n-1}+b}{2}.$$

Then by (19) we have

$$0 \leqslant \frac{1}{b-a}\sum_{i=1}^{n}\Phi\left(\frac{x_i+x_{i-1}}{2}\right)(x_i-x_{i-1}) - \Phi\left(\frac{a+b}{2}\right) \tag{44}$$
$$\leqslant \frac{\Phi'_-\left(\frac{x_{n-1}+b}{2}\right)-\Phi'_+\left(\frac{x_1+a}{2}\right)}{\frac{x_{n-1}+b}{2}-\frac{x_1+a}{2}}\left(\frac{x_{n-1}-a}{2}\right)\left(\frac{b-x_1}{2}\right)$$
$$\leqslant \frac{1}{4}\left(\frac{x_{n-1}+b}{2}-\frac{x_1+a}{2}\right)\left(\Phi'_-\left(\frac{x_{n-1}+b}{2}\right)-\Phi'_+\left(\frac{x_1+a}{2}\right)\right),$$

while from (24) we have

$$0 \leqslant \frac{1}{b-a}\sum_{i=1}^{n}\Phi\left(\frac{x_i+x_{i-1}}{2}\right)(x_i-x_{i-1}) - \Phi\left(\frac{a+b}{2}\right) \tag{45}$$
$$\leqslant \left[\frac{\Phi\left(\frac{x_1+a}{2}\right)+\Phi\left(\frac{x_{n-1}+b}{2}\right)}{2} - \Phi\left(\frac{a+x_1+x_{n-1}+b}{4}\right)\right]$$
$$\times \left(1+\left|\frac{a+b-x_1-x_n}{x_{n-1}+b-x_1-a}\right|\right)$$
$$\leqslant 2\left[\frac{\Phi\left(\frac{x_1+a}{2}\right)+\Phi\left(\frac{x_{n-1}+b}{2}\right)}{2} - \Phi\left(\frac{a+x_1+x_{n-1}+b}{4}\right)\right].$$

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