Some Hermite-Hadamard type Inequalities for Functions whose Partial Derivatives in Absloute Value are Preinvex on the Co-oordinates

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SOME HERMITE-HADAMARD TYPE INEQUALITIES FOR FUNCTIONS WHOSE PARTIAL DERIVATIVES IN ABSOLUTE VALUE ARE PREINVEX ON THE CO-ORDINATES

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Abstract. In this paper we point out some inequalities of Hermite-Hadamard type for double integrals of functions whose partial derivatives in absolute value are preinvex on the co-ordinates on rectangle from the plane. Our established results generalize some recent results for functions whose partial derivatives in absolute value are convex on the co-ordinates on the rectangle from the plane.

1. Introduction

The following definition is well known in literature:

A function $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on I if the inequality

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$.

Many important inequalities have been established for the class of convex functions but the most famous is the Hermite-Hadamard's inequality. This double inequality is stated as:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$ is a convex function, $a, b \in I$ with $a < b$. The inequalities in (1.1) are in reversed order if f is a concave function.

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The inequalities (1.1) have become an important cornerstone in mathematical analysis and optimization and many uses of these inequalities have been discovered in a variety of settings. The Hermite-Hadamard inequality (1.1) has been extended, refined and generalized in a number of ways, see for instance [6, 7, 9, 20, 24, 29, 32, 34] and the references therein.

In recent years, lot of efforts have been made by many mathematicians to generalize the classical convexity. These studies include among others the work of Hanson in [12], Ben-Israel and Mond [6], Pini [26], M.A.Noor [21, 22], Yang and Li [35] and Weir [34]. Mond [6], Weir [34] and Noor [21, 22], have studied the basic properties of the preinvex functions and their role in optimization, variational inequalities and equilibrium problems. In [12], Hanson introduced invex functions as a significant generalization of convex functions. Ben-Israel and Mond [6], gave the concept of preinvex function which is special case of invexity. Pini [26], introduced the concept of prequasiinvex functions as a generalization of invex functions.

Let us recall some known results concerning invexity and preinvexity

Let K be a closed set in $\mathbb{R}^n$ and let $f : K \to \mathbb{R}$ and $\eta : K \times K \to \mathbb{R}^n$ be continuous functions. Let $x \in K$, then the set K is said to be invex at x with respect to $\eta(\cdot,\cdot)$, if

$$x + t\eta(y,x) \in K, \forall x, y \in K, t \in [0,1].$$

K is said to be an invex set with respect to η if K is invex at each $x \in K$. The invex set K is also called a η-connected set.

Definition 1.1. [34]The function f on the invex set K is said to be preinvex with respect to η, if

$$f(u + t\eta(v,u)) \le (1-t) f(u) + t f(v), \forall u, v \in K, t \in [0,1].$$

The function f is said to be preconcave if and only if $-f$ is preinvex.

It is to be noted that every convex function is preinvex with respect to the map $\eta(x,y) = x - y$ but the converse is not true see for instance [34].

In the recent paper, Noor [20] has obtained the following Hermite-Hadamard inequalities for the preinvex functions:

Theorem 1.1. *[20]Let $f : [a, a+\eta(b,a)] \to (0,\infty)$ be a preinvex function on the interval of the real numbers K° (the interior of K) and $a, b \in K^\circ$ with $a < a+\eta(b,a)$. Then the following inequality holds:*

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \le \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx \le \frac{f(a)+f(b)}{2}. \tag{1.2}$$

Barani, Ghazanfari and Dragomir in [2], presented the following estimates of the right-side of a Hermite- Hadamard type inequality in which some preinvex functions are involved.

Theorem 1.2. *[2]Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. Assume $p \in \mathbb{R}$ with $p > 1$. If $\left|f'\right|^{\frac{p}{p-1}}$ is preinvex on K then, for every $a, b \in K$ with $\eta(b,a) \neq 0$,*

$$\left|\frac{f(a) + f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \leq \frac{\eta(b,a)}{2(1+p)^{\frac{1}{p}}}\left[\frac{\left|f'(a)\right|^{\frac{p}{p-1}} + \left|f'(b)\right|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}}. \tag{1.3}$$

Theorem 1.3. *[2]Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. If $\left|f'\right|$ is preinvex on K then, for every $a, b \in K$ with $\eta(b,a) \neq 0$,*

$$\left|\frac{f(a) + f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \leq \frac{\eta(b,a)}{8}\left(\left|f'(a)\right| + \left|f'(b)\right|\right). \tag{1.4}$$

For more recent results on Hermite-Hadamard type inequalities for preinvex, log-preinvex functions, we refer the readers to the latest papers of M. Z. Sarikaya et. al , [30].

Let us consider now a bidimensional interval $\Delta =: [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. A mapping $f : \Delta \to \mathbb{R}$ is said to be convex on Δ if the inequality

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x,y) + (1-\lambda) f(z,w),$$

holds for all $(x,y),(z,w) \in \Delta$ and $\lambda \in [0,1]$.

A modification for convex functions on Δ, known as co-ordinated convex functions, was introduced by S. S. Dragomir [10] as follows:

A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ if the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ are convex where defined for all $x \in [a,b]$, $y \in [c,d]$.

A formal definition for co-ordinated convex functions may be stated as follows:

Definition 1.2. [15] A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ if the inequality

$$f(tx + (1-t)y, su + (1-s)w) \leq tsf(x,u) + t(1-s)f(x,w) + s(1-t)f(y,u) + (1-t)(1-s)f(y,w),$$

holds for all $t, s \in [0,1]$ and $(x,u),(y,w) \in \Delta$.

Clearly, every convex mapping $f : \Delta \rightarrow \mathbb{R}$ is convex on the co-ordinates but converse may not be true [10].

The following Hermite-Hadamard type for co-ordinated convex functions on the rectangle from the plane $\mathbb{R}^2$ were established in [10]:

Theorem 1.4. *[10] Suppose that $f : \Delta \rightarrow \mathbb{R}$ is co-ordinated convex on Δ, then*

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right] \tag{1.5}$$
$$\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx$$
$$\leq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b [f(x,c)+f(x,d)]\,dx + \frac{1}{d-c}\int_c^d [f(a,y)+f(b,y)]\,dy\right]$$
$$\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.$$

The above inequalities are sharp.

For several recent results on Hermite-Hadamard type inequalities for functions that satisfy different kinds of convexity on the co-ordinates on the rectangle from the plane $\mathbb{R}^2$ we refer the reader to [1]-[4], [10]-[11], [15]-[17], [25]-[28] and [33].

By using the following lemma:

Lemma 1.1. [21, Lemma 1] *Let $f : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ be a partial differentiable mapping on $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$, $c < d$. If $\frac{\partial^2 f}{\partial t \partial s} \in L(\Delta)$, then the following equality holds:*

$$\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \tag{1.6}$$
$$-\frac{1}{2}\left[\frac{1}{b-a}\int_a^b [f(x,c)+f(x,d)]\,dx + \frac{1}{d-c}\int_c^d [f(a,y)+f(b,y)]\,dy\right]$$
$$= \frac{(b-a)(d-c)}{4}\int_0^1\int_0^1 (1-2t)(1-2s)\frac{\partial^2 f(ta+(1-t)b, sc+(1-s)d)}{\partial t \partial s}dtds.$$

Sarikaya, et. al [33], proved the following Hermite-Hadamard type inequalities for differentiable co-ordinated convex functions:

Theorem 1.5. [21, Theorem 2, Page 4] *Let $f : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ be a partial differentiable mapping on $\Delta := [a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a < b$, $c < d$. If $\left|\frac{\partial^2 f}{\partial t \partial s}\right|$ is convex on the co-ordinates* *on Δ, then one has the inequalities:*

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx-A\right| \le \frac{(b-a)(d-c)}{16}\left(\frac{\left|\frac{\partial^2 f(a,c)}{\partial t\partial s}\right|+\left|\frac{\partial^2 f(a,d)}{\partial t\partial s}\right|+\left|\frac{\partial^2 f(b,c)}{\partial t\partial s}\right|+\left|\frac{\partial^2 f(b,d)}{\partial t\partial s}\right|}{4}\right), \tag{1.7}$$

where

$$A=\frac{1}{2}\left[\frac{1}{b-a}\int_a^b[f(x,c)+f(x,d)]\,dx+\frac{1}{d-c}\int_c^d[f(a,y)+f(b,y)]\,dy\right].$$

Theorem 1.6. [33, Theorem 3, Page 6-7] *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$, $c<d$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$, $q>1$, is convex on the co-ordinates on Δ, then one has the inequalities:*

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx-A\right| \le \frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}}\left(\frac{\left|\frac{\partial^2 f(a,c)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2 f(a,d)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2 f(b,c)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2 f(b,d)}{\partial t\partial s}\right|^q}{4}\right)^{\frac{1}{q}}, \tag{1.8}$$

where

$$A=\frac{1}{2}\left[\frac{1}{b-a}\int_a^b[f(x,c)+f(x,d)]\,dx+\frac{1}{d-c}\int_c^d[f(a,y)+f(b,y)]\,dy\right]$$

and $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 1.7. *[33, Theorem 4, Page 8-9]Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$, $c<d$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$, $q\ge 1$, is convex on the co-ordinates on Δ, then one has the inequalities:*

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx-A\right| \le \frac{(b-a)(d-c)}{16}\left(\frac{\left|\frac{\partial^2 f(a,c)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2 f(a,d)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2 f(b,c)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2 f(b,d)}{\partial t\partial s}\right|^q}{4}\right)^{\frac{1}{q}}, \tag{1.9}$$

where

$$A=\frac{1}{2}\left[\frac{1}{b-a}\int_a^b[f(x,c)+f(x,d)]\,dx+\frac{1}{d-c}\int_c^d[f(a,y)+f(b,y)]\,dy\right].$$

In a recent paper, M. Matloka [19] introduced a new class of functions which are (h_1, h_2)-preinvex on the co-ordinates and established some Hermite-Hadamard and Fejér type inequalities for this class of functions.

Motivated by the results established in [19], the main aim of the present paper is to define preinvex functions on the co-ordinates and to establish some Hermite-Hadamard type inequalities for functions whose second order partial derivatives in absolute value are preinvex on the co-ordinates. Our established results generalize those result proved above in Theorem 1.5-Theorem 1.7.

2. Main Results

In this section we first give notion of preinvex functions on the co-ordinates which generalize the classical convexity on the co-ordinates and then we prove some inequalities of Hermite-Hadamard type for such functions.

Definition 2.1. [19] Let K_1 and K_2 be non-empty subsets of $\mathbb{R}^n$ and let $\eta_1 : K_1 \times K_1 \to \mathbb{R}^n$ and $\eta_2 : K_2 \times K_2 \to \mathbb{R}^n$. We say $K_1 \times K_2$ is invex with respect to η_1 and η_2 at $(u, v) \in K_1 \times K_2$ if for each $(x, y) \in K_1 \times K_2$ and $t, s \in [0, 1]$, we have

$$\left(u + t\eta_1(x, u), v + s\eta_2(y, v)\right) \in K_1 \times K_2.$$

$K_1 \times K_2$ is said to be invex set with respect to η_1 and η_2 if $K_1 \times K_2$ is invex at each $(u, v) \in K_1 \times K_2$.

Definition 2.2. Let $K_1 \times K_2$ is invex set with respect to $\eta_1 : K_1 \times K_1 \to \mathbb{R}^n$ and $\eta_2 : K_2 \times K_2 \to \mathbb{R}^n$. A function $f : K_1 \times K_2 \to \mathbb{R}$ is said to be preinvex if for every $(x, y), (u, v) \in K_1 \times K_2$ and $t \in [0, 1]$, we have

$$f\left(u + t\eta_1(x, u), v + t\eta_2(y, v)\right) \le (1 - t)\, f(x, y) + t f(u, v).$$

Definition 2.3. Let $K_1 \times K_2$ be an invex set with respect to $\eta_1 : K_1 \times K_1 \to \mathbb{R}^n$ and $\eta_2 : K_2 \times K_2 \to \mathbb{R}^n$. A function $f : K_1 \times K_2 \to \mathbb{R}$ is said to preinvex on the co-ordinates if the partial mappings $f_y : K_1 \to \mathbb{R}$, $f_y(u) = f(u, y)$ and $f_x : K_2 \to \mathbb{R}$, $f_x(v) = f(x, v)$ are preinvex with respect to η_1 and η_2 respectively for all $y \in K_2$ and $x \in K_1$.

Remark 2.1. If $\eta_1(x, u) = x - u$ and $\eta_2(y, v) = y - v$ then f will be a convex function on the coordinates.

Remark 2.2. From the Definition 2.3 it follows that if f is preinvex on the co-ordinates on $K_1 \times K_2$ then

$$\begin{aligned} f\left(u + t\eta_1(x, u), v + s\eta_2(y, v)\right) &\le (1 - t)(1 - s)\, f(u, v) + (1 - t)\, s f(u, y) \\ &\quad + (1 - s)\, t f(x, v) + ts f(x, y). \end{aligned}$$

Remark 2.3. Every convex function on the co-ordinates is preinvex on the co-ordinates but the converse in not true. For example the function $f(u,v) = -|u||v|$ is not convex on the co-ordinates but it is a preinvex function on the co-ordinates with respect to the mappings

$$\eta_1(u,z) = \begin{cases} u-z, & u \geq 0, z \geq 0 \text{ and } u \leq 0, z \leq 0 \\ z-u, & \text{otherwise} \end{cases}$$

and

$$\eta_2(v,w) = \begin{cases} v-w, & v \geq 0, w \geq 0 \text{ and } v \leq 0, w \leq 0 \\ w-v, & \text{otherwise} \end{cases}.$$

The following Lemma is essential to establish our results:

Lemma 2.1. *Let $K_1 \times K_2$ be an open invex subset of $\mathbb{R}^2$ with respect to the mappings $\eta_1 : K_1 \times K_1 \to \mathbb{R}$ and $\eta_2 : K_2 \times K_2 \to \mathbb{R}$. Suppose $f : K_1 \times K_2 \to \mathbb{R}$ be a twice partial differentiable mapping such that $\frac{\partial^2 f}{\partial t \partial s} \in L([a, a+t\eta_1(b,a)] \times [c, c+s\eta_2(d,c)])$ with $\eta_1(b,a) \neq 0$, $\eta_2(d,c) \neq 0$, where $a, b \in K_1$ and $c, d \in K_2$. Then the following equality holds:*

$$\begin{aligned} &\frac{f(a,c) + f(a, c+\eta_2(d,c)) + f(a+\eta_1(b,a), c) + f(a+\eta_1(b,a), c+\eta_2(d,c))}{4} \\ &\quad + \frac{1}{\eta_1(b,a)\eta_2(d,c)} \int_a^{a+\eta_1(b,a)} \int_c^{c+\eta_2(d,c)} f(x,y)\,dydx - A' \\ &\qquad = \frac{\eta_1(b,a)\eta_2(d,c)}{4} \int_0^1 \int_0^1 (1-2t)(1-2s) \\ &\qquad\qquad \times \frac{\partial^2 f(a+t\eta_1(b,a), c+s\eta_2(d,c))}{\partial t \partial s} dtds, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} A' = &\frac{1}{2\eta_1(b,a)} \int_a^{a+\eta_1(b,a)} [f(x,c) + f(x, c+\eta_2(d,c))]\,dx \\ &+ \frac{1}{2\eta_2(d,c)} \int_c^{c+\eta_2(d,c)} [f(a,y) + f(a+\eta_1(b,a), y)]\,dy. \end{aligned}$$

Proof. By integration by parts with respect to t, we have

$$\frac{\eta_1(b,a)\eta_2(d,c)}{4}\int_0^1(1-2s)\left[\frac{-\frac{\partial f(a+\eta_1(b,a),c+s\eta_2(d,c))}{\partial s}-\frac{\partial f(a,c+s\eta_2(d,c))}{\partial s}}{\eta_1(b,a)}\right.$$
$$\left.+\frac{2}{\eta_1(b,a)}\int_0^1\frac{\partial f(a+t\eta_1(b,a),c+s\eta_2(d,c))}{\partial s}dt\right]ds$$
$$=-\frac{\eta_2(d,c)}{4}\int_0^1(1-2s)\frac{\partial f(a+\eta_1(b,a),c+s\eta_2(d,c))}{\partial s}ds$$
$$-\frac{\eta_2(d,c)}{4}\int_0^1(1-2s)\frac{\partial f(a,c+s\eta_2(d,c))}{\partial s}ds$$
$$+\frac{\eta_2(d,c)}{2}\int_0^1\int_0^1(1-2s)\frac{\partial f(a+t\eta_1(b,a),c+s\eta_2(d,c))}{\partial s}dsdt. \tag{2.2}$$

Integrating each integral on right hand side of (2.2) by parts with respect to s and using the substitution $x = a + t\eta_1(b,a)$ and $y = c + s\eta_2(d,c)$, we get the desired identity. This completes the proof of the lemma. □

Theorem 2.1. *Let $K_1 \times K_2$ be an open invex subset of $\mathbb{R}^2$ with respect to the mappings $\eta_1 : K_1 \times K_1 \to \mathbb{R}$ and $\eta_2 : K_2 \times K_2 \to \mathbb{R}$. Suppose $f : K_1 \times K_2 \to \mathbb{R}$ be a twice partial differentiable mapping such that $\frac{\partial^2 f}{\partial t\partial s} \in L([a,a+t\eta_1(b,a)]\times[c,c+s\eta_2(d,c)])$ with $\eta_1(b,a) > 0$, $\eta_2(d,c) > 0$, where $a, b \in K_1$ and $c, d \in K_2$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ is preinvex on the co-ordinates on $K_1 \times K_2$, then the following inequality holds:*

$$\left|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}\right.$$
$$\left.+\frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)}f(x,y)\,dydx-A'\right|$$
$$\le\frac{\eta_1(b,a)\eta_2(d,c)}{16}\left(\frac{\left|\frac{\partial^2 f(a,c)}{\partial t\partial s}\right|+\left|\frac{\partial^2 f(a,d)}{\partial t\partial s}\right|+\left|\frac{\partial^2 f(b,c)}{\partial t\partial s}\right|+\left|\frac{\partial^2 f(b,d)}{\partial t\partial s}\right|}{4}\right), \tag{2.3}$$

where A' is as defined in Lemma 2.1.

Proof. From Lemma 2.1 we have:

$$\left|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}\right.$$
$$\left.+\frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)}f(x,y)\,dydx-A'\right|$$
$$\le\frac{\eta_1(b,a)\eta_2(d,c)}{4}\int_0^1\int_0^1|1-2t||1-2s|\left|\frac{\partial^2 f(a+t\eta_1(b,a),c+s\eta_2(d,c))}{\partial t\partial s}\right|dtds \tag{2.4}$$

By preinvexity of $\left|\frac{\partial^2 f}{\partial t \partial s}\right|$ on the co-ordinates on $K_1 \times K_2$, we have

$$\begin{aligned} (2.5) \quad & \left| \frac{f(a,c) + f(a, c+\eta_2(d,c)) + f(a+\eta_1(b,a), c) + f(a+\eta_1(b,a), c+\eta_2(d,c))}{4} \right. \\ & \left. + \frac{1}{\eta_1(b,a)\eta_2(d,c)} \int_a^{a+\eta_1(b,a)} \int_c^{c+\eta_2(d,c)} f(x,y)\,dydx - A' \right| \\ & \leq \frac{\eta_1(b,a)\eta_2(d,c)}{4} \left[\left| \frac{\partial^2 f(a,c)}{\partial t \partial s} \right| \int_0^1 \int_0^1 |1-2t|\,|1-2s|\,(1-t)(1-s)\,dsdt \right. \\ & + \left| \frac{\partial f(a,d)}{\partial t \partial s} \right| \int_0^1 \int_0^1 |1-2t|\,|1-2s|\,(1-t)\,sdsdt \\ & + \left| \frac{\partial f(b,c)}{\partial t \partial s} \right| \int_0^1 \int_0^1 |1-2t|\,|1-2s|\,t(1-s)\,dsdt \\ & \left. + \left| \frac{\partial f(b,d)}{\partial t \partial s} \right| \int_0^1 \int_0^1 |1-2t|\,|1-2s|\,tsdsdt \right]. \end{aligned}$$

Since

$$\begin{aligned} \int_0^1 |1-2t|\,(1-t)\,dt &= \int_0^{\frac{1}{2}} (1-2t)(1-t)\,dt - \int_{\frac{1}{2}}^1 (1-2t)(1-t)\,dt \\ &= \tfrac{1}{4} \end{aligned}$$

and

$$\int_0^1 |1-2t|\,tdt = \int_0^{\frac{1}{2}} (1-2t)\,tdt - \int_{\frac{1}{2}}^1 (1-2t)\,tdt = \frac{1}{4}.$$

Making use the above in (2.5), we get the inequality (2.3). This completes the proof of the theorem. □

Theorem 2.2. *Let $K_1 \times K_2$ be an open invex subset of $\mathbb{R}^2$ with respect to the mappings $\eta_1 : K_1 \times K_1 \to \mathbb{R}$ and $\eta_2 : K_2 \times K_2 \to \mathbb{R}$. Suppose $f : K_1 \times K_2 \to \mathbb{R}$ be a twice partial differentiable mapping such that $\frac{\partial^2 f}{\partial t \partial s} \in L([a, a+t\eta_1(b,a)] \times [c, c+s\eta_2(d,c)])$ with $\eta_1(b,a) > 0$, $\eta_2(d,c) > 0$, where $a, b \in K_1$ and $c, d \in K_2$. If $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q$ is preinvex on the co-ordinates on $K_1 \times K_2$, $q \in (1,\infty)$, then the following inequality holds:*

$$\begin{aligned} (2.6) \quad & \left| \frac{f(a,c) + f(a, c+\eta_2(d,c)) + f(a+\eta_1(b,a), c) + f(a+\eta_1(b,a), c+\eta_2(d,c))}{4} \right. \\ & \left. + \frac{1}{\eta_1(b,a)\eta_2(d,c)} \int_a^{a+\eta_1(b,a)} \int_c^{c+\eta_2(d,c)} f(x,y)\,dydx - A' \right| \\ & \leq \frac{\eta_1(b,a)\eta_2(d,c)}{4(p+1)^{\frac{2}{p}}} \left(\frac{\left|\frac{\partial^2 f(a,c)}{\partial t \partial s}\right|^q + \left|\frac{\partial^2 f(a,d)}{\partial t \partial s}\right|^q + \left|\frac{\partial^2 f(b,c)}{\partial t \partial s}\right|^q + \left|\frac{\partial^2 f(b,d)}{\partial t \partial s}\right|^q}{4} \right)^{\frac{1}{q}}, \end{aligned}$$

where $A^{'}$ is as defined in Lemma 2.1 and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2.1 and Hölder's integral inequality, we have:

$$\begin{aligned}(2.7)\quad &\left|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}\right. \\ &\left.+\frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)}f(x,y)\,dydx-A^{'}\right| \\ &\leq \frac{\eta_1(b,a)\eta_2(d,c)}{4}\left(\int_0^1\int_0^1|1-2t|^p|1-2s|^p\,dsdt\right)^{\frac{1}{p}} \\ &\times\left(\int_0^1\int_0^1\left|\frac{\partial^2 f(a+t\eta_1(b,a),c+s\eta_2(d,c))}{\partial t\partial s}\right|^q dtds\right)^{\frac{1}{q}}\end{aligned}$$

Since $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ is preinvex on the co-ordinates on $K_1\times K_2$, we have

$$\begin{aligned}(2.8)\quad &\int_0^1\int_0^1\left|\frac{\partial^2 f(a+t\eta_1(b,a),c+s\eta_2(d,c))}{\partial t\partial s}\right|^q dtds \\ &\leq\left|\frac{\partial^2 f(a,c)}{\partial t\partial s}\right|^q\int_0^1\int_0^1(1-t)(1-s)\,dtds+\left|\frac{\partial^2 f(a,d)}{\partial t\partial s}\right|^q\int_0^1\int_0^1(1-t)\,sdtds \\ &+\left|\frac{\partial^2 f(b,c)}{\partial t\partial s}\right|^q\int_0^1\int_0^1(1-s)\,tdtds+\left|\frac{\partial^2 f(b,d)}{\partial t\partial s}\right|^q\int_0^1\int_0^1 tsdtds \\ &=\frac{1}{4}\left[\left|\frac{\partial^2 f(a,c)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2 f(a,d)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2 f(b,c)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2 f(b,d)}{\partial t\partial s}\right|^q\right].\end{aligned}$$

Using (2.8) and

$$\int_0^1\int_0^1|1-2t|^p|1-2s|^p\,dtds=\frac{1}{(p+1)^2}$$

in (2.7) gives us the desired inequality (2.6). This completes the proof of the theorem. □

Theorem 2.3. *Let $K_1\times K_2$ be an open invex subset of $\mathbb{R}^2$ with respect to the mappings $\eta_1: K_1\times K_1\to\mathbb{R}$ and $\eta_2: K_2\times K_2\to\mathbb{R}$. Suppose $f: K_1\times K_2\to\mathbb{R}$ be a twice partial differentiable mapping such that*

$$\frac{\partial^2 f}{\partial t\partial s}\in L\left([a,a+t\eta_1(b,a)]\times[c,c+s\eta_2(d,c)]\right)$$

with $\eta_1(b,a)>0$, $\eta_2(d,c)>0$, where $a,b\in K_1$ and $c,d\in K_2$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ is preinvex on the *co-ordinates on* $K_1 \times K_2$, $q \in [1, \infty)$, *then the following inequality holds:*

$$\left|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}\right. \tag{2.9}$$
$$\left.+\frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)\,dydx - A'\right|$$
$$\leq \frac{\eta_1(b,a)\eta_2(d,c)}{16}\left(\frac{\left|\frac{\partial^2 f(a,c)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2 f(a,d)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2 f(b,c)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2 f(b,d)}{\partial t\partial s}\right|^q}{4}\right)^{\frac{1}{q}},$$

where A' *is as defined in Lemma 2.1.*

Proof. For $q=1$, the proof is similar to that of Theorem 2.1. Suppose now that $q>1$ then from Lemma 2.1 and the power-mean integral inequality, we have:

$$\left|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}\right. \tag{2.10}$$
$$\left.+\frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)\,dydx - A'\right|$$
$$\leq \frac{\eta_1(b,a)\eta_2(d,c)}{4}\left(\int_0^1\int_0^1 |1-2t||1-2s|\,dsdt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\int_0^1 |1-2t||1-2s|\left|\frac{\partial^2 f(a+t\eta_1(b,a),c+s\eta_2(d,c))}{\partial t\partial s}\right|^q dtds\right)^{\frac{1}{q}}$$

Since $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ is preinvex on the co-ordinates on $K_1 \times K_2$, we have

$$\int_0^1\int_0^1 |1-2t||1-2s|\left|\frac{\partial^2 f(a+t\eta_1(b,a),c+s\eta_2(d,c))}{\partial t\partial s}\right|^q dtds \tag{2.11}$$
$$\leq \left|\frac{\partial^2 f(a,c)}{\partial t\partial s}\right|^q\int_0^1\int_0^1 (1-t)(1-s)|1-2t||1-2s|\,dtds$$
$$+\left|\frac{\partial^2 f(a,d)}{\partial t\partial s}\right|^q\int_0^1\int_0^1 (1-t)s|1-2t||1-2s|\,dtds$$
$$+\left|\frac{\partial^2 f(b,c)}{\partial t\partial s}\right|^q\int_0^1\int_0^1 (1-s)t|1-2t||1-2s|\,dtds$$
$$+\left|\frac{\partial^2 f(b,d)}{\partial t\partial s}\right|^q\int_0^1\int_0^1 ts|1-2t||1-2s|\,dtds$$
$$=\frac{1}{16}\left[\left|\frac{\partial^2 f(a,c)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2 f(a,d)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2 f(b,c)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2 f(b,d)}{\partial t\partial s}\right|^q\right].$$

Using (2.11) and

$$\int_0^1\int_0^1 |1-2t||1-2s|\,dtds = \frac{1}{4}$$

in (2.10) gives us the desired results. □

Remark 2.4. Since $\frac{1}{4} < \frac{1}{(1+p)^{\frac{2}{p}}} < 1$, if $p > 1$; the estimation given in Theorem 2.2 is better than the one given in Theorem 2.3.

Remark 2.5. If $\eta_1(b,a) = b - a$ and $\eta_2(d,c) = d - c$, then we get those results proved in Theorem 1.5-Theorem 1.7 from [21]. This also reveals that our results are more general than those proved in [21].

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