Some inequalities for operator weighted geometric mean

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SOME INEQUALITIES FOR OPERATOR WEIGHTED GEOMETRIC MEAN

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Abstract

In this paper, by the use of some recent Young's type scalar inequalities we obtain some inequalities for the weighted geometric mean of two positive operators on a complex Hilbert space.

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1 Introduction

The famous *Young inequality* for scalars says that if a, $b>0$ and $\nu\in[0,1]$, then

$$a^{1-\nu}b^{\nu}\leq(1-\nu)\,a+\nu b \tag{1}$$

with equality if and only if $a=b$. The inequality (1) is also called ν*-weighted arithmetic-geometric mean inequality.*

We recall that *Specht's ratio* is defined by [7]

$$S\left(h\right):=\begin{cases}\dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)}\text{ if }h\in(0,1)\cup(1,\infty)\,,\\ \\ 1\text{ if }h=1.\end{cases} \tag{2}$$

It is well known that $\lim_{h\to1}S\left(h\right)=1$, $S\left(h\right)=S\left(\frac{1}{h}\right)>1$ for $h>0$, $h\neq1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

The following inequality provides a refinement and a multiplicative reverse for Young's inequality

$$S\left(\left(\frac{a}{b}\right)^{r}\right)a^{1-\nu}b^{\nu}\leq(1-\nu)\,a+\nu b\leq S\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}, \tag{3}$$

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where $a,\ b>0,\ \nu\in[0,1],\ r=\min\{1-\nu,\nu\}$.

The second inequality in (3) is due to Tominaga [8] while the first one is due to Furuichi [3].

We also consider the *Kantorovich's ratio* defined by

$$K(h):=\frac{(h+1)^2}{4h},\ \ h>0. \tag{4}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h)\geq 1$ for any $h>0$ and $K(h)=K\left(\frac{1}{h}\right)$ for any $h>0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's ratio holds

$$K^r\left(\frac{a}{b}\right)a^{1-\nu}b^\nu\leq(1-\nu)a+\nu b\leq K^R\left(\frac{a}{b}\right)a^{1-\nu}b^\nu \tag{5}$$

where $a,b>0,\ \nu\in[0,1],\ r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

The first inequality in (5) was obtained by Zou et al. in [9] while the second by Liao et al. [6].

In [9] the authors also showed that $K^r(h)\geq S(h^r)$ for $h>0$ and $r\in\left[0,\frac{1}{2}\right]$ implying that the lower bound in (5) is better than the lower bound from (3).

In the recent paper [1] we obtained the following reverse of Young's

$$1\leq\frac{(1-\nu)a+\nu b}{a^{1-\nu}b^\nu}\leq\exp\left[4\nu(1-\nu)\left(K\left(\frac{a}{b}\right)-1\right)\right], \tag{6}$$

where $a,\ b>0,\ \nu\in[0,1]$.

It has been shown in [1] that there is no ordering for the upper bounds of the quantity $\frac{(1-\nu)a+\nu b}{a^{1-\nu}b^\nu}$ incorporated in the inequalities (3), (5) and (6).

In [2] we obtained the following refinement and reverse of Young's inequality:

$$\begin{aligned}&\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^2\right] \\ &\leq\frac{(1-\nu)a+\nu b}{a^{1-\nu}b^\nu} \\ &\leq\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2\right],\end{aligned} \tag{7}$$

for any $a,\ b>0$ and $\nu\in[0,1]$.

It has been shown in [2] that there is no ordering between the upper bounds of the quantity $\frac{(1-\nu)a+\nu b}{a^{1-\nu}b^\nu}$ provided by (6) and (7).

In this paper, by the use of some Young's type scalar inequalities (3), (5), (6) and (7) we obtain some inequalities for the weighted geometric mean of two positive operators on a complex Hilbert space.

2 Operator Inequalities

Throughout this paper A, B are positive operators on a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. We use the following notations for operators

$$A\sharp_{\nu}B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{\nu}A^{1/2}, \text{ the } \textit{weighted geometric mean,}$$

where $\nu \in [0,1]$. When $\nu = \frac{1}{2}$ we write $A\sharp B$ for brevity.

The definition of the weighted geometric mean can be extended for any real number $\nu \in \mathbb{R}$ and positive operators A, B on the Hilbert space $(H, \langle \cdot, \cdot \rangle)$.

We observe that if $\nu \in [0,1]$, then

$$\begin{aligned} A\sharp_{\nu-1}B &:= A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{\nu-1}A^{1/2} = A^{1/2}\left[\left(A^{-1/2}BA^{-1/2}\right)^{-1}\right]^{1-\nu}A^{1/2} \\ &= A^{1/2}\left(A^{1/2}B^{-1}A^{1/2}\right)^{1-\nu}A^{1/2} = A^{-1}\sharp_{1-\nu}B^{-1}. \end{aligned}$$

Theorem 1. *Let A, B be positive operators and such that there exists the positive numbers $0 < m < M$ with the property*

$$mI \le A^{-1/2}BA^{-1/2} \le MI, \tag{8}$$

then we have

$$\min_{x\in[m,M]} S\left(x^r\right)A \le (1-\nu)\,A\sharp_{\nu}B + \nu A\sharp_{\nu-1}B \le \max_{x\in[m,M]} S\left(x\right)A \tag{9}$$

for any $\nu \in [0,1]$, where $r := \min\{1-\nu, \nu\}$.

In particular,

$$\min_{x\in[m,M]} S\left(\sqrt{x}\right)A \le \frac{1}{2}\left(A\sharp B + A^{-1}\sharp B^{-1}\right) \le \max_{x\in[m,M]} S\left(x\right)A. \tag{10}$$

Proof. From (3) we have

$$S\left(x^r\right)x^{1-\nu} \le (1-\nu)\,x + \nu \le S\left(x\right)x^{1-\nu}, \tag{11}$$

where $x > 0$, $\nu \in [0,1]$ and $r = \min\{1-\nu, \nu\}$.

If we divide (11) by $x^{1-\nu} > 0$ then we get

$$S\left(x^r\right) \le (1-\nu)\,x^{\nu} + \nu x^{\nu-1} \le S\left(x\right), \tag{12}$$

for any $x > 0$.

If $x \in [m,M]$ then by (12) we get

$$\min_{x\in[m,M]} S\left(x^r\right) \le (1-\nu)\,x^{\nu} + \nu x^{\nu-1} \le \max_{x\in[m,M]} S\left(x\right). \tag{13}$$

Using the functional calculus for continuous functions we have by (13) that

$$\min_{x\in[m,M]} S\left(x^r\right)I \le (1-\nu)\,X^{\nu} + \nu X^{\nu-1} \le \max_{x\in[m,M]} S\left(x\right)I \tag{14}$$

for any selfadjoint operator X with $Sp(X) \subset [m, M]$.

Now, if we write the inequality (14) for $X = A^{-1/2}BA^{-1/2}$, then we get

$$\min_{x\in[m,M]} S(x^r) I \leq (1-\nu)\left(^{-1/2}BA^{-1/2}\right)^{\nu} + \nu\left(^{-1/2}BA^{-1/2}\right)^{\nu-1} \leq \max_{x\in[m,M]} S(x) I. \tag{15}$$

By multiplying both sides of (15) by $A^{1/2}$ we get (9). □

We have:

Corollary 1. *If either $0 < mI \leq A \leq m'I < M'I \leq B \leq MI$ for positive real numbers m, m', M, M' or $0 < mI \leq B \leq m'I < M'I \leq A \leq MI$, then by putting $h := \frac{M}{m}$, $h' := \frac{M'}{m'}$ we have*

$$S\left((h')^r\right) A \leq (1-\nu) A\sharp_\nu B + \nu A\sharp_{\nu-1} B \leq S(h) A, \tag{16}$$

for any $\nu \in [0,1]$, where $r =: \min\{1-\nu, \nu\}$.

In particular,

$$S\left(\sqrt{h'}\right) A \leq \frac{1}{2}\left(A\sharp B + A^{-1}\sharp B^{-1}\right) \leq S(h) A. \tag{17}$$

Proof. If $0 < mI \leq A \leq m'I < M'I \leq B \leq MI$, then we have

$$0 < h'I \leq A^{-1/2}BA^{-1/2} \leq hI.$$

Since $h > h' > 1$ then $\max_{x\in[h',h]} S(x) = S(h)$ and $\min_{x\in[h',h]} S(x^r) = S((h')^r)$ and by (9) we have (16).

If $0 < mI \leq B \leq m'I < M'I \leq A \leq MI$, then we have

$$0 < \frac{1}{h} I \leq A^{-1/2}BA^{-1/2} \leq \frac{1}{h'} I.$$

Since $\frac{1}{h} < \frac{1}{h'} < 1$ then $\max_{x\in\left[\frac{1}{h},\frac{1}{h'}\right]} S(x) = S\left(\frac{1}{h}\right) = S(h)$ and $\min_{x\in\left[\frac{1}{h},\frac{1}{h'}\right]} S(x^r) = S\left(\left(\frac{1}{h'}\right)^r\right) = S((h')^r)$ and by (9) we also have (16). □

Theorem 2. *With the assumptions of Theorem 1, we have*

$$\min_{x\in[m,M]} K^r(x) A \leq (1-\nu) A\sharp_\nu B + \nu A\sharp_{\nu-1} B \leq \max_{x\in[m,M]} K^R(x) A \tag{18}$$

for any $\nu \in [0,1] \setminus \left\{\frac{1}{2}\right\}$, where $r := \min\{1-\nu, \nu\}$ and $R := \max\{1-\nu, \nu\}$.

Proof. From (5) we have

$$K^r(x) x^{1-\nu} \leq (1-\nu) x + \nu 1 \leq K^R(x) x^{1-\nu} \tag{19}$$

where $x > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

If we divide (19) by $x^{1-\nu} > 0$, then we get

$$K^r(x) \leq (1-\nu)x^\nu + \nu x^{\nu-1} \leq K^R(x), \tag{20}$$

for any $x > 0$.

This inequality implies that

$$\min_{x\in[m,M]} K^r(x) \leq (1-\nu)x^\nu + \nu x^{\nu-1} \leq \max_{x\in[m,M]} K^R(x), \tag{21}$$

for any $x \in [m, M]$.

Using the functional calculus for continuous functions we have by (21) that

$$\min_{x\in[m,M]} K^r(x)\, I \leq (1-\nu)X^\nu + \nu X^{\nu-1} \leq \max_{x\in[m,M]} K^R(x)\, I \tag{22}$$

for any selfadjoint operator X with $Sp(X) \subset [m, M]$, where $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

Now, if we write the inequality (14) for $X = A^{-1/2}BA^{-1/2}$ then we get

$$\begin{aligned} \min_{x\in[m,M]} K^r(x)\, I &\leq (1-\nu)\left({}^{-1/2}BA^{-1/2}\right)^\nu + \nu\left({}^{-1/2}BA^{-1/2}\right)^{\nu-1} \\ &\leq \max_{x\in[m,M]} K^R(x)\, I. \end{aligned} \tag{23}$$

By multiplying both sides of (23) by $A^{1/2}$ we get (18). $\square$

Corollary 2. *If either $0 < mI \leq A \leq m'I < M'I \leq B \leq MI$ for positive real numbers m, m', M, M' or $0 < mI \leq B \leq m'I < M'I \leq A \leq MI$, then by putting $h := \frac{M}{m}$, $h' := \frac{M'}{m'}$ we have*

$$K^r(h')\, A \leq (1-\nu)A\sharp_\nu B + \nu A\sharp_{\nu-1}B \leq K^R(h)\, A, \tag{24}$$

for any $\nu \in [0,1] \setminus \left\{\frac{1}{2}\right\}$, where $r := \min\{1-\nu, \nu\}$ and $R := \max\{1-\nu, \nu\}$.

The proof is similar to the one from Corollary 1 and we omit the details.

The following result concerning another upper bound for

$$(1-\nu)A\sharp_\nu B + \nu A\sharp_{\nu-1}B$$

also holds.

Theorem 3. *With the assumptions of Theorem 1, we have*

$$(A \leq)\, (1-\nu)A\sharp_\nu B + \nu A\sharp_{\nu-1}B \leq \exp\left[4\nu(1-\nu)\left(\max_{x\in[m,M]} K(x) - 1\right)\right] A \tag{25}$$

for any $\nu \in [0,1]$.

In particular, we have

$$(A \leq)\, \frac{1}{2}\left(A\sharp B + A^{-1}\sharp B^{-1}\right) \leq \exp\left(\max_{x\in[m,M]} K(x) - 1\right) A. \tag{26}$$

Proof. From the inequality (6) we have

$$(1 \leq)\left(1-\nu\right)x^{\nu}+\nu x^{\nu-1}\leq \exp\left[4\nu\left(1-\nu\right)\left(K\left(x\right)-1\right)\right], \tag{27}$$

for any $x>0$.

This implies that

$$\begin{aligned}(1-\nu)\,x^{\nu}+\nu x^{\nu-1} &\leq \max_{x\in[m,M]}\left\{\exp\left[4\nu\left(1-\nu\right)\left(K\left(x\right)-1\right)\right]\right\} \\ &= \exp\left[4\nu\left(1-\nu\right)\left(\max_{x\in[m,M]}K\left(x\right)-1\right)\right]\end{aligned} \tag{28}$$

for any $x\in[m,M]$.

Using the functional calculus for continuous functions we have by (28) that

$$(1-\nu)\,X^{\nu}+\nu X^{\nu-1}\leq \exp\left[4\nu\left(1-\nu\right)\left(\max_{x\in[m,M]}K\left(x\right)-1\right)\right]$$

for any selfadjoint operator X with $Sp(X)\subset[m,M]$, where $\nu\in[0,1]$.

The proof follows now in a similar way as above and we omit the details. □

Corollary 3. *With the assumptions of Corollary 1, we have*

$$(A\leq)\left(1-\nu\right)A\sharp_{\nu}B+\nu A\sharp_{\nu-1}B\leq \exp\left[4\nu\left(1-\nu\right)\left(K\left(h\right)-1\right)\right]A \tag{29}$$

for any $\nu\in[0,1]$.

In particular, we have

$$(A\leq)\,\frac{1}{2}\left(A\sharp B+A^{-1}\sharp B^{-1}\right)\leq \exp\left(K\left(h\right)-1\right)A. \tag{30}$$

Finally, we have:

Theorem 4. *With the assumptions of Theorem 1, we have*

$$\begin{aligned}&\exp\left[\frac{1}{2}\nu\left(1-\nu\right)\left(1-\frac{\min\{1,M\}}{\max\{1,m\}}\right)^{2}\right]A \\ &\leq (1-\nu)\,A\sharp_{\nu}B+\nu A\sharp_{\nu-1}B \\ &\leq \exp\left[\frac{1}{2}\nu\left(1-\nu\right)\left(\frac{\max\{1,M\}}{\min\{1,m\}}-1\right)^{2}\right]A\end{aligned} \tag{31}$$

and, in particular

$$\begin{aligned}\exp\left[\frac{1}{8}\left(1-\frac{\min\{1,M\}}{\max\{1,m\}}\right)^{2}\right]A &\leq \frac{1}{2}\left(A\sharp B+A^{-1}\sharp B^{-1}\right) \\ &\leq \exp\left[\frac{1}{8}\left(\frac{\max\{1,M\}}{\min\{1,m\}}-1\right)^{2}\right]A.\end{aligned} \tag{32}$$

Proof. From the inequality (7) we have

$$\begin{aligned}
&\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{\min\{1,x\}}{\max\{1,x\}}\right)^2\right] \\
&\le (1-\nu)x^{\nu}+\nu x^{\nu-1} \\
&\le \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{1,x\}}{\min\{1,x\}}-1\right)^2\right]
\end{aligned} \tag{33}$$

for any $x>0$ and any $\nu\in[0,1]$.

If $x\in[m,M]\subset(0,\infty)$ then

$$0\le\frac{\max\{1,x\}}{\min\{1,x\}}-1\le\frac{\max\{1,M\}}{\min\{1,m\}}-1$$

and

$$0\le 1-\frac{\min\{1,M\}}{\max\{1,m\}}\le 1-\frac{\min\{1,x\}}{\max\{1,x\}},$$

which implies that

$$\begin{aligned}
&\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{1,x\}}{\min\{1,x\}}-1\right)^2\right] \\
&\le \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{1,M\}}{\min\{1,m\}}-1\right)^2\right]
\end{aligned}$$

and

$$\begin{aligned}
&\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{\min\{1,M\}}{\max\{1,m\}}\right)^2\right] \\
&\le \exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{\min\{1,x\}}{\max\{1,x\}}\right)^2\right].
\end{aligned}$$

By (33) we then have

$$\begin{aligned}
&\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{\min\{1,M\}}{\max\{1,m\}}\right)^2\right] \\
&\le (1-\nu)x^{\nu}+\nu x^{\nu-1} \\
&\le \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{1,M\}}{\min\{1,m\}}-1\right)^2\right]
\end{aligned} \tag{34}$$

for any $x\in[m,M]$ and any $\nu\in[0,1]$.

Using the functional calculus for continuous functions we have by (34) that

$$\begin{aligned}
&\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{\min\{1,M\}}{\max\{1,m\}}\right)^2\right] \qquad (35)\\
&\leq (1-\nu)X^{\nu}+\nu X^{\nu-1}\\
&\leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{1,M\}}{\min\{1,m\}}-1\right)^2\right]
\end{aligned}$$

for any selfadjoint operator X with $Sp(X) \subset [m, M]$, where $\nu \in [0, 1]$.

The proof follows now in a similar way as above and we omit the details. □

Corollary 4. *With the assumptions of Corollary 1 we have*

$$\begin{aligned}
\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{h'-1}{h'}\right)^2\right] &\leq (1-\nu)A\sharp_{\nu}B+\nu A\sharp_{\nu-1}B \qquad (36)\\
&\leq \exp\left[\frac{1}{2}\nu(1-\nu)(h-1)^2\right]
\end{aligned}$$

and, in particular,

$$\exp\left[\frac{1}{8}\left(\frac{h'-1}{h'}\right)^2\right] \leq \frac{1}{2}\left(A\sharp B+A^{-1}\sharp B^{-1}\right) \leq \exp\left[\frac{1}{8}(h-1)^2\right]. \qquad (37)$$

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