Some Hermite-Hadamard type integral inequalities for convex functions defined on convex bodies in Rn\mathbb{R}^{n}

Research Article

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Some Hermite–Hadamard type integral inequalities for convex functions defined on convex bodies in $\mathbb{R}^n$

https://doi.org/10.1515/jaa-2020-2005
Received July 9, 2019; accepted November 4, 2019

Abstract: In this paper, by the use of the divergence theorem, we establish some integral inequalities of Hermite–Hadamard type for convex functions of several variables defined on closed and bounded convex bodies in the Euclidean space $\mathbb{R}^n$ for any $n \geq 2$.

Keywords: Hermite–Hadamard inequality, multiple integral inequalities, Green identity, Gauss–Ostrogradsky identity, divergence theorem

MSC 2010: 26D15

Dedicated to Audrey and Sienna

1 Introduction

In the following, consider a closed and bounded convex subset D of $\mathbb{R}^2$. Define by

$$A_D := \iint_D dxdy$$

the *area* of D and by $(\overline{x_D}, \overline{y_D})$ the *center of mass* for D, where

$$\overline{x_D} := \frac{1}{A_D} \iint_D xdxdy, \quad \overline{y_D} := \frac{1}{A_D} \iint_D ydxdy.$$

Consider the function of two variables $f = f(x, y)$ and denote by $\frac{\partial f}{\partial x}$ the partial derivative with respect to the variable x and by $\frac{\partial f}{\partial y}$ the partial derivative with respect to the variable y.

In the recent paper [9], we obtained the following Hermite–Hadamard type inequalities.

Theorem 1.1. *Let $f : D \to \mathbb{R}$ be a differentiable convex function on D, a closed and bounded convex subset of $\mathbb{R}^2$ surrounded by the smooth curve ∂D. Then for all $(u, v) \in D$ we have*

$$\begin{aligned}\frac{\partial f}{\partial x}(u, v)(\overline{x_D} - u) + \frac{\partial f}{\partial y}(u, v)(\overline{y_D} - v) + f(u, v) &\leq \frac{1}{A_D} \iint_D f(x, y)dxdy \\ &\leq \frac{1}{3} f(u, v) + \frac{1}{3A_D} \oint_{\partial D} [(v - y)f(x, y)dx + (x - u)f(x, y)dy].\end{aligned}$$

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In particular,

$$f(\overline{x_D}, \overline{y_D}) \le \frac{1}{A_D} \iint_D f(x,y)dxdy$$
$$\le \frac{1}{3} f(\overline{x_D}, \overline{y_D}) + \frac{1}{3A_D} \oint_{\partial D} [(\overline{y_D} - y) f(x,y)dx + (x - \overline{x_D}) f(x,y)dy].$$

We also have the following corollary.

Corollary 1.2. *With the assumptions of Theorem 1.1, we have*

$$f(\overline{x_D}, \overline{y_D}) \le \frac{1}{A_D} \iint_D f(x,y)dxdy$$
$$\le \frac{1}{2A_D} \oint_{\partial D} [(\overline{y_D} - y) f(x,y)dx + (x - \overline{x_D}) f(x,y)dy].$$

Some examples for rectangles and disks on the plane were also provided in [9].

The case of a convex function defined on a convex body in the space was considered in [10], where we obtained the following result.

Theorem 1.3. *Let B be a convex body in the three-dimensional space $\mathbb{R}^3$ bounded by an orientable closed surface ∂B and let $f : B \to \mathbb{C}$ be a continuously differentiable function defined on an open set containing B. If f is convex on B, then for any $(u, v, w) \in B$ we have*

$$f(u,v,w) + (\overline{x_B} - u)\frac{\partial f(u,v,w)}{\partial x} + (\overline{y_B} - v)\frac{\partial f(u,v,w)}{\partial y} + (\overline{z_B} - w)\frac{\partial f(u,v,w)}{\partial z}$$
$$\le \frac{1}{V(B)} \iiint_B f(x,y,z)dxdydz$$
$$\le \frac{1}{4} f(u,v,w) + \frac{1}{4}\frac{1}{V(B)}\Bigg[\iint_{\partial B} (x-u) f(x,y,z) dy \wedge dz$$
$$+ \iint_{\partial B} (y-v) f(x,y,z) dz \wedge dx + \iint_{\partial B} (z-w) f(x,y,z) dx \wedge dy\Bigg],$$

where

$$\overline{x_B} := \frac{1}{V(B)} \iiint_B x dxdydz,$$
$$\overline{y_B} := \frac{1}{V(B)} \iiint_B y dxdydz,$$
$$\overline{z_B} := \frac{1}{V(B)} \iiint_B z dxdydz.$$

In particular, we have

$$f(\overline{x_B}, \overline{y_B}, \overline{z_B}) \le \frac{1}{V(B)} \iiint_B f(x,y,z)dxdydz$$
$$\le \frac{1}{4} f(\overline{x_B}, \overline{y_B}, \overline{z_B}) + \frac{1}{4}\frac{1}{V(B)}\Bigg[\iint_{\partial B} (x-\overline{x_B}) f(x,y,z) dy \wedge dz$$
$$+ \iint_{\partial B} (y-\overline{y_B}) f(x,y,z) dz \wedge dx + \iint_{\partial B} (z-\overline{z_B}) f(x,y,z) dx \wedge dy\Bigg].$$

We also have the following corollary.

Corollary 1.4. *With the assumptions of Theorem 1.3,*

$$\frac{1}{V(B)}\iiint_B f(x,y,z)dxdydz$$
$$\le \frac{1}{3}\frac{1}{V(B)}\Big[\iint_S (x-\overline{x_B})f(x,y,z)dy\wedge dz + \iint_S (y-\overline{y_B})f(x,y,z)dz\wedge dx + \iint_S (z-\overline{z_B})f(x,y,z)dx\wedge dy\Big].$$

Examples for three-dimensional balls and spheres were also considered in [10].

For other Hermite–Hadamard type integral inequalities for multiple integrals, see [2–8, 11–15, 17–19].

Motivated by the above results, in this paper, by the use of the divergence theorem, we establish some integral inequalities of Hermite–Hadamard type for convex functions of several variables defined on closed and bounded convex bodies in the Euclidean space $\mathbb{R}^n$ for any $n \ge 2$.

2 Some preliminary facts

Let B be a bounded open subset of $\mathbb{R}^n$ ($n \ge 2$) with smooth (or piecewise smooth) boundary ∂B. Assume that $F = (F_1, \ldots, F_n)$ is a smooth vector field defined in $\mathbb{R}^n$, or at least in $B \cup \partial B$. Let $\mathbf{n}$ be the unit outward-pointing normal of ∂B. Then the *divergence theorem* states (see for instance [16]):

$$\int_B \operatorname{div} F dV = \int_{\partial B} F\cdot n dA, \tag{2.1}$$

where

$$\operatorname{div} F = \nabla\cdot F = \sum_{k=1}^n \frac{\partial F_k}{\partial x_k},$$

dV is the element of volume in $\mathbb{R}^n$ and dA is the element of surface area on ∂B.

If $\mathbf{n} = (\mathbf{n}_1, \ldots, \mathbf{n}_n)$, $x = (x_1, \ldots, x_n) \in B$ and using the notation dx for dV, we can write (2.1) more explicitly as

$$\sum_{k=1}^n \int_B \frac{\partial F_k(x)}{\partial x_k} dx = \sum_{k=1}^n \int_{\partial B} F_k(x) n_k(x) dA. \tag{2.2}$$

By taking the real and imaginary parts, we can extend the above equality for complex-valued functions F_k, $k \in \{1, \ldots, n\}$, defined on B.

If $n = 2$, the normal is obtained by rotating the tangent vector by 90° (in the correct direction so that it points out). The quantity tds can be written (dx_1, dx_2) along the surface, so that

$$ndA := nds = (dx_2, -dx_1).$$

Here t is the tangent vector along the boundary curve and ds is the element of arc-length.

From (2.2) we get for $B \subset \mathbb{R}^2$ that

$$\int_B \frac{\partial F_1(x_1,x_2)}{\partial x_1} dx_1 dx_2 + \int_B \frac{\partial F_2(x_1,x_2)}{\partial x_2} dx_1 dx_2 = \int_{\partial B} F_1(x_1,x_2)dx_2 - \int_{\partial B} F_2(_1,x_2)dx_1,$$

which is *Green's theorem* in the plane.

If $n = 3$ and if ∂B is described as a level-set of a function of three variables, i.e.,

$$\partial B = \{x_1, x_2, x_3 \in \mathbb{R}^3 \mid G(x_1,x_2,x_3) = 0\},$$

then a vector pointing in the direction of $\mathbf{n}$ is grad G. We shall use the case where $G(x_1,x_2,x_3) = x_3 - g(x_1,x_2)$, $(x_1,x_2) \in D$, and a domain in $\mathbb{R}^2$ for some differentiable function g on D and B corresponds to the inequality

$x_3 < g(x_1, x_2)$, namely

$$B = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 < g(x_1, x_2)\}.$$

Then

$$\begin{aligned} \mathbf{n} &= \frac{(-g_{x_1}, -g_{x_2}, 1)}{(1 + g_{x_1}^2 + g_{x_2}^2)^{1/2}}, \\ dA &= (1 + g_{x_1}^2 + g_{x_2}^2)^{1/2} dx_1 dx_2, \\ \mathbf{n} dA &= (-g_{x_1}, -g_{x_2}, 1) dx_1 dx_2. \end{aligned}$$

From (2.2) we get

$$\begin{aligned} &\int_B \Big(\frac{\partial F_1(x_1, x_2, x_3)}{\partial x_1} + \frac{\partial F_2(x_1, x_2, x_3)}{\partial x_2} + \frac{\partial F_3(x_1, x_2, x_3)}{\partial x_3} \Big) dx_1 dx_2 dx_3 \\ &\quad = -\int_D F_1(x_1, x_2, g(x_1, x_2)) g_{x_1}(x_1, x_2) dx_1 dx_2 - \int_D F_1(x_1, x_2, g(x_1, x_2)) g_{x_2}(x_1, x_2) dx_1 dx_2 \\ &\qquad + \int_D F_3(x_1, x_2, g(x_1, x_2)) dx_1 dx_2, \end{aligned} \tag{2.3}$$

which is the *Gauss–Ostrogradsky theorem* in the space.

Following Apostol [1], we can also consider a surface described by the vector equation

$$r(u, v) = x_1(u, v) \vec{i} + x_2(u, v) \vec{j} + x_3(u, v) \vec{k},$$

where $(u, v) \in [a, b] \times [c, d]$.

If x_1, x_2, x_3 are differentiable on $[a, b] \times [c, d]$, we consider the two vectors

$$\frac{\partial r}{\partial u} = \frac{\partial x_1}{\partial u} \vec{i} + \frac{\partial x_2}{\partial u} \vec{j} + \frac{\partial x_3}{\partial u} \vec{k}$$

and

$$\frac{\partial r}{\partial v} = \frac{\partial x_1}{\partial v} \vec{i} + \frac{\partial x_2}{\partial v} \vec{j} + \frac{\partial x_3}{\partial v} \vec{k}.$$

The *cross product* of these two vectors $\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}$ will be referred to as the fundamental vector product of the representation r. Its components can be expressed as *Jacobian determinants*. In fact, we have (see [1, p. 420])

$$\begin{aligned} \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} &= \begin{vmatrix} \frac{\partial x_2}{\partial u} & \frac{\partial x_3}{\partial u} \\ \frac{\partial x_2}{\partial v} & \frac{\partial x_3}{\partial v} \end{vmatrix} \vec{i} + \begin{vmatrix} \frac{\partial x_3}{\partial u} & \frac{\partial x_1}{\partial u} \\ \frac{\partial x_3}{\partial v} & \frac{\partial x_1}{\partial v} \end{vmatrix} \vec{j} + \begin{vmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_2}{\partial u} \\ \frac{\partial x_1}{\partial v} & \frac{\partial x_2}{\partial v} \end{vmatrix} \vec{k} \\ &= \frac{\partial(x_2, x_3)}{\partial(u, v)} \vec{i} + \frac{\partial(x_3, x_1)}{\partial(u, v)} \vec{j} + \frac{\partial(x_1, x_2)}{\partial(u, v)} \vec{k}. \end{aligned}$$

Let $\partial B = r(T)$ be a parametric surface described by a vector-valued function r defined on the box $T = [a, b] \times [c, d]$. The area of ∂B denoted by $A_{\partial B}$ is defined by the double integral (see [1, pp. 424–425])

$$\begin{aligned} A_{\partial B} &= \int_a^b \int_c^d \Big\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \Big\| du dv \\ &= \int_a^b \int_c^d \sqrt{\Big(\frac{\partial(x_2, x_3)}{\partial(u, v)} \Big)^2 + \Big(\frac{\partial(x_3, x_1)}{\partial(u, v)} \Big)^2 + \Big(\frac{\partial(x_1, x_2)}{\partial(u, v)} \Big)^2} du dv. \end{aligned}$$

We define surface integrals in terms of a parametric representation for the surface. One can prove that under certain general conditions the value of the integral is independent of the representation.

Let $\partial B = r(T)$ be a parametric surface described by a vector-valued differentiable function r defined on the box $T = [a, b] \times [c, d]$ and let $f : \partial B \to \mathbb{C}$ be defined and bounded on ∂B. The surface integral of f over ∂B

is defined by (see [1, p. 430])

$$\iint_{\partial B} f dA = \int_a^b \int_c^d f(x_1, x_2, x_3) \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| du dv$$
$$= \int_a^b \int_c^d f(x_1(u,v), x_2(u,v), x_3(u,v)) \sqrt{\left(\frac{\partial(x_2, x_3)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(x_3, x_1)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(x_1, x_2)}{\partial(u,v)}\right)^2} du dv.$$

If $\partial B = r(T)$ is a parametric surface, the fundamental vector product $N = \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}$ is normal to ∂B at each regular point of the surface. At each such point there are two unit normals, a unit normal $\mathbf{n}_1$, which has the same direction as N, and a unit normal $\mathbf{n}_2$, which has the opposite direction. Thus

$$\mathbf{n}_1 = \frac{N}{\|N\|} \quad \text{and} \quad \mathbf{n}_2 = -\mathbf{n}_1.$$

Let $\mathbf{n}$ be one of the two normals $\mathbf{n}_1$ or $\mathbf{n}_2$. Let also F be a vector field defined on ∂B and assume that the surface integral

$$\iint_{\partial B} (F \cdot \mathbf{n}) dA,$$

called the flux surface integral, exists. Here $F \cdot \mathbf{n}$ is the dot or inner product.

We can write (see [1, p. 434])

$$\iint_{\partial B} (F \cdot \mathbf{n}) dA = \pm \int_a^b \int_c^d F(r(u,v)) \cdot \left(\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right) du dv,$$

where the sign "+" is used if $\mathbf{n} = \mathbf{n}_1$ and the "–" sign is used if $\mathbf{n} = \mathbf{n}_2$.

If

$$F(x_1, x_2, x_3) = F_1(x_1, x_2, x_3)\vec{i} + F_2(x_1, x_2, x_3)\vec{j} + F_3(x_1, x_2, x_3)\vec{k}$$

and

$$r(u,v) = x_1(u,v)\vec{i} + x_2(u,v)\vec{j} + x_3(u,v)\vec{k}, \quad \text{where } (u,v) \in [a,b] \times [c,d],$$

then the flux surface integral for $\mathbf{n} = \mathbf{n}_1$ can be explicitly calculated as (see [1, p. 435])

$$\iint_{\partial B} (F \cdot \mathbf{n}) dA = \int_a^b \int_c^d F_1(x_1(u,v), x_2(u,v), x_3(u,v)) \frac{\partial(x_2, x_3)}{\partial(u,v)} du dv$$
$$+ \int_a^b \int_c^d F_2(x_1(u,v), x_2(u,v), x_3(u,v)) \frac{\partial(x_3, x_1)}{\partial(u,v)} du dv$$
$$+ \int_a^b \int_c^d F_3(x_1(u,v), x_2(u,v), x_3(u,v)) \frac{\partial(x_1, x_2)}{\partial(u,v)} du dv.$$

The sum of the double integrals on the right-hand side is often written more briefly as (see [1, p. 435])

$$\iint_{\partial B} F_1(x_1, x_2, x_3) dx_2 \wedge dx_3 + \iint_{\partial B} F_2(x_1, x_2, x_3) dx_3 \wedge dx_1 + \iint_{\partial B} F_3(x_1, x_2, x_3) dx_1 \wedge dx_2.$$

Let $B \subset \mathbb{R}^3$ be a solid in the 3-space bounded by an orientable closed surface ∂B, and let $\mathbf{n}$ be the unit outer normal to ∂B. If F is a continuously differentiable vector field defined on B, we have the *Gauss–Ostrogradsky identity*

$$\iiint_B (\operatorname{div} F) dV = \iint_{\partial B} (F \cdot \mathbf{n}) dA.$$

If we express

$$F(x_1, x_2, x_3) = F_1(x_1, x_2, x_3)\overrightarrow{i} + F_2(x_1, x_2, x_3)\overrightarrow{j} + F_3(x_1, x_2, x_3)\overrightarrow{k},$$

then (2.3) can be written as

$$\iiint_B \Big(\frac{\partial F_1(x_1, x_2, x_3)}{\partial x_1} + \frac{\partial F_2(x_1, x_2, x_3)}{\partial x_2} + \frac{\partial F_3(x_1, x_2, x_3)}{\partial x_3}\Big)dx_1 dx_2 dx_3$$
$$= \iint_{\partial B} F_1(x_1, x_2, x_3)dx_2 \wedge dx_3 + \iint_{\partial B} F_2(x_1, x_2, x_3)dx_3 \wedge dx_1 + \iint_{\partial B} F_3(x_1, x_2, x_3)dx_1 \wedge dx_2.$$

3 General identities

We have the following identity of interest.

Lemma 3.1. *Let B be a bounded open subset of $\mathbb{R}^n$ $(n \geq 2)$ with smooth (or piecewise smooth) boundary ∂B. Let f be a continuously differentiable function defined in $\mathbb{R}^n$, or at least in $B \cup \partial B$ and with complex values. If $\alpha_k, \beta_k \in \mathbb{C}$ for $k \in \{1, \ldots, n\}$ with $\sum_{k=1}^n \alpha_k = 1$, then*

$$\int_B f(x)dx = \sum_{k=1}^n \int_B (\beta_k - \alpha_k x_k)\frac{\partial f(x)}{\partial x_k}dx + \sum_{k=1}^n \int_{\partial B} (\alpha_k x_k - \beta_k)f(x)n_k(x)dA. \tag{3.1}$$

We also have

$$\int_B f(x)dx = \frac{1}{n}\sum_{k=1}^n \int_B (\gamma_k - x_k)\frac{\partial f(x)}{\partial x_k}dx + \frac{1}{n}\sum_{k=1}^n \int_{\partial B} (x_k - \gamma_k)f(x)n_k(x)dA \tag{3.2}$$

for all $\gamma_k \in \mathbb{C}$ where $k \in \{1, \ldots, n\}$.

Proof. Let $x = (x_1, \ldots, x_n) \in B$. We consider

$$F_k(x) = (\alpha_k x_k - \beta_k)f(x), \quad k \in \{1, \ldots, n\},$$

and take the partial derivatives $\frac{\partial F_k(x)}{\partial x_k}$ to get

$$\frac{\partial F_k(x)}{\partial x_k} = \alpha_k f(x) + (\alpha_k x_k - \beta_k)\frac{\partial f(x)}{\partial x_k}, \quad k \in \{1, \ldots, n\}.$$

If we sum this equality over k from 1 to n we get

$$\sum_{k=1}^n \frac{\partial F_k(x)}{\partial x_k} = \sum_{k=1}^n \alpha_k f(x) + \sum_{k=1}^n (\alpha_k x_k - \beta_k)\frac{\partial f(x)}{\partial x_k} = f(x) + \sum_{k=1}^n (\alpha_k x_k - \beta_k)\frac{\partial f(x)}{\partial x_k} \tag{3.3}$$

for all $x = (x_1, \ldots, x_n) \in B$.

Now, if we take the integral in equality (3.3) over $(x_1, \ldots, x_n) \in B$, we get

$$\int_B \Big(\sum_{k=1}^n \frac{\partial F_k(x)}{\partial x_k}\Big)dx = \int_B f(x)dx + \sum_{k=1}^n \int_B \Big[(\alpha_k x_k - \beta_k)\frac{\partial f(x)}{\partial x_k}\Big]dx. \tag{3.4}$$

By the divergence theorem (2.2), we also have

$$\int_B \Big(\sum_{k=1}^n \frac{\partial F_k(x)}{\partial x_k}\Big)dx = \sum_{k=1}^n \int_{\partial B} (\alpha_k x_k - \beta_k)f(x)n_k(x)dA. \tag{3.5}$$

By making use of (3.4) and (3.5), we get

$$\int_B f(x)dx + \sum_{k=1}^n \int_B \Big[(\alpha_k x_k - \beta_k)\frac{\partial f(x)}{\partial x_k}\Big]dx = \sum_{k=1}^n \int_{\partial B} (\alpha_k x_k - \beta_k)f(x)n_k(x)dA,$$

which gives the desired representation (3.1).

Identity (3.2) follows by (3.1) for $\alpha_k = \frac{1}{n}$ and $\beta_k = \frac{1}{n}\gamma_k$, $k \in \{1, \ldots, n\}$. □

For the body B we consider the coordinates for the *center of gravity*

$$G_B := G(\overline{x_{B,1}}, \ldots, \overline{x_{B,n}})$$

defined by

$$\overline{x_{B,k}} := \frac{1}{V(B)} \int_B x_k dx, \quad k \in \{1, \ldots, n\},$$

where

$$V(B) := \int_B x dx$$

is the volume of B.

Corollary 3.2. *With the assumptions of Lemma 3.1, we have*

$$\int_B f(x)dx = \sum_{k=1}^{n} \int_B \alpha_k(\overline{x_{B,k}} - x_k)\frac{\partial f(x)}{\partial x_k} dx + \sum_{k=1}^{n} \int_{\partial B} \alpha_k(x_k - \overline{x_{B,k}})f(x)n_k(x)dA$$

and, in particular,

$$\int_B f(x)dx = \frac{1}{n}\sum_{k=1}^{n} \int_B (\overline{x_{B,k}} - x_k)\frac{\partial f(x)}{\partial x_k} dx + \frac{1}{n}\sum_{k=1}^{n} \int_{\partial B} (x_k - \overline{x_{B,k}})f(x)n_k(x)dA.$$

The proof follows by (3.1) on taking $\beta_k = \alpha_k \overline{x_{B,k}}$, $k \in \{1, \ldots, n\}$.

For a function f as in Lemma 3.1 above, we define the points

$$x_{B,\partial f,k} := \frac{\int_B x_k \frac{\partial f(x)}{\partial x_k} dx}{\int_B \frac{\partial f(x)}{\partial x_k} dx}, \quad k \in \{1, \ldots, n\},$$

provided that all denominators are not zero.

Corollary 3.3. *With the assumptions of Lemma 3.1, we have*

$$\int_B f(x)dx = \sum_{k=1}^{n} \int_{\partial B} \alpha_k(x_k - x_{B,\partial f,k})f(x)n_k(x)dA$$

and, in particular,

$$\int_B f(x)dx = \frac{1}{n}\sum_{k=1}^{n} \int_{\partial B} (x_k - x_{B,\partial f,k})f(x)n_k(x)dA.$$

The proof follows by (3.1) on taking $\beta_k = \alpha_k x_{B,\partial f,k}$, $k \in \{1, \ldots, n\}$, and observing that

$$\sum_{k=1}^{n} \int_B (\beta_k - \alpha_k x_k)\frac{\partial f(x)}{\partial x_k} dx = \sum_{k=1}^{n} \alpha_k \int_B (x_{B,\partial f,k} - x_k)\frac{\partial f(x)}{\partial x_k} dx = 0.$$

For a function f as in Lemma 3.1 above, we define the points

$$x_{\partial B,f,k} := \frac{\int_{\partial B} x_k f(x)n_k(x)dA}{\int_{\partial B} f(x)n_k(x)dA}, \quad k \in \{1, \ldots, n\},$$

provided that all denominators are not zero.

Corollary 3.4. *With the assumptions of Lemma 3.1, we have*

$$\int_B f(x)dx = \sum_{k=1}^{n} \int_B \alpha_k(x_{\partial B,f,k} - x_k)\frac{\partial f(x)}{\partial x_k} dx$$

and, in particular,

$$\int_B f(x)dx = \frac{1}{n}\sum_{k=1}^{n} \int_B (x_{\partial B,f,k} - x_k)\frac{\partial f(x)}{\partial x_k} dx.$$

The proof follows by (3.1) on taking $\beta_k = \alpha_k x_{\partial B,f,k}$, $k \in \{1, \dots, n\}$, and observing that

$$\sum_{k=1}^{n} \int_{\partial B} (\alpha_k x_k - \beta_k) f(x) n_k(x) dA = \sum_{k=1}^{n} \alpha_k \int_{\partial B} (x_k - x_{\partial B,f,k}) f(x) n_k(x) dA = 0.$$

4 Inequalities for convex functions

We have the following result that generalizes the inequalities from Section 1.

Theorem 4.1. *Let B be a bounded convex and closed subset of $\mathbb{R}^n$ $(n \geq 2)$ with smooth (or piecewise smooth) boundary ∂B. Let f be a continuously differentiable convex function defined on an open neighborhood of B. Then for all $y \in B$ we have*

$$f(y) + \sum_{k=1}^{n} \frac{\partial f(y)}{\partial x_k} (\overline{x_{B,k}} - y_k) \leq \frac{1}{V(B)} \int_B f(x) dx \leq \frac{1}{n+1} f(y) + \frac{1}{n+1} \sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} (x_k - y_k) f(x) n_k(x) dA. \tag{4.1}$$

In particular,

$$f(G_B) \leq \frac{1}{V(B)} \int_B f(x) dx \leq \frac{1}{n+1} f(G_B) + \frac{1}{n+1} \sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} (x_k - \overline{x_{B,k}}) f(x) n_k(x) dA, \tag{4.2}$$

where $G_B \in B$ is the center of gravity for B, *i.e.*, $G_B := G(\overline{x_{B,1}}, \dots, \overline{x_{B,n}})$.

Proof. Since $f : B \to \mathbb{R}$ is a differentiable convex function on B, for all $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in B$ we have the *gradient inequalities*

$$\sum_{k=1}^{n} \frac{\partial f(y)}{\partial x_k} (x_k - y_k) \leq f(x) - f(y) \leq \sum_{k=1}^{n} \frac{\partial f(x)}{\partial x_k} (x_k - y_k). \tag{4.3}$$

Taking the integral mean $\frac{1}{V(B)} \int_B$ in (4.3) over the variable $x \in B$, we deduce

$$\sum_{k=1}^{n} \frac{\partial f(y)}{\partial x_k} \left(\frac{1}{V(B)} \int_B x_k dx - y_k \right) \leq \frac{1}{V(B)} \int_B f(x) dx - f(y) \leq \sum_{k=1}^{n} \frac{1}{V(B)} \int_B \frac{\partial f(x)}{\partial x_k} (x_k - y_k) dx. \tag{4.4}$$

From equality (3.2) we get for $y_k = y_k$, $k \in \{1, \dots, n\}$, that

$$\int_B f(x) dx = \frac{1}{n} \sum_{k=1}^{n} \int_B (y_k - x_k) \frac{\partial f(x)}{\partial x_k} dx + \frac{1}{n} \sum_{k=1}^{n} \int_{\partial B} (x_k - y_k) f(x) n_k(x) dA,$$

namely

$$\sum_{k=1}^{n} \int_B (x_k - y_k) \frac{\partial f(x)}{\partial x_k} dx = \sum_{k=1}^{n} \int_{\partial B} (x_k - y_k) f(x) n_k(x) dA - n \int_B f(x) dx.$$

Since

$$\sum_{k=1}^{n} \frac{\partial f(y)}{\partial x_k} \left(\frac{1}{V(B)} \int_B x_k dx - y_k \right) = \sum_{k=1}^{n} \frac{\partial f(y)}{\partial x_k} (\overline{x_{B,k}} - y_k)$$

and

$$\sum_{k=1}^{n} \frac{1}{V(B)} \int_B \frac{\partial f(x)}{\partial x_k} (x_k - y_k) dx = \sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} (x_k - y_k) f(x) n_k(x) dA - n \frac{1}{V(B)} \int_B f(x) dx,$$

by (4.4) we get

$$\sum_{k=1}^{n} \frac{\partial f(y)}{\partial x_k} (\overline{x_{B,k}} - y_k) \leq \frac{1}{V(B)} \int_B f(x) dx - f(y) \leq \sum_{k=1}^{n} \frac{1}{V(B)} \int_{\partial B} (x_k - y_k) f(x) n_k(x) dA - n \frac{1}{V(B)} \int_B f(x) dx. \tag{4.5}$$

Now, from the first inequality in (4.5) we get the first inequality in (4.1).

The second inequality in (4.5) can be written as

$$\frac{1}{V(B)}\int_B f(x)dx + \frac{n}{V(B)}\int_B f(x)dx \le f(y) + \sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(x_k - y_k)f(x)n_k(x)dA,$$

which is equivalent to the second part of (4.1). □

Corollary 4.2. *With the assumptions of Theorem 4.1, we have*

$$\frac{1}{V(B)}\int_B f(x)dx \le \frac{1}{n}\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(x_k - \overline{x_{B,k}})f(x)n_k(x)dA. \tag{4.6}$$

Proof. From (4.2) we have

$$\frac{1}{V(B)}\int_B f(x)dx \le \frac{1}{n+1}f(G_B) + \frac{1}{n+1}\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(x_k - \overline{x_{B,k}})f(x)n_k(x)dA. \tag{4.7}$$

Since

$$f(G_B) \le \frac{1}{V(B)}\int_B f(x)dx,$$

we obtain

$$\begin{aligned}&\frac{1}{n+1}f(G_B) + \frac{1}{n+1}\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(x_k - \overline{x_{B,k}})f(x)n_k(x)dA\\ &\quad\le \frac{1}{n+1}\frac{1}{V(B)}\int_B f(x)dx + \frac{1}{n+1}\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(x_k - \overline{x_{B,k}})f(x)n_k(x)dA. \end{aligned}\tag{4.8}$$

By (4.7) and (4.8), we get

$$\frac{1}{V(B)}\int_B f(x)dx \le \frac{1}{n+1}\frac{1}{V(B)}\int_B f(x)dx + \frac{1}{n+1}\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(x_k - \overline{x_{B,k}})f(x)n_k(x)dA,$$

which is equivalent to (4.6). □

Corollary 4.3. *With the assumptions of Theorem 4.1 and for* $(x_{\partial B,f,1},\dots,x_{\partial B,f,n}) \in B$, *we have*

$$f(x_{\partial B,f,1},\dots,x_{\partial B,f,n}) + \sum_{k=1}^{n}\frac{\partial f(y)}{\partial x_k}(\overline{x_{B,k}} - x_{\partial B,f,k}) \le \frac{1}{V(B)}\int_B f(x)dx \le \frac{1}{n+1}f(x_{\partial B,f,1},\dots,x_{\partial B,f,n}).$$

The proof follows by (4.1) observing that

$$\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(x_k - x_{\partial B,f,k})f(x)n_k(x)dA = 0.$$

We also have the following result.

Corollary 4.4. *With the assumptions of Theorem 4.1, if we define*

$$\overline{s_{\partial B,k}} := \frac{1}{A(\partial B)}\int_{\partial B} y_k dS, \quad k \in \{1,\dots,n\}, \tag{4.9}$$

where $A(\partial B)$ *is the area of the surface* ∂B, *then we have the inequality*

$$\begin{aligned}&\frac{1}{A(\partial B)}\int_{\partial B} f(y)dS + \sum_{k=1}^{n}\frac{1}{A(\partial B)}\int_{\partial B}\frac{\partial f(y)}{\partial x_k}(\overline{x_{B,k}} - y_k)dS\\ &\quad\le \frac{1}{V(B)}\int_B f(x)dx\\ &\quad\le \frac{1}{n+1}\frac{1}{A(\partial B)}\int_{\partial B} f(y)dS + \frac{1}{n+1}\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(x_k - \overline{s_{\partial B,k}})f(x)n_k(x)dA. \end{aligned}\tag{4.10}$$

Proof. If we take the integral mean $\frac{1}{A(\partial B)}\int_{\partial B}(\cdot)dS$ over the variable $y \in \partial B$, then we get

$$\begin{aligned}
&\frac{1}{A(\partial B)}\int_{\partial B} f(y)dS + \frac{1}{A(\partial B)}\int_{\partial B}\Big(\sum_{k=1}^{n}\frac{\partial f(y)}{\partial x_k}(\overline{x_{B,k}} - y_k)\Big)dS \\
&\quad\le \frac{1}{V(B)}\int_B f(x)dx \\
&\quad\le \frac{1}{n+1}\frac{1}{A(\partial B)}\int_{\partial B} f(y)dS + \frac{1}{n+1}\frac{1}{A(\partial B)}\int_{\partial B}\Big(\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(x_k - y_k)f(x)n_k(x)dA\Big)dS. \qquad (4.11)
\end{aligned}$$

Now, observe that

$$\frac{1}{A(\partial B)}\int_{\partial B}\Big(\sum_{k=1}^{n}\frac{\partial f(y)}{\partial x_k}(\overline{x_{B,k}} - y_k)\Big)dS = \sum_{k=1}^{n}\frac{1}{A(\partial B)}\int_{\partial B}\frac{\partial f(y)}{\partial x_k}(\overline{x_{B,k}} - y_k)dS$$

and

$$\begin{aligned}
&\frac{1}{A(\partial B)}\int_{\partial B}\Big(\sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(x_k - y_k)f(x)n_k(x)dA\Big)dS \\
&\quad= \sum_{k=1}^{n}\frac{1}{V(B)}\frac{1}{A(\partial B)}\int_{\partial B}\Big(\int_{\partial B}(x_k - y_k)f(x)n_k(x)dA\Big)dS \\
&\quad= \sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}\Big(x_k - \frac{1}{A(\partial B)}\int_{\partial B} y_k dS\Big)f(x)n_k(x)dA \qquad \text{(by Fubini's theorem)} \\
&\quad= \sum_{k=1}^{n}\frac{1}{V(B)}\int_{\partial B}(x_k - \overline{s_{\partial B,k}})f(x)n_k(x)dA \qquad \text{(by (4.9))}.
\end{aligned}$$

By making use of inequality (4.11), we then obtain the desired result (4.10). □

Remark. By taking $n = 2$ in the above inequalities, we recapture some results from [9], while for $n = 3$ we obtain results from [10]. The details are omitted.

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