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Research Article

Some New Inequalities Involving κ-Fractional Integral for Certain Classes of Functions and Their Applications

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In this article, we present several new inequalities involving the κ-fractional integral for the integrable function $\mathscr{F}$ which satisfies one of the following conditions: (a) $|\mathscr{F}|^q$ is preinvex for some $q > 1$; (b) $\mathscr{F}'$ is bounded; (c) $\mathscr{F}'$ is a Lipschitz function. As applications, we establish new inequalities for the weighted arithmetic and generalized logarithmic means.

1. Introduction

Let $E \subseteq \mathbb{R}$ be a nonempty interval. Then, a real-valued function $f: E \longrightarrow \mathbb{R}$ is said to be convex (concave) on E if the inequality

$$f[\lambda\sigma + (1-\lambda)\tau] \leq (\geq) \lambda f(\sigma) + (1-\lambda) f(\tau), \quad (1)$$

takes place for any $\sigma, \tau \in E$ and $0 \leq \lambda \leq 1$.

We all know that the convexity theory has penetrated into every branch of pure and applied mathematics [1–20], and it has more and more practical applications in physics, mechanics, statistics, operations research, and even in economics and meteorology [21–40]. Many remarkable inequalities in mathematics, control theory, and game theory can be found in the literature [41–60] by use of the convexity theory. In the past half century, to research the generalizations and variants for the convexity has always been a hot topic for mathematicians and physicists as well as engineers. Recently, a great deal of generalizations and variants has been made for the convexity, for example, the GA-convexity and GG-convexity [61], s-convexity [62, 63], preinvex convexity [64], strong convexity [65–68], and Schur convexity [69].

When we talk about convex functions, we have to mention a classical and most important inequality, which is the well-known Hermite–Hadamard inequality [70] which states that the double inequality

$$\mathscr{F}\left(\frac{\varepsilon+\zeta}{2}\right) \leq (\geq) \frac{1}{\zeta-\varepsilon}\int_{\varepsilon}^{\zeta} \mathscr{F}(x)\mathrm{d}x \leq (\geq) \frac{\mathscr{F}(\varepsilon)+\mathscr{F}(\zeta)}{2}, \quad (2)$$

holds for all $\varepsilon, \zeta \in J$ with $\varepsilon \neq \zeta$ if $\mathscr{F}: J \longrightarrow \mathbb{R}$ is a convex (concave) function on J and $J \subseteq \mathbb{R}$ is a nonempty interval. For a long time, numerous researchers have been devoted to the generalizations, improvements, refinements, and variations for inequality (2) [71–73].

The aim of this article is to provide new Hermite–Hadamard-type inequalities for certain classes of functions via the κ-fractional integral and give their applications to the bivariate means.

In order to clearly describe and prove our main results in the next sections, we have to recall some definitions which we present in this section.

Definition 1. Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty set and $\eta: \Omega \times \Omega \longrightarrow \mathbb{R}^n$ be a mapping. Then, Ω is said to be an invex set with respect to the mapping η if

$$\varepsilon + \theta\eta(\zeta, \varepsilon) \in \Omega, \tag{3}$$

for all $\varepsilon, \zeta \in \Omega$ and $\theta \in [0, 1]$.

Definition 2. Let $\Omega \subseteq \mathbb{R}^n$ be an invex set with respect to the mapping $\eta: \Omega \times \Omega \longrightarrow \mathbb{R}^n$. Then, the mapping $\varphi: \Omega \longrightarrow \mathbb{R}$ is said to be preinvex with respect to the mapping η if the inequality

$$\varphi(\varepsilon + \theta\eta(\zeta, \varepsilon)) \leq (1 - \theta)\varphi(\varepsilon) + \theta\varphi(\zeta), \tag{4}$$

holds for all $\varepsilon, \zeta \in \Omega$ and $\theta \in [0, 1]$.

Definition 3 (see [74]). Let $\beta, \kappa > 0$, $\varepsilon, \zeta \in \mathbb{R}$ with $\varepsilon < \zeta$, and $\mathcal{F} \in L[\varepsilon, \zeta]$. Then, the β order κ-fractional integral operators ${}_{\kappa}J^{\beta}_{\varepsilon^+}$ and ${}_{\kappa}J^{\beta}_{\zeta^-}$ of $\mathcal{F}$ are defined by

$$\begin{aligned} {}_{\kappa}\mathcal{T}^{\beta}_{\varepsilon^+}\mathcal{F}(x) &= \frac{1}{\kappa\Gamma_{\kappa}(\beta)} \int_{\varepsilon}^{x} (x - \tau)^{(\beta/\kappa)-1}\mathcal{F}(\tau)\mathrm{d}\tau, \\ {}_{\kappa}\mathcal{T}^{\beta}_{\zeta^-}\mathcal{F}(x) &= \frac{1}{\kappa\Gamma_{\kappa}(\beta)} \int_{x}^{\zeta} (\tau - x)^{(\beta/\kappa)-1}\mathcal{F}(\tau)\mathrm{d}\tau, \end{aligned} \tag{5}$$

respectively, where

$$\Gamma_{\kappa}(x) = \int_0^{\infty} \tau^{x-1}e^{-(\tau^{\kappa}/\kappa)}\mathrm{d}\tau, \tag{6}$$

is the κ-gamma function.

2. Main Results

Throughout this section, we always assume that $\mathbb{Z}^*$ is the set of positive integers, $\beta, \kappa > 0$, $\theta \in [0, 1]$, $\Omega \subseteq \mathbb{R}$ is an open invex set with respect to the mapping $\eta: \Omega \times \Omega \longrightarrow \mathbb{R}\backslash\{0\}$, $\varepsilon, \zeta \in \Omega$ with $\varepsilon < \zeta$, and $\mathcal{F}: \Omega \longrightarrow \mathbb{R}$ is a differentiable mapping such that $\mathcal{F}'$ is integrable on $[\varepsilon, \varepsilon + \eta(\zeta, \varepsilon)]$ for $\eta(\zeta, \varepsilon) > 0$, and

$$\begin{aligned} \Lambda_{\eta}(\beta, \kappa, \theta; x) = {} & (1 - \theta)\frac{(\eta(x, \varepsilon))^{\beta/\kappa}\{\mathcal{F}(\varepsilon) + \mathcal{F}(\varepsilon + \eta(x, \varepsilon))\} + (\eta(\zeta, x))^{\beta/\kappa}\{\mathcal{F}(x) + \mathcal{F}(x + \eta(\zeta, x))\}}{\eta(\zeta, \varepsilon)} \\ & + \frac{\theta}{\eta(\zeta, \varepsilon)}\left[(\eta(x, \varepsilon))^{\beta/\kappa}\left\{\mathcal{F}\left(\varepsilon + \frac{1}{n+1}\eta(x, \varepsilon)\right) + \mathcal{F}\left(\varepsilon + \frac{n}{n+1}\eta(x, \varepsilon)\right)\right\}\right. \\ & \left. + (\eta(\zeta, x))^{\beta/\kappa}\left\{\mathcal{F}\left(x + \frac{n}{n+1}\eta(\zeta, x)\right) + \mathcal{F}\left(x + \frac{1}{n+1}\eta(\zeta, x)\right)\right\}\right] \\ & - \frac{(n+1)^{\beta/\kappa}\Gamma_{\kappa}(\beta + \kappa)}{\eta(\zeta, \varepsilon)}\left[{}_{\kappa}\mathcal{T}^{\beta}_{(\varepsilon+\eta(x,\varepsilon))^-}\mathcal{F}\left(\varepsilon + \frac{n}{n+1}\eta(x, \varepsilon)\right) + {}_{\kappa}\mathcal{T}^{\beta}_{\varepsilon^+}\mathcal{F}\left(\varepsilon + \frac{1}{n+1}\eta(x, \varepsilon)\right)\right. \\ & \left. + {}_{\kappa}\mathcal{T}^{\beta}_{x^+}\mathcal{F}\left(x + \frac{1}{n+1}\eta(\zeta, x)\right) + {}_{\kappa}\mathcal{T}^{\beta}_{(x+\eta(\zeta,x))^-}\mathcal{F}\left(x + \frac{n}{n+1}\eta(\zeta, x)\right)\right]. \end{aligned} \tag{7}$$

Lemma 1. *Let $\Lambda_{\eta}(\beta, \kappa, \theta; x)$ be defined by (7). Then, we have the identity*

$$\begin{aligned} \Lambda_{\eta}(\beta, \kappa, \theta; x) = {} & \frac{(\eta(x, \varepsilon))^{\beta/\kappa}}{(n+1)\eta(\zeta, \varepsilon)}\left\{\int_0^1\left(\tau^{\beta/\kappa} - \theta\right)\mathcal{F}'\left(\varepsilon + \frac{n+\tau}{n+1}\eta(x, \varepsilon)\right)\mathrm{d}\tau - \int_0^1\left(\tau^{\beta/\kappa} - \theta\right)\mathcal{F}'\left(\varepsilon + \frac{1-\tau}{n+1}\eta(x, \varepsilon)\right)\mathrm{d}\tau\right. \\ & - \frac{(\eta(\zeta, x))^{\beta/\kappa}}{(n+1)\eta(\zeta, \varepsilon)}\left\{\int_0^1\left(\tau^{\beta/\kappa} - \theta\right)\mathcal{F}'\left(x + \frac{n+\tau}{n+1}\eta(\zeta, x)\right)\mathrm{d}\tau\right. \\ & \left. - \int_0^1\left(\tau^{\beta/\kappa} - \theta\right)\mathcal{F}'\left(x + \frac{1-\tau}{n+1}\eta(\zeta, x)\right)\mathrm{d}\tau\right\}. \end{aligned} \tag{8}$$

Proof. Making use of integration by parts and variable transformation, one has

$$\begin{aligned}I_1 &= \int_0^1 \left(\tau^{\beta/\kappa} - \theta\right)\mathcal{F}'\left(\varepsilon + \frac{n+\tau}{n+1}\eta(x,\varepsilon)\right)d\tau \\ &= \frac{(n+1)}{\eta(x,\varepsilon)}\left[\left(\tau^{\beta/\kappa} - \theta\right)\mathcal{F}\left(\varepsilon + \frac{n+\tau}{n+1}\eta(x,\varepsilon)\right)\Big|_0^1 - \frac{\beta(n+1)}{\kappa\eta(x,\varepsilon)}\int_0^1 \tau^{\beta/\kappa}\mathcal{F}\left(\varepsilon + \frac{n+\tau}{n+1}\eta(x,\varepsilon)\right)\mathrm{d}\tau\right] \\ &= \frac{(n+1)}{\eta(x,\varepsilon)}\left[(1-\theta)\mathcal{F}(\varepsilon + \eta(x,\varepsilon)) + \theta\mathcal{F}\left(\varepsilon + \frac{n}{n+1}\eta(x,\varepsilon)\right)\right. \\ &\quad \left. - \frac{(n+1)^{(\beta/\kappa)+1}\Gamma_\kappa(\beta+\kappa)}{(\eta(x,\varepsilon))^{(\beta/\kappa)+1}}\,{}_\kappa\mathcal{T}^{\beta}_{(\varepsilon+\eta(x,\varepsilon))^-}\mathcal{F}\left(\varepsilon + \frac{n}{n+1}\eta(x,\varepsilon)\right)\right].\end{aligned} \tag{9}$$

Analogously, we also have

$$\begin{aligned}I_2 = &-\frac{(n+1)}{\eta(x,\varepsilon)}\left[(1-\theta)\mathcal{F}(\varepsilon) + \theta\mathcal{F}\left(\varepsilon + \frac{1}{n+1}\eta(x,\varepsilon)\right)\right. \\ &\left. - \frac{(n+1)^{\beta/\kappa} + \Gamma_\kappa(\beta+\kappa)}{(\eta(x,\varepsilon))^{(\beta/\kappa)+1}}\,{}_\kappa\mathcal{T}^{\beta}_{\varepsilon^+}\mathcal{F}\left(\varepsilon + \frac{1}{n+1}\eta(x,\varepsilon)\right)\right],\end{aligned} \tag{10}$$

$$\begin{aligned}I_3 = &\frac{(n+1)}{\eta(\zeta,x)}\left[(1-\theta)\mathcal{F}(x + \eta(\zeta,x)) + \theta\mathcal{F}\left(x + \frac{n}{n+1}\eta(\zeta,x)\right)\right. \\ &\left. - \frac{(n+1)^{(\beta/\kappa)+1}\Gamma_\kappa(\beta+\kappa)}{(\eta(\zeta,x))^{(\beta/\kappa)+1}}\,{}_\kappa\mathcal{T}^{\beta}_{(x+\eta(\zeta,x))^-}\mathcal{F}\left(\varepsilon + \frac{n}{n+1}\eta(\zeta,x)\right)\right],\end{aligned} \tag{11}$$

$$\begin{aligned}I_4 = &-\frac{(n+1)}{\eta(\zeta,x)}\left[(1-\theta)\mathcal{F}(x + \eta(\zeta,x)) + \theta\mathcal{F}\left(x + \frac{n}{n+1}\eta(\zeta,x)\right)\right. \\ &\left. - \frac{(n+1)^{(\beta/\kappa)+1}\Gamma_\kappa(\beta+\kappa)}{(\eta(\zeta,x))^{(\beta/\kappa)+1}}\,{}_\kappa\mathcal{T}^{\beta}_{x^+}\mathcal{F}\left(\varepsilon + \frac{1}{n+1}\eta(\zeta,x)\right)\right].\end{aligned} \tag{12}$$

Therefore, identity (8) follows from multiplying (9) and (10) by $(\eta(x,\varepsilon))^{\beta/\kappa+2}/[(n+1)\eta(\zeta,\varepsilon)]$, multiplying (11) and (12) by $(\eta(\zeta,x))^{\beta/\kappa+2}/[(n+1)\eta(\zeta,\varepsilon)]$, and then adding them. □

Remark 1. Lemma 1 leads to the conclusions as follows:

(i) Let $n = 1$. Then, Lemma 1 leads to Lemma 2.1 of [75].

(ii) Let $\eta(\omega, \upsilon) = \omega - \upsilon$. Then, one has

$$\begin{aligned}\Lambda(\beta,\kappa,\theta;x) = &(1-\theta)\frac{(x-\varepsilon)^{\beta/\kappa}\{\mathcal{F}(\varepsilon) + \mathcal{F}(x)\} + (\zeta - x)^{\beta/\kappa}\{\mathcal{F}(x) + \mathcal{F}(\zeta)\}}{(\zeta-\varepsilon)} \\ &+ \frac{\theta}{(\zeta-\varepsilon)}\left[(x-\varepsilon)^{\beta/\kappa}\left\{\mathcal{F}\left(\frac{n\varepsilon + x}{n+1}\right) + \mathcal{F}\left(\frac{nx+\varepsilon}{n+1}\right)\right\} + (\zeta - x)^{\beta/\kappa}\left\{\mathcal{F}\left(\frac{x+n\zeta}{n+1}\right) + \mathcal{F}\left(\frac{nx+\zeta}{n+1}\right)\right\}\right] \\ &- \frac{(n+1)^{\beta/\kappa}\Gamma_\kappa(\beta+\kappa)}{(\zeta-\varepsilon)}\left[{}_\kappa\mathcal{T}^{\beta}_{x^-}\mathcal{F}\left(\frac{nx+\varepsilon}{n+1}\right) + {}_\kappa\mathcal{T}^{\beta}_{\varepsilon^+}\mathcal{F}\left(\frac{n\varepsilon+x}{n+1}\right){}_\kappa\mathcal{T}^{\beta}_{x^+}\mathcal{F}\left(\frac{nx+\zeta}{n+1}\right) + {}_\kappa\mathcal{T}^{\beta}_{\zeta^-}\mathcal{F}\left(\frac{x+n\zeta}{n+1}\right)\right].\end{aligned} \tag{13}$$

(iii) If $\theta = 0$ and $\beta = \kappa = 1$, then (13) reduces to

$$\Lambda(1,1,0;x) = \mathcal{F}(x) + \frac{(\zeta - x)\mathcal{F}(\zeta) + (x-\varepsilon)\mathcal{F}(\varepsilon)}{\zeta - \varepsilon} - \frac{(n+1)}{\zeta-\varepsilon}\int_{\varepsilon}^{\zeta}\mathcal{F}(z)\mathrm{d}z. \tag{14}$$

(iv) If $\theta = 1$ and $\beta = \kappa = 1$, then (13) becomes

$$\Lambda(1,1,1;x) = \frac{(x-\varepsilon)\{\mathcal{F}((n\varepsilon + x)/(n+1)) + \mathcal{F}((\varepsilon + nx)/(n+1))\} + (\zeta - x)\{\mathcal{F}((n\zeta + x)/(n+1)) + \mathcal{F}((\zeta + nx)/(n+1))\}}{\zeta-\varepsilon} - \frac{(n+1)}{\zeta-\varepsilon}\int_{\varepsilon}^{\zeta}\mathcal{F}(z)\mathrm{d}z. \tag{15}$$

(v) If $\eta(\omega, \upsilon) = \omega - \upsilon$ and $n = 1$, then Corollary 2.1 of [75] can be derived from Lemma 1.

Theorem 1. *Let $p, q > 1$ such that $1/p + 1/q = 1$ and $|\mathcal{F}|^q$ be preinvex on Ω. Then, the inequality*

$$\begin{aligned}\left|\Lambda_\eta(\beta,\kappa,\theta;x)\right| \le \Psi^{1/p}(\beta,\kappa,\theta,p)\Bigg[&\frac{|(\eta(x,\varepsilon))|^{(\beta/\kappa)+1}}{(n+1)|\eta(\zeta,\varepsilon)|}\left\{\left(\frac{\left|\mathcal{F}'(\varepsilon)\right|^q + (2n+1)\left|\mathcal{F}'(x)\right|^q}{2(n+1)}\right)^{1/q}\right.\\ &\left.+\left(\frac{(2n+1)\left|\mathcal{F}'(\varepsilon)\right|^q + \left|\mathcal{F}'(x)\right|^q}{2(n+1)}\right)^{1/q}\right\}\\ &+\frac{(\eta(\zeta,x))^{(\beta/\kappa)+1}}{(n+1)\eta(\zeta,\varepsilon)}\left\{\left(\frac{\left|\mathcal{F}'(x)\right|^q + (2n+1)\left|\mathcal{F}'(\zeta)\right|^q}{2(n+1)}\right)^{1/q}\right.\\ &\left.+\left(\frac{(2n+1)\left|\mathcal{F}'(x)\right|^q + \left|\mathcal{F}'(\zeta)\right|^q}{2(n+1)}\right)^{1/q}\right\}\Bigg],\end{aligned} \tag{16}$$

holds for all $x \in [\varepsilon, \varepsilon + \eta(\zeta,\varepsilon)]$, where

$$\Psi(\beta,\kappa,\theta,p) = \begin{cases}\dfrac{\kappa}{\beta p + \kappa}, & \theta = 0,\\[2ex] \dfrac{\kappa\left(1 - 2\theta^{\beta p + \kappa/\beta}\right)}{\beta p + \kappa} + 2\theta^{\beta p + \kappa/\beta} - \theta^p, & 0 < \theta < 1,\\[2ex] \dfrac{\Gamma(p+1)\Gamma((\beta/\kappa)+1)}{\Gamma((\beta/\kappa)+1+p)}, & \theta = 1.\end{cases} \tag{17}$$

Proof. It follows from Lemma 1 and the preinvexity of $|\mathcal{F}|^q$ together with the Hölder's inequality that

$$\begin{aligned}\left|\lambda_\eta(\beta,\kappa,\theta;x)\right| \le{} & \frac{(\eta(x,\varepsilon))^{(\beta/\kappa)+1}}{(n+1)(\eta(\zeta,\varepsilon))}\left[\int_0^1\left|\tau^{(\beta/\kappa)}-\theta\right|^p \mathrm{d}\tau\right]^{1/p} \\ & \left[\int_0^1\left|\mathcal{F}'\left(\varepsilon+\frac{n+\tau}{n+1}\eta(x,\varepsilon)\right)\right|^q \mathrm{d}\tau+\int_0^1\left|\mathcal{F}'\left(\varepsilon+\frac{1-\tau}{n+1}\eta(x,\varepsilon)\right)\right|^q \mathrm{d}\tau\right]^{1/q} \\ & -\frac{(\eta(\zeta,x))^{(\beta/\kappa)+1}}{(n+1)\eta(\zeta,\varepsilon)}\left[\int_0^1\left|\tau^{\beta/\kappa}-\theta\right|^p \mathrm{d}\tau\right]^{1/p} \\ & \times\left[\int_0^1\left|\mathcal{F}'\left(x+\frac{n+\tau}{n+1}\eta(\zeta,x)\right)\right|^q \mathrm{d}\tau+\int_0^1\left|\mathcal{F}'\left(x+\frac{1-\tau}{n+1}\eta(\zeta,x)\right)\right|^q \mathrm{d}\tau\right]^{1/q}.\end{aligned} \tag{18}$$

Note that

$$\begin{aligned}&\int_0^1\left|\mathcal{F}'\left(\varepsilon+\frac{n+\tau}{n+1}\eta(x,\varepsilon)\right)\right|^q \mathrm{d}\tau \\ &\le \int_0^1\left(\frac{1-\tau}{n+1}\left|\mathcal{F}'(\varepsilon)\right|^q+\frac{n+\tau}{n+1}\left|\mathcal{F}'(x)\right|^q\right)\mathrm{d}\tau \\ &= \frac{\left|\mathcal{F}'(\varepsilon)\right|^q+(2n+1)\left|\mathcal{F}'(x)\right|^q}{2(n+1)}.\end{aligned} \tag{19}$$

Analogously, we have

$$\begin{aligned}&\int_0^1\left|\mathcal{F}'\left(\varepsilon+\frac{1-\tau}{n+1}\eta(x,\varepsilon)\right)\right|^q \mathrm{d}\tau \le \frac{(2n+1)\left|\mathcal{F}'(\varepsilon)\right|^q+\left|\mathcal{F}'(x)\right|^q}{2(n+1)}, \\ &\int_0^1\left|\mathcal{F}'\left(x+\frac{n+\tau}{n+1}\eta(\zeta,x)\right)\right|^q \mathrm{d}\tau \le \frac{\left|\mathcal{F}'(x)\right|^q+(2n+1)\left|\mathcal{F}'(\zeta)\right|^q}{2(n+1)}, \\ &\int_0^1\left|\mathcal{F}'\left(x+\frac{1-\tau}{n+1}\eta(\zeta,x)\right)\right|^q \mathrm{d}\tau \le \frac{(2n+1)\left|\mathcal{F}'(x)\right|^q+\left|\mathcal{F}'(\zeta)\right|^q}{2(n+1)}.\end{aligned} \tag{20}$$

We clearly see that

$$\int_0^1\left|\tau^{\beta/\kappa}-\theta\right|^p \mathrm{d}\tau=\frac{\kappa}{\beta p+\kappa}, \tag{21}$$

for $\theta=0$,

$$\int_0^1\left|\tau^{\beta/\kappa}-\theta\right|^p \mathrm{d}\tau=\frac{\Gamma(p+1)\Gamma((\beta/\kappa)+1)}{\Gamma((\beta/\kappa)+1+p)}, \tag{22}$$

for $\theta=1$, and

$$\int_0^1\left|\tau^{\beta/\kappa}-\theta\right|^p \mathrm{d}\tau=\frac{\kappa\left(1-2\theta^{(\beta p+\kappa)/\beta}\right)}{\beta p+\kappa}+2\theta^{(\beta p+\kappa)/\beta}-\theta^p, \tag{23}$$

for $0<\theta<1$.

Therefore, inequality (16) can be derived from the above inequalities and identities. □

Remark 2. Theorem 1 leads to the conclusion as follows:

(i) Theorem 2.1 of [75] can be obtained from Theorem 1 if we take $n=1$.

(ii) If $\theta=\kappa=1$ and $\eta(\omega,\upsilon)=\omega-\upsilon$, Theorem 1 reduces to

$$\begin{aligned}|\Lambda(\beta,1,1:x)| ={}& \left|\frac{(x-\varepsilon)^\beta\{\mathcal{F}((n\varepsilon+x)/(n+1))+\mathcal{F}((n\varepsilon+x)/(n+1))\}+(\zeta-x)^\beta\{\mathcal{F}((n\zeta+x)/(n+1))+\mathcal{F}((nx+\zeta)/(n+1))\}}{\zeta-\varepsilon}\right. \\ & \left.-\frac{(n+1)^\beta\Gamma(\beta+1)}{(\zeta-\varepsilon)}\left[{}_\kappa\mathcal{T}^\beta_{x^-}\mathcal{F}\left(\frac{nx+\varepsilon}{n+1}\right)+{}_\kappa\mathcal{T}^\beta_{\varepsilon^+}\mathcal{F}\left(\frac{n\varepsilon+x}{n+1}\right)+{}_\kappa\mathcal{T}^\beta_{x^+}\mathcal{F}\left(\frac{n\zeta+x}{n+1}\right)+{}_\kappa\mathcal{T}^\beta_{\zeta^-}\mathcal{F}\left(\frac{nx+\zeta}{n+1}\right)\right]\right| \\ \le{}& \frac{\Gamma(p+1)\Gamma(\beta+1)}{\Gamma(p+\beta+1)(n+1)^{(1/q)+1}2^{(1/q)}}\left\{\frac{(x-\varepsilon)^{\beta+1}}{\zeta-\varepsilon}\left\{\left(|\mathcal{F}(\varepsilon)|^q+(2n+1)|\mathcal{F}(x)|^q\right)^{1/q}+\left((2n+1)|\mathcal{F}(\varepsilon)|^q+|\mathcal{F}(x)|^q\right)^{1/q}\right\}\right. \\ & \left.+\frac{(\zeta-x)^{\beta+1}}{\zeta-\varepsilon}\left\{\left(|\mathcal{F}(\zeta)|^q+(2n+1)|\mathcal{F}(x)|^q\right)^{1/q}+\left((2n+1)|\mathcal{F}(x)|^q+|\mathcal{F}(\zeta)|^q\right)^{1/q}\right\}\right\}.\end{aligned} \tag{24}$$

(iii) Let $n=1$. Then, (24) leads to Corollary 2.2 of [75].

Theorem 2. *Let $q>1$ and $|\mathscr{F}|^q$ be preinvex on Ω. Then, the inequality*

$$\begin{aligned}
&\left|\Lambda_\eta(\beta,\kappa,\theta;x)\right| \\
&\leq \Phi_1^{1-(1/q)}\left[\frac{(\eta(x,\varepsilon))^{(\beta/\kappa)+1}}{(n+1)(\eta(\zeta,\varepsilon))}\left[\left\{\frac{1}{n+1}\left((\Phi_1-\Phi_2)\left|\mathscr{F}'(\varepsilon)\right|^q\right.\right.\right.\right. \\
&\quad\left.\left.+(\Phi_1+n\Phi_2)\left|\mathscr{F}'(x)\right|^q\right)\right\}^{1/q} \\
&\quad\left.+\left\{\frac{1}{n+1}\left((n\Phi_1+\Phi_2)\left|\mathscr{F}'(\varepsilon)\right|^q+(\Phi_1-\Phi_2)\left|\mathscr{F}'(x)\right|^q\right)\right\}^{1/q}\right] \\
&\quad+\frac{(\eta(x,\varepsilon))^{(\beta/\kappa)+1}}{(n+1)(\eta(\zeta,\varepsilon))}\left[\left\{\frac{1}{n+1}\left((\Phi_1-\Phi_2)\left|\mathscr{F}'(x)\right|^q\right.\right.\right. \\
&\quad\left.\left.+(\Phi_1+n\Phi_2)\left|\mathscr{F}'(\zeta)\right|^q\right)\right\}^{1/q} \\
&\quad\left.\left.+\left\{\frac{1}{n+1}\left((n\Phi_1+\Phi_2)\left|\mathscr{F}'(x)\right|^q+(\Phi_1-\Phi_2)\left|\mathscr{F}'(\zeta)\right|^q\right)\right\}^{1/q}\right]\right]
\end{aligned} \tag{25}$$

holds for $x\in[\varepsilon,\varepsilon+\eta(\zeta,\varepsilon)]$, where

$$\begin{aligned}
\Phi_1 &= \int_0^1\left|\tau^{\beta/\kappa}-\theta\right|\mathrm{d}\tau=\frac{\left(\kappa+2\beta\theta^{(\beta+\kappa)/\beta}\right)}{\beta+\kappa}-\theta, \\
\Phi_2 &= \int_0^1\tau\left|\tau^{\beta/\kappa}-\theta\right|\mathrm{d}\tau=\frac{\left(\kappa+\beta\theta^{(\beta+2\kappa)/\beta}\right)}{\beta+2\kappa}-\frac{\theta}{2}.
\end{aligned} \tag{26}$$

Proof. It follows from Lemma 1 and the preinvexity of $|\mathscr{F}|^q$ together with the power-mean inequality that

$$\begin{aligned}
\left|\Lambda_\eta(\beta,\kappa,\theta;x)\right| \leq{}& \frac{(\eta(x,\varepsilon))^{(\beta/\kappa)+1}}{(n+1)(\eta(\zeta,\varepsilon))}\left\{\int_0^1\left|\left(\tau^{\beta/\kappa}-\theta\right)\right|\mathrm{d}\tau\right\}^{1-(1/q)}\left[(\chi_1)^{1/q}+(\chi_2)^{1/q}\right] \\
&-\frac{(\eta(\zeta,x))^{(\beta/\kappa)+1}}{(n+1)\eta(\zeta,\varepsilon)}\left\{\int_0^1\left|\left(\tau^{\beta/\kappa}-\theta\right)\right|\mathrm{d}\tau\right\}^{1-(1/q)}\left[(\chi_3)^{1/q}+(\chi_4)^{1/q}\right],
\end{aligned} \tag{27}$$

where

$$\begin{aligned}
\chi_1 &= \int_0^1\left|\left(\tau^{\beta/\kappa}-\theta\right)\right|\left|\mathscr{F}'\left(\varepsilon+\frac{n+\tau}{n+1}\eta(x,\varepsilon)\right)\right|^q\mathrm{d}\tau \\
&\leq \int_0^1\left|\tau^{\beta/\kappa}-\theta\right|\left(\frac{1-\tau}{n+1}\left|\mathscr{F}'(\varepsilon)\right|^q\right)+\frac{n+\tau}{n+1}\left|\mathscr{F}'(x)\right|^q\mathrm{d}\tau \\
&= \frac{1}{n+1}\left[\left(\int_0^1\left|\left(\tau^{\beta/\kappa}-\theta\right)\right|\mathrm{d}\tau-\int_0^1\tau\left|\left(\tau^{\beta/\kappa}-\theta\right)\right|\mathrm{d}\tau\right)\left|\mathscr{F}'(\varepsilon)\right|^q+\left(\int_0^1\left|\left(\tau^{\beta/\kappa}-\theta\right)\right|\mathrm{d}\tau+n\int_0^1\tau\left|\left(\tau^{\beta/\kappa}-\theta\right)\right|\mathrm{d}\tau\right)\left|\mathscr{F}'(x)\right|^q\right], \\
\chi_2 &= \int_0^1\left|\left(\tau^{\beta/\kappa}-\theta\right)\right|\left|\mathscr{F}'\left(\varepsilon+\frac{1-\tau}{n+1}\eta(x,\varepsilon)\right)\right|^q\mathrm{d}\tau \\
&\leq \frac{1}{n+1}\left[\left(n\int_0^1\left|\left(\tau^{\beta/\kappa}-\theta\right)\right|\mathrm{d}\tau+\int_0^1\tau\left|\left(\tau^{\beta/\kappa}-\theta\right)\right|\mathrm{d}\tau\right)\left|\mathscr{F}'(\varepsilon)\right|^q+\left(\int_0^1\left|\left(\tau^{\beta/\kappa}-\theta\right)\right|\mathrm{d}\tau-\int_0^1\tau\left|\left(\tau^{\beta/\kappa}-\theta\right)\right|\mathrm{d}\tau\right)\left|\mathscr{F}'(x)\right|^q\right], \\
\chi_3 &= \int_0^1\left|\left(\tau^{\beta/\kappa}-\theta\right)\right|\left|\mathscr{F}'\left(x+\frac{n+\tau}{n+1}\eta(\zeta,x)\right)\right|^q\mathrm{d}\tau \\
&\leq \frac{1}{n+1}\left[\left(\int_0^1\left|\left(\tau^{\beta/\kappa}-\theta\right)\right|\mathrm{d}\tau-\int_0^1\tau\left|\tau^{\beta/\kappa}-\theta\right|\mathrm{d}\tau\right)\left|\mathscr{F}'(x)\right|^q+\left(\int_0^1\left|\left(\tau^{\beta/\kappa}-\theta\right)\right|\mathrm{d}\tau+n\int_0^1\tau\left|\left(\tau^{\beta/\kappa}-\theta\right)\right|\mathrm{d}\tau\right)\left|\mathscr{F}'(\zeta)\right|^q\right], \\
\chi_4 &= \int_0^1\left|\left(\tau^{\beta/\kappa}-\theta\right)\right|\left|\mathscr{F}'\left(x+\frac{1-\tau}{n+1}\eta(\zeta,x)\right)^q\right|\mathrm{d}\tau \\
&\leq \frac{1}{n+1}\left[\left(n\int_0^1\left(\tau^{\beta/\kappa}-\theta\right)\mathrm{d}\tau+\int_0^1\tau\left|\left(\tau^{\beta/\kappa}-\theta\right)\right|\mathrm{d}\tau\right)\left|\mathscr{F}'(x)\right|^q+\left(\int_0^1\left|\left(\tau^{\beta/\kappa}-\theta\right)\right|\mathrm{d}\tau-\int_0^1\tau\left|\left(\tau^{\beta/\kappa}-\theta\right)\right|\mathrm{d}\tau\right)\left|\mathscr{F}'(\zeta)\right|^q\right].
\end{aligned} \tag{28}$$

Substituting the above inequalities in (27), we get inequality (25). □

Remark 3. From Theorem 2, we have two conclusions as follows:

(i) If $n = 1$, then we get Theorem 2.2 of [76].

(ii) If $\theta = \kappa = 1$ and $\eta(\omega, \upsilon) = \omega - \upsilon$, then

$$|\Lambda(\beta, 1, 1; x)| \leq \frac{1}{(2(\beta+2))^{1/q}(\zeta-\varepsilon)}\left(\frac{\beta}{(n+1)(\beta+1)}\right)^{1+(1/q)} \times\left[(x-\varepsilon)^{\beta}\left\{\left((\beta+2)\left|\mathcal{F}'(\varepsilon)\right|^{q}+(\beta(n+2)+(n+4))\left|\mathcal{F}'(x)\right|^{q}\right)^{1/q}+\left((\beta+2)\left|\mathcal{F}'(x)\right|^{q}+(\beta(n+2)+(n+4))\left|\mathcal{F}'(\varepsilon)\right|^{q}\right)^{1/q}\right\}\right. \left.+(\zeta-x)^{\beta}\left\{\left((\beta+2)\left|\mathcal{F}'(\zeta)\right|^{q}+(\beta(n+2)+(n+4))\left|\mathcal{F}'(x)\right|^{q}\right)^{1/q}+\left((\beta+2)\left|\mathcal{F}'(x)\right|^{q}+(\beta(n+2)+(n+4))\left|\mathcal{F}'(\zeta)\right|^{q}\right)^{1/q}\right\}\right]. \tag{29}$$

Theorem 3. *If $r, \mathcal{R} \in (0,\infty)$ with $r < \mathcal{R}$, and $r \leq \mathcal{F}'(y) \leq \mathcal{R}$ for all $y \in [\varepsilon, \varepsilon + \eta(\zeta, \varepsilon)]$, then we have*

$$\left|\Lambda_{\eta}(\beta, \kappa, \theta; x)\right| \leq \frac{\left[(\eta(x,\varepsilon))^{\beta/\kappa}+(\eta(\zeta,x))^{\beta/\kappa}\right](\mathcal{R}-r)}{(n+1)(\eta(\zeta,\varepsilon))} \cdot\left[\frac{\left(\kappa+2\beta\theta^{(\beta+\kappa/\beta)}\right)}{\beta+\kappa}-\theta\right]. \tag{30}$$

Proof. It follows from Lemma 1 that

$$\begin{aligned}\Lambda_{\eta}(\beta,\kappa,\theta;x) = {} & \frac{(\eta(x,\varepsilon))^{\beta/\kappa}}{(n+1)\eta(\zeta,\varepsilon)}\left\{\int_0^1\left(\tau^{\beta/\kappa}-\theta\right)\left[\mathcal{F}'\left(\varepsilon+\frac{n+\tau}{n+1}\eta(x,\varepsilon)\right)-\frac{r+\mathcal{R}}{2}\right]d\tau\right. \\ & \left.-\int_0^1\left(\tau^{\beta/\kappa}-\theta\right)\left[\mathcal{F}'\left(\varepsilon+\frac{1-\tau}{n+1}\eta(x,\varepsilon)\right)-\frac{r+\mathcal{R}}{2}\right]d\tau\right\} \\ & -\frac{(\eta(\zeta,x))^{\beta/\kappa}}{(n+1)\eta(\zeta,\varepsilon)}\left\{\int_0^1\left(\tau^{\beta/\kappa}-\theta\right)\left[\mathcal{F}'\left(x+\frac{n+\tau}{n+1}\eta(\zeta,x)\right)-\frac{r+\mathcal{R}}{2}\right]d\tau\right. \\ & \left.-\int_0^1\left(\tau^{\beta/\kappa}-\theta\right)\left[\mathcal{F}'\left(x+\frac{1-\tau}{n+1}\eta(\zeta,x)\right)-\frac{r+\mathcal{R}}{2}\right]d\tau\right\}.\end{aligned} \tag{31}$$

Making use of the fact that

$$r-\frac{r+\mathcal{R}}{2} \leq \mathcal{F}'\left(\varepsilon+\frac{1-\tau}{n+1}\eta(\varepsilon,x)\right)-\frac{r+\mathcal{R}}{2} \leq \mathcal{R}-\frac{r+\mathcal{R}}{2}, \tag{32}$$

one has

$$\left|\mathcal{F}'\left(\varepsilon+\frac{n+\tau}{n+1}\eta(x,\varepsilon)\right)-\frac{r+\mathcal{R}}{2}\right| \leq \frac{\mathcal{R}-r}{2}. \tag{33}$$

Similarly, we have

$$\begin{aligned}&\left|\mathcal{F}'\left(\varepsilon+\frac{1-\tau}{n+1}\eta(x,\varepsilon)\right)-\frac{r+\mathcal{R}}{2}\right| \leq \frac{\mathcal{R}-r}{2}, \\ &\left|\mathcal{F}'\left(x+\frac{n+\tau}{n+1}\eta(\zeta,x)\right)-\frac{r+\mathcal{R}}{2}\right| \leq \frac{\mathcal{R}-r}{2}, \\ &\left|\mathcal{F}'\left(x+\frac{1-\tau}{n+1}\eta(\zeta,x)\right)-\frac{r+\mathcal{R}}{2}\right| \leq \frac{\mathcal{R}-r}{2}.\end{aligned} \tag{34}$$

Therefore,

$$\begin{aligned}
\left|\Lambda_\eta(\beta,\kappa,\theta;x)\right| &\le \frac{(\eta(x,\varepsilon))^{\beta/\kappa}}{(n+1)(\eta(\zeta,\varepsilon))}\left\{\int_0^1\left|\tau^{\beta/\kappa}-\theta\right|\left|\mathscr{F}'\left(\varepsilon+\frac{n+\tau}{n+1}\eta(x,\varepsilon)\right)-\frac{r+\mathscr{R}}{2}\right|\mathrm{d}\tau\right.\\
&\quad\left.-\int_0^1\left|\tau^{\beta/\kappa}-\theta\right|\left|\mathscr{F}'\left(\varepsilon+\frac{1-\tau}{n+1}\eta(x,\varepsilon)\right)-\frac{r+\mathscr{R}}{2}\right|\mathrm{d}\tau\right\}\\
&\quad-\frac{(\eta(\zeta,x))^{\beta/\kappa}}{(n+1)\eta(\zeta,\varepsilon)}\left\{\int_0^1\left|\tau^{\beta/\kappa}-\theta\right|\left|\mathscr{F}'\left(x+\frac{n+\tau}{n+1}\eta(\zeta,x)\right)-\frac{r+\mathscr{R}}{2}\right|\mathrm{d}\tau\right.\\
&\quad\left.-\int_0^1\left|\tau^{\beta/\kappa}-\theta\right|\left|\mathscr{F}'\left(x+\frac{1-\tau}{n+1}\eta(\zeta,x)\right)-\frac{r+\mathscr{R}}{2}\right|\mathrm{d}\tau\right\}\\
&\le\frac{(\eta(x,\varepsilon))^{\beta/\kappa}}{(n+1)(\eta(\zeta,\varepsilon))}\frac{\mathscr{R}-r}{2}\int_0^1\left|\tau^{\beta/\kappa}-\theta\right|\mathrm{d}\tau+\frac{(\eta(\zeta,x))^{\beta/\kappa}}{(n+1)(\eta(\zeta,\varepsilon))}\frac{\mathscr{R}-r}{2}\int_0^1\left|\tau^{\beta/\kappa}-\theta\right|\mathrm{d}\tau\\
&=\frac{\left[(\eta(x,\varepsilon))^{\beta/\kappa}+(\eta(\zeta,x))^{\beta/\kappa}\right](\mathscr{R}-r)}{(n+1)(\eta(\zeta,\varepsilon))}\Phi_1,
\end{aligned}\tag{35}$$

which completes the proof of Theorem 3. □

Remark 4. From Theorem 3, we get two conclusions as follows:

(i) Let $n=1$. Then, Theorem 3.1 of [75] can be derived from Theorem 3.

(ii) If $\beta=\kappa=1$ and $\eta(\omega,v)=\omega-v$, then Theorem 3 leads to

$$\begin{aligned}
\left|\Lambda_\eta(1,1,\theta;x)\right| &= \left|(1-\theta)\frac{(x-\varepsilon)\{\mathscr{F}(\varepsilon)+\mathscr{F}(x)\}+(\zeta-x)\{\mathscr{F}(\zeta)+\mathscr{F}(x)\}}{\zeta-\varepsilon}\right.\\
&\quad+\theta\frac{(x-\varepsilon)\{\mathscr{F}((n\varepsilon+x)/(n+1))+\mathscr{F}((\varepsilon+nx)/(n+1))\}+(\zeta-x)\{\mathscr{F}((n\zeta+x)/(n+1))+\mathscr{F}((\zeta+nx)/(n+1))\}}{(\zeta-\varepsilon)}\\
&\quad\left.-\frac{n+1}{\zeta-\varepsilon}\int_\varepsilon^\zeta\mathscr{F}(z)\mathrm{d}z\right|\le\frac{(\mathscr{R}+r)\left((x-\varepsilon)^2+(\zeta-x)^2\right)}{(n+1)(\zeta-\varepsilon)}\left(\frac{2\theta^2-2\theta+1}{2}\right).
\end{aligned}\tag{36}$$

(iii) If $n=1$, then (36) becomes Corollary 3.1 of [75].

(iv) If $\theta=0$, then (36) leads to

$$\begin{aligned}
&\left|\frac{(x-\varepsilon)\{\mathscr{F}(\varepsilon)+\mathscr{F}(x)\}+(\zeta-x)\{\mathscr{F}(\zeta)+\mathscr{F}(x)\}}{\zeta-\varepsilon}-\frac{n+1}{\zeta-\varepsilon}\int_\varepsilon^\zeta\mathscr{F}(z)\mathrm{d}z\right|\\
&\le\frac{(\mathscr{R}+r)\left((x-\varepsilon)^2+(\zeta-x)^2\right)}{2(n+1)(\zeta-\varepsilon)}.
\end{aligned}\tag{37}$$

(v) If $\theta = 1/2$, then (36) leads to the conclusion that

$$\left|\frac{(x-\varepsilon)\{\mathcal{F}(\varepsilon)+\mathcal{F}(x)\}+(\zeta-x)\{\mathcal{F}(\zeta)+\mathcal{F}(x)\}}{2(\zeta-\varepsilon)}\right.$$
$$+\frac{(x-\varepsilon)\mathcal{F}((n\varepsilon+x)/(n+1))+\mathcal{F}((\varepsilon+nx)/(n+1))+(\zeta-x)\{\mathcal{F}((n\zeta+x)/(n+1))+\mathcal{F}((\zeta+nx)/(n+1))\}}{2(\zeta-\varepsilon)}$$
$$\left.-\frac{n+1}{\zeta-\varepsilon}\int_{\varepsilon}^{\zeta}\mathcal{F}(z)\mathrm{d}z\right|\leq\frac{(\mathcal{R}+r)\left((x-\varepsilon)^2+(\zeta-x)^2\right)}{4(n+1)(\zeta-\varepsilon)}. \quad (38)$$

(vi) If $\theta = 1$, then (36) gives

$$\left|\frac{(x-\varepsilon)\{\mathcal{F}((n\varepsilon+x)/(n+1))+\mathcal{F}((\varepsilon+nx)/(n+1))\}+(\zeta-x)\{\mathcal{F}((n\zeta+x)/(n+1))+\mathcal{F}((\zeta+nx)/(n+1))\}}{(\zeta-\varepsilon)}\right.$$
$$\left.-\frac{n+1}{\zeta-\varepsilon}\int_{\varepsilon}^{\zeta}\mathcal{F}(z)\mathrm{d}z\right|\leq\frac{(\mathcal{R}+r)\left((x-\varepsilon)^2+(\zeta-x)^2\right)}{2(n+1)(\zeta-\varepsilon)}. \quad (39)$$

Theorem 4. *If $\mathcal{F}'$ is a Lipschitz function on Ω with the Lipschitz constant $\mathcal{L}>0$, then the fractional integral inequality*

$$\left|\Lambda_{\eta}(\beta,\kappa,\theta;x)\right|\leq\mathcal{L}\left[\frac{(\eta(x,\varepsilon))^{(\beta/\kappa)+2}+\mathcal{L}(\eta(\zeta,x))^{(\beta/\kappa)+2}}{(n+1)(\eta(\zeta,\varepsilon))}\right]$$
$$\times\left[\frac{n-1}{n+1}\left(\frac{\left(\kappa+2\beta\theta^{(\beta+\kappa)/\beta}\right)}{\beta+\kappa}-\theta\right)\right.$$
$$\left.+\frac{2}{n+1}\left(\frac{\left(\kappa+\beta\theta^{(\beta+2\kappa)/\beta}\right)}{\beta+2\kappa}-\frac{\theta}{2}\right)\right], \quad (40)$$

holds for $x\in[\varepsilon,\varepsilon+\eta(\zeta,\varepsilon)]$.

Proof. It follows from Lemma 1 that

$$\Lambda_{\eta}(\beta,\kappa,\theta;x)=\frac{(\eta(x,\varepsilon))^{(\beta/\kappa)+1}}{(n+1)\eta(\zeta,\varepsilon)}\left\{\int_0^1\left(\tau^{\beta/\kappa}-\theta\right)\left\{\mathcal{F}'\left(\varepsilon+\frac{n+\tau}{n+1}\eta(x,\varepsilon)\right)\right.\right.$$
$$\left.\left.-\mathcal{F}'\left(\varepsilon+\frac{1-\tau}{n+1}\eta(x,\varepsilon)\right)\right\}\mathrm{d}\tau\right\}-\frac{(\eta(\zeta,x))^{(\beta/\kappa)+1}}{(n+1)\eta(\zeta,\varepsilon)}\left\{\int_0^1\left(\tau^{\beta/\kappa}-\theta\right)\left\{\mathcal{F}'\left(x+\frac{n+\tau}{n+1}\eta(\zeta,x)\right)\right.\right. \quad (41)$$
$$\left.\left.-\mathcal{F}'\left(x+\frac{1-\tau}{n+1}\eta(\zeta,x)\right)\right\}\mathrm{d}\tau\right\}.$$

Since $\mathcal{F}'$ is a Lipschitz function on $[\varepsilon, \varepsilon + \eta(\zeta, \varepsilon)]$ with Lipschitz constant $\mathcal{L} > 0$, we get

$$\left|\mathcal{F}'\left(\varepsilon + \frac{n+\tau}{n+1}\eta(x,\varepsilon)\right) - \mathcal{F}'\left(\varepsilon + \frac{1-\tau}{n+1}\eta(x,\varepsilon)\right)\right| \le \mathcal{L}\left(\frac{2\tau+n-1}{n+1}\right)(\eta(x,\varepsilon)). \tag{42}$$

Similarly, we have

$$\left|\mathcal{F}'\left(x + \frac{n+\tau}{n+1}\eta(\zeta,x)\right) - \mathcal{F}'\left(x + \frac{1-\tau}{n+1}\eta(\zeta,x)\right)\right| \le \mathcal{L}\left(\frac{2\tau+n-1}{n+1}\right)(\eta(\zeta,x)). \tag{43}$$

Therefore, one has

$$\begin{aligned}
&\left|\Lambda_\eta(\beta,\kappa,\theta;x)\right| \\
&\le \frac{(\eta(x,\varepsilon))^{(\beta/\kappa)+1}}{(n+1)(\eta(\zeta,\varepsilon))}\left[\int_0^1\left|\tau^{\beta/\kappa}-\theta\right|\left|\mathcal{F}'\left(\varepsilon+\frac{n+\tau}{n+1}\eta(x,\varepsilon)\right) - \mathcal{F}'\left(\varepsilon+\frac{1-\tau}{n+1}\eta(x,\varepsilon)\right)\right|d\tau\right] \\
&- \frac{(\eta(\zeta,x))^{(\beta/\kappa)+1}}{(n+1)(\eta(\zeta,\varepsilon))}\left[\int_0^1\left|\tau^{\beta/\kappa}-\theta\right|\left|\mathcal{F}'\left(x+\frac{n+\tau}{n+1}\eta(\zeta,x)\right) - \mathcal{F}'\left(x+\frac{1-\tau}{n+1}\eta(\zeta,x)\right)\right|d\tau\right] \\
&\le \mathcal{L}\left(\frac{2\tau+n-1}{n+1}\right)\left[\frac{(\eta(x,\varepsilon))^{(\beta/\kappa)+2}+(\eta(\zeta,x))^{(\beta/\kappa)+2}}{(n+1)(\eta(\zeta,\varepsilon))}\right] \\
&\quad\left\{\frac{n-1}{n+1}\int_0^1\left|\tau^{\beta/\kappa}-\theta\right|d\tau + \frac{2}{n+1}\int_0^1\tau\left|\tau^{\beta/\kappa}-\theta\right|d\tau\right. \\
&= \mathcal{L}\left[\frac{(\eta(x,\varepsilon))^{(\beta/\kappa)+2}+(\eta(\zeta,x))^{(\beta/\kappa)+2}}{(n+1)(\eta(\zeta,\varepsilon))}\right] \\
&\quad\left[\frac{n-1}{n+1}\left(\frac{\left(\kappa+2\beta\theta^{(\beta+\kappa)/\beta}\right)}{\beta+\kappa}-\theta\right) + \frac{2}{n+1}\left(\frac{\left(\kappa+\beta\theta^{(\beta+2\kappa)/\beta}\right)}{\beta+2\kappa}-\frac{\theta}{2}\right)\right].
\end{aligned} \tag{44}$$

□

Remark 5. From Theorem 4, we clearly see that the following six results are true:

(i) If $n = 1$, then we get

$$\left|\Lambda_\eta(\beta,\kappa,\theta;x)\right| \le \mathcal{L}\left[\frac{(\eta(x,\varepsilon))^{(\beta/\kappa)+2}+(\eta(\zeta,x))^{(\beta/\kappa)+2}}{2(\eta(\zeta,\varepsilon))}\right]\left[\left(\frac{\left(\kappa+\beta\theta^{(\beta+2\kappa)/\beta}\right)}{\beta+2\kappa}-\frac{\theta}{2}\right)\right]. \tag{45}$$

(ii) If $\eta(\omega, v) = \omega - v$, then (45) becomes

$$|\Lambda(\beta,\kappa,\theta;x)| \le \mathcal{L}\left[\frac{(x-\varepsilon)^{(\beta/\kappa)+2}+(\zeta-x)^{(\beta/\kappa)+2}}{2(\zeta-\varepsilon)}\right]\left[\left(\frac{\left(\kappa+\beta\theta^{(\beta+2\kappa)/\beta}\right)}{\beta+2\kappa}-\frac{\theta}{2}\right)\right]. \tag{46}$$

(iii) If $\beta = \kappa = 1$, then (46) reduces to

$$|\Lambda(1,1,\theta;x)| \le \mathcal{L}\left[\frac{(x-\varepsilon)^3+(\zeta-x)^3}{(\zeta-\varepsilon)}\right]\left(\frac{2\theta^3-3\theta+2}{6}\right). \tag{47}$$

(iv) If $\theta = 0$, then inequality (47) leads to

$$|\Lambda(1,1,0;x)| = \left|\frac{(\zeta-x)\mathcal{F}(\zeta)+(x-\varepsilon)\mathcal{F}(\varepsilon)}{\zeta-\varepsilon} - \frac{2}{\zeta-\varepsilon}\int_\varepsilon^\zeta\mathcal{F}(z)dz\right| \le \mathcal{L}\left[\frac{(x-\varepsilon)^3+(\zeta-x)^3}{3(\zeta-\varepsilon)}\right]. \tag{48}$$

(v) If $\theta = 1/2$, then inequality (47) gives

$$\begin{aligned}
\left|\Lambda\left(1,1,\frac{1}{2};x\right)\right| &= \left|\frac{(x-\varepsilon)\{\mathcal{F}(\varepsilon)+\mathcal{F}(x)\}+(\zeta-x)\{\mathcal{F}(\zeta)+\mathcal{F}(x)\}}{2(\zeta-\varepsilon)}\right. \\
&\quad\left.+\frac{(x-\varepsilon)\{\mathcal{F}((\varepsilon+x)/2)+(\zeta-x)\{\mathcal{F}((\zeta+x)/2)\}\}}{2(\zeta-\varepsilon)} - \frac{2}{\zeta-\varepsilon}\int_\varepsilon^\zeta\mathcal{F}(z)dz\right| \\
&\le \mathcal{L}\left[\frac{\left|(x-\varepsilon)^3\right|+\left|(\zeta-x)^3\right|}{6|(\zeta-\varepsilon)|}\right].
\end{aligned} \tag{49}$$

(vi) If $\theta = 1$, then inequality (47) becomes

$$\Lambda(1,1,1;x) = \frac{(x-\varepsilon)\mathcal{F}((\varepsilon+x)/2)(\zeta-x)\mathcal{F}((\zeta+x)/2)}{\zeta-\varepsilon} - \frac{1}{\zeta-\varepsilon}\int_{\varepsilon}^{\zeta}\mathcal{F}(z)dz \le \mathcal{L}\left[\frac{(x-\varepsilon)^3+(\zeta-x)^3}{12(\zeta-\varepsilon)}\right]. \tag{50}$$

3. Applications to Special Bivariate Means

A bivariate function $\Upsilon\colon (0,\infty)\times(0,\infty)\longrightarrow(0,\infty)$ is said to be a mean if $\min\{\varepsilon,\zeta\}\le\Upsilon(\varepsilon,\zeta)\le\max\{\varepsilon,\zeta\}$ for all $\varepsilon,\zeta\in(0,\infty)$. It is well known that the bivariate means are closely related to the special functions. Recently, the inequalities between different bivariate means have attracted the attention of many researchers [77–82].

In this section, we use our obtained results in Section 2 to provide two new inequalities between the weighted arithmetic mean

$$\mathcal{A}(\varepsilon,\zeta;w_1,w_2) = \frac{w_1\varepsilon+w_2\zeta}{w_1+w_2} \quad (\varepsilon,\zeta,w_1,w_2>0) \tag{51}$$

and sth generalized logarithmic mean

$$\Lambda_s(\varepsilon,\zeta) = \left[\frac{\zeta^{s+1}-\varepsilon^{s+1}}{(s+1)(\zeta-\varepsilon)}\right]^{1/s} \quad (\varepsilon,\zeta>0,\varepsilon\ne\zeta,s\in\mathbb{Z}\backslash\{-1,0\}). \tag{52}$$

Theorem 5. *Let $p,q>1$ with $1/p+1/q=1$, $\varepsilon,\zeta>0$ with $\varepsilon\ne\zeta$, and $s\ge2$ be a positive integer. Then, one has*

$$\begin{aligned}&\left|\Lambda\left(\mathcal{A}^s(\varepsilon,\zeta,3,1)\right|,\mathcal{A}^s(\varepsilon,\zeta,1,3)\right)-\Lambda_s^s(\varepsilon,\zeta)\Big|\\ &\le\frac{s(\zeta-\varepsilon)}{8}\left(\frac{1}{2}\right)^{1+(2/q)}\left(\frac{1}{1+p}\right)^{1/p}\\ &\times\left[\mathcal{A}^{1/q}\left(|\varepsilon|^{q(s-1)},\left|\frac{\varepsilon+\zeta}{2}\right|^{q(s-1)};1,3\right)\right.\\ &+\mathcal{A}^{1/q}\left(|\varepsilon|^{q(s-1)},\left|\frac{\varepsilon+\zeta}{2}\right|^{q(s-1)};3,1\right)\\ &+\mathcal{A}^{1/q}\left(\left|\frac{\varepsilon+\zeta}{2}\right|^{q(s-1)},|\zeta|^{q(s-1)};1,3\right)\\ &\left.+\mathcal{A}^{1/q}\left(\left|\frac{\varepsilon+\zeta}{2}\right|^{q(s-1)},|\zeta|^{q(s-1)};3,1\right)\right].\end{aligned} \tag{53}$$

Proof. Let $\beta=\kappa=\theta=1$, $\eta(\omega,\upsilon)=\omega-\upsilon$, and $\mathcal{F}(x)=x^s$. Then, Theorem 5 follows from Theorem 1. □

Theorem 6. *Let $p,q>1$ with $1/p+1/q=1$, $\varepsilon,\zeta>0$ with $\varepsilon\ne\zeta$, and $s\ge2$ be a positive integer. Then, one has*

$$\begin{aligned}&\left|\Lambda\left(\mathcal{A}^s(\varepsilon,\zeta,3,1)\right|,\mathcal{A}^s(\varepsilon,\zeta,1,3)\right)-\Lambda_s^s(\varepsilon,\zeta)\Big|\le\frac{1}{4}\left(\frac{11}{24}\right)^{1/q}\\ &\times\left[\mathcal{A}^{1/q}\left(|\varepsilon|^{q(s-1)},\left|\frac{\varepsilon+\zeta}{2}\right|^{q(s-1)};3,8\right)\right.\\ &+\mathcal{A}^{1/q}\left(|\varepsilon|^{q(s-1)},\left|\frac{\varepsilon+\zeta}{2}\right|^{q(s-1)};8,3\right)\\ &+\mathcal{A}^{1/q}\left(\left|\frac{\varepsilon+\zeta}{2}\right|^{q(s-1)},|\zeta|^{q(s-1)};3,8\right)\\ &\left.+\mathcal{A}^{1/q}\left(\left|\frac{\varepsilon+\zeta}{2}\right|^{q(s-1)},|\zeta|^{q(s-1)};8,3\right)\right].\end{aligned} \tag{54}$$

Proof. Theorem 6 follows directly from Theorem 2 if we take $\beta=\kappa=\theta=1$ and $\eta(\omega,\upsilon)=\omega-\upsilon$ together with $\mathcal{F}(x)=x^s$. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally to writing of this paper. All authors read and approved the final manuscript.

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