Some inequalities for weighted and integral means of convex functions on linear spaces

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SOME INEQUALITIES FOR WEIGHTED AND INTEGRAL MEANS OF CONVEX FUNCTIONS ON LINEAR SPACES

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Abstract. Let f be a convex function on a convex subset C of a linear space and x, $y \in C$, with $x \neq y$. If $p : [0,1] \to \mathbb{R}$ is a Lebesgue integrable and symmetric function, namely $p(1-t) = p(t)$ for all $t \in [0,1]$ and such that the condition

$$0 \leq \int_0^{\tau} p(s)\,ds \leq \int_0^1 p(s)\,ds \text{ for all } \tau \in [0,1]$$

holds, then we have

$$\left| \frac{1}{\int_0^1 p(\tau)\,d\tau} \int_0^1 p(\tau) f((1-\tau)x + \tau y)\,d\tau - \int_0^1 f((1-\tau)x + \tau y)\,d\tau \right|$$
$$\leq \frac{1}{\int_0^1 p(\tau)\,d\tau} \int_0^1 \left(\int_0^{\tau} p(s)\,ds \right) (1-\tau)\,d\tau \left[\nabla_- f_y(y-x) - \nabla_+ f_x(y-x) \right]$$
$$\leq \frac{1}{2} \left[\nabla_- f_y(y-x) - \nabla_+ f_x(y-x) \right].$$

Some applications for norms and semi-inner products are also provided.

1. Introduction

Let X be a real linear space, x, $y \in X$, $x \neq y$ and let

$$[x,y] := \{(1-\lambda)x + \lambda y,\ \lambda \in [0,1]\}$$

be the *segment* generated by x and y. We consider the function $f : [x,y] \to \mathbb{R}$ and the attached function $\varphi_{(x,y)} : [0,1] \to \mathbb{R}$, $\varphi_{(x,y)}(t) := f[(1-t)x + ty]$, $t \in [0,1]$.

It is well known that f is convex on $[x,y]$ iff $\varphi(x,y)$ is convex on $[0,1]$, and the following lateral derivatives exist and satisfy

(i) $\varphi'_{\pm(x,y)}(s) = \nabla_{\pm} f_{(1-s)x+sy}(y-x)$, $s \in [0,1)$,
(ii) $\varphi'_{+(x,y)}(0) = \nabla_+ f_x(y-x)$,
(iii) $\varphi'_{-(x,y)}(1) = \nabla_- f_y(y-x)$,

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where $\nabla_{\pm}f_x(y)$ are the *Gâteaux lateral derivatives,* we recall that

$$\begin{aligned}\nabla_{+}f_x(y) &:= \lim_{h\to 0+}\frac{f(x+hy)-f(x)}{h},\\ \nabla_{-}f_x(y) &:= \lim_{k\to 0-}\frac{f(x+ky)-f(x)}{k},\quad x,\ y\in X.\end{aligned}$$

The following inequality is the well-known Hermite-Hadamard integral inequality for convex functions defined on a segment $[x,y]\subset X$:

$$f\left(\frac{x+y}{2}\right)\le\int_0^1 f[(1-t)x+ty]\,dt\le\frac{f(x)+f(y)}{2},\tag{HH}$$

which easily follows by the classical Hermite-Hadamard inequality for the convex function $\varphi(x,y):[0,1]\to\mathbb{R}$

$$\varphi_{(x,y)}\left(\frac{1}{2}\right)\le\int_0^1\varphi_{(x,y)}(t)\,dt\le\frac{\varphi_{(x,y)}(0)+\varphi_{(x,y)}(1)}{2}.$$

For other related results see the monograph on line [8]. For some recent results in linear spaces see [1], [2] and [9]-[12].

In the recent paper [7] we established the following refinements and reverses of Féjer's inequality for functions defined on linear spaces:

Theorem 1.1. *Let f be an convex function on C and $x,\ y\in C$ with $x\neq y$. If $p:[0,1]\to[0,\infty)$ is Lebesgue integrable and symmetric, namely $p(1-t)=p(t)$ for all $t\in[0,1]$, then*

$$\begin{aligned}0&\le\frac{1}{2}\left[\nabla_{+}f_{\frac{x+y}{2}}(y-x)-\nabla_{-}f_{\frac{x+y}{2}}(y-x)\right]\int_0^1\left|t-\frac{1}{2}\right|p(t)\,dt\\&\le\int_0^1 f((1-t)x+ty)\,p(t)\,dt-f\left(\frac{x+y}{2}\right)\int_0^1 p(t)\,dt\\&\le\frac{1}{2}\left[\nabla_{-}f_y(y-x)-\nabla_{+}f_x(y-x)\right]\left(\int_0^1\left|t-\frac{1}{2}\right|p(t)\,dt\right)\end{aligned}\tag{1.1}$$

and

$$\begin{aligned}0&\le\frac{1}{2}\left[\nabla_{+}f_{\frac{x+y}{2}}(y-x)-\nabla_{-}f_{\frac{x+y}{2}}(y-x)\right]\int_0^1\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)p(t)\,dt\\&\le\frac{f(x)+f(y)}{2}\int_0^1 p(t)\,dt-\int_0^1 f((1-t)x+ty)\,p(t)\,dt\\&\le\frac{1}{2}\left[\nabla_{-}f_y(y-x)-\nabla_{+}f_x(y-x)\right]\int_0^1\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)p(t)\,dt.\end{aligned}\tag{1.2}$$

If we take $p\equiv 1$ in (1.1), then we get

$$\begin{aligned}0&\le\frac{1}{8}\left[\nabla_{+}f_{\frac{x+y}{2}}(y-x)-\nabla_{-}f_{\frac{x+y}{2}}(y-x)\right]\\&\le\int_0^1 f[(1-t)x+ty]\,dt-f\left(\frac{x+y}{2}\right)\\&\le\frac{1}{8}\left[\nabla_{-}f_y(y-x)-\nabla_{+}f_x(y-x)\right]\end{aligned}\tag{1.3}$$

that was firstly obtained in [4], while from (1.2) we recapture the result obtained in [5]

$$0 \le \frac{1}{8}\left[\nabla_+ f_{\frac{x+y}{2}}(y-x) - \nabla_- f_{\frac{x+y}{2}}(y-x)\right] \tag{1.4}$$
$$\le \frac{f(x)+f(y)}{2} - \int_0^1 f[(1-t)x+ty]\,dt$$
$$\le \frac{1}{8}\left[\nabla_- f_y(y-x) - \nabla_+ f_x(y-x)\right].$$

Motivated by the above results, we establish in this paper some upper and lower bounds for the difference

$$\int_0^1 p(\tau) f((1-\tau)x+\tau y)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)x+\tau y)\,d\tau$$

where f is a convex function on C and $x,\ y \in C$, with $x \neq y$ while $p:[0,1]\to\mathbb{R}$ is a Lebesgue integrable function such that

$$0 \le \int_0^\tau p(s)\,ds \le \int_0^1 p(s)\,ds \text{ for all } \tau\in[0,1].$$

Some applications for norms and semi-inner products are also provided.

2. Main Results

We start to the following identity that is of interest in itself as well:

Lemma 2.1. *Let f be a convex function on C and $x,\ y\in C$, with $x\neq y$. If $g:[0,1]\to\mathbb{C}$ is a Lebesgue integrable function, then we have the equality*

$$\int_0^1 g(\tau)\varphi_{(x,y)}(\tau)\,d\tau - \int_0^1 g(\tau)\,d\tau\int_0^1 \varphi_{(x,y)}(\tau)\,d\tau \tag{2.1}$$
$$= \int_0^1\left(\int_\tau^1 g(s)\,ds\right)\tau\varphi'_{(x,y)}(\tau)\,d\tau$$
$$+ \int_0^1\left(\int_0^\tau g(s)\,ds\right)(\tau-1)\,\varphi'_{(x,y)}(\tau)\,d\tau.$$

Proof. Integrating by parts in the Lebesgue integral, we have

$$\int_0^\tau t\varphi'_{(x,y)}(t)\,dt + \int_\tau^1 (t-1)\,\varphi'_{(x,y)}(t)\,dt$$
$$= \tau\varphi_{(x,y)}(\tau) - \int_0^\tau \varphi_{(x,y)}(t)\,dt - (\tau-1)\,\varphi_{(x,y)}(\tau) - \int_\tau^1 \varphi_{(x,y)}(t)\,dt$$
$$= \varphi_{(x,y)}(\tau) - \int_0^1 \varphi_{(x,y)}(t)\,dt$$

that holds for all $\tau\in[0,1]$.

If we multiply this identity by $g(\tau)$ and integrate over τ in $[0,1]$, then we get

$$\begin{aligned}&\int_0^1 g(\tau)\varphi_{(x,y)}(\tau)\,d\tau-\int_0^1 g(\tau)\,d\tau\int_0^1\varphi_{(x,y)}(t)\,dt \qquad (2.2)\\ &=\int_0^1 g(\tau)\left(\int_0^\tau t\varphi'_{(x,y)}(t)\,dt\right)d\tau+\int_0^1 g(\tau)\left(\int_\tau^1(t-1)\varphi'_{(x,y)}(t)\,dt\right)d\tau.\end{aligned}$$

Using integration by parts, we get

$$\begin{aligned}&\int_0^1 g(\tau)\left(\int_0^\tau t\varphi'_{(x,y)}(t)\,dt\right)d\tau \qquad (2.3)\\ &=\int_0^1\left(\int_0^\tau t\varphi'_{(x,y)}(t)\,dt\right)d\left(\int_0^\tau g(s)\,ds\right)\\ &=\left(\int_0^\tau g(s)\,ds\right)\left(\int_0^\tau t\varphi'_{(x,y)}(t)\,dt\right)\Big|_0^1\\ &-\int_0^1\left(\int_0^\tau g(s)\,ds\right)\tau\varphi'_{(x,y)}(\tau)\,d\tau\\ &=\left(\int_0^1 g(s)\,ds\right)\left(\int_0^1 t\varphi'_{(x,y)}(t)\,dt\right)\\ &-\int_0^1\left(\int_0^\tau g(s)\,ds\right)\tau\varphi'_{(x,y)}(\tau)\,d\tau\\ &=\int_0^1\left(\int_0^1 g(s)\,ds-\int_0^\tau g(s)\,ds\right)\tau\varphi'_{(x,y)}(\tau)\,d\tau\\ &=\int_0^1\left(\int_\tau^1 g(s)\,ds\right)\tau\varphi'_{(x,y)}(\tau)\,d\tau\end{aligned}$$

and

$$\begin{aligned}&\int_0^1 g(\tau)\left(\int_\tau^1(t-1)\varphi'_{(x,y)}(t)\,dt\right)d\tau \qquad (2.4)\\ &=\int_0^1\left(\int_\tau^1(t-1)\varphi'_{(x,y)}(t)\,dt\right)d\left(\int_0^\tau g(s)\,ds\right)\\ &=\left(\int_\tau^1(t-1)\varphi'_{(x,y)}(t)\,dt\right)\left(\int_0^\tau g(s)\,ds\right)\Big|_0^1\\ &+\int_0^1\left(\int_0^\tau g(s)\,ds\right)(\tau-1)\varphi'_{(x,y)}(\tau)\,d\tau\\ &=\int_0^1\left(\int_0^\tau g(s)\,ds\right)(\tau-1)\varphi'_{(x,y)}(\tau)\,d\tau,\end{aligned}$$

which proves the identity in (2.1). □

Theorem 2.1. *Let f be an operator convex function on C and $x,\ y\in C$, with $x\neq y$. If $p:[0,1]\to\mathbb{R}$ is a Lebesgue integrable function such that*

$$0\le\int_0^\tau p(s)\,ds\le\int_0^1 p(s)\,ds \text{ for all } \tau\in[0,1], \qquad (2.5)$$

then we have the inequalities

$$\begin{aligned}
&\int_0^1\left(\int_\tau^1 p(s)\,ds\right)\tau d\tau\nabla_+f_x(y-x) \\
&-\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau\nabla_-f_y(y-x) \\
&\le\int_0^1 p(\tau)\,f((1-\tau)x+\tau y)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1 f((1-\tau)x+\tau y)\,d\tau \\
&\le\int_0^1\left(\int_\tau^1 p(s)\,ds\right)\tau d\tau\nabla_-f_y(y-x) \\
&-\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau\nabla_+f_x(y-x)
\end{aligned}\tag{2.6}$$

or, equivalently,

$$\begin{aligned}
&\int_0^1(1-\tau)\left(\int_0^\tau[p(1-s)\nabla_-f_y(y-x)-p(s)\nabla_+f_x(y-x)]\,ds\right)d\tau \\
&\le\int_0^1 p(\tau)\,f((1-\tau)x+\tau y)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1 f((1-\tau)x+\tau y)\,d\tau \\
&\le\int_0^1(1-\tau)\left(\int_0^\tau[p(1-s)\nabla_+f_x(y-x)-p(s)\nabla_-f_y(y-x)]\,ds\right)d\tau.
\end{aligned}\tag{2.7}$$

Proof. We have for $\varphi_{(x,y)}$ and $p:[0,1]\to\mathbb{R}$ a Lebesgue integrable function that

$$\begin{aligned}
&\int_0^1 p(\tau)\,\varphi_{(x,y)}(\tau)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1\varphi_{(x,y)}(\tau)\,d\tau \\
&=\int_0^1\left(\int_\tau^1 p(s)\,ds\right)(\tau)\,\varphi'_{(x,y)}(\tau)\,d\tau \\
&-\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,\varphi'_{(x,y)}(\tau)\,d\tau.
\end{aligned}\tag{2.8}$$

By the gradient inequalities for $\varphi_{(x,y)}$ we have

$$\tau\nabla_-f_y(y-x)\ge\tau\varphi'_{(x,y)}(\tau)\ge\tau\nabla_+f_x(y-x)\tag{2.9}$$

and

$$(1-\tau)\nabla_-f_y(y-x)\ge(1-\tau)\,\varphi'_{(x,y)}(\tau)\ge(1-\tau)\nabla_+f_x(y-x)\tag{2.10}$$

for all $\tau\in(0,1)$.

From

$$\int_0^\tau p(s)\,ds\le\int_0^1 p(s)\,ds=\int_0^\tau p(s)\,ds+\int_\tau^1 p(s)\,ds,$$

we get that $\int_\tau^1 p(s)\,ds\ge 0$ for all $\tau\in(0,1)$.

From (2.9) we derive that

$$\begin{aligned}
\left(\int_\tau^1 p(s)\,ds\right)\tau\nabla_-f_y(y-x)&\ge\left(\int_\tau^1 p(s)\,ds\right)\tau\varphi'_{(x,y)}(\tau) \\
&\ge\left(\int_\tau^1 p(s)\,ds\right)\tau\nabla_+f_x(y-x)
\end{aligned}$$

and from (2.10) that

$$
\begin{aligned}
-\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\nabla_+ f_x(y-x) &\le -\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\varphi'_{(x,y)}(\tau) \\
&\le -\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\nabla_- f_y(y-x)
\end{aligned}
$$

all $\tau \in (0,1)$.

If we integrate these inequalities over $\tau \in [0,1]$ and add the obtained results, then we get

$$
\begin{aligned}
&\int_0^1 \left(\int_\tau^1 p(s)\,ds\right)\tau d\tau \nabla_- f_y(y-x) - \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \nabla_+ f_x(y-x) \\
&\ge \int_0^1 \left(\int_\tau^1 p(s)\,ds\right)\tau \varphi'_{(x,y)}(\tau)\,d\tau - \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\varphi'_{(x,y)}(\tau)\,d\tau \\
&\ge \int_0^1 \left(\int_\tau^1 p(s)\,ds\right)\tau d\tau \nabla_+ f_x(y-x) - \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \nabla_- f_y(y-x).
\end{aligned}
$$

By using the equality (2.1) we obtain

$$
\begin{aligned}
&\int_0^1 \left(\int_\tau^1 p(s)\,ds\right)\tau d\tau \nabla_+ f_x(y-x) \\
&- \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \nabla_- f_y(y-x) \\
&\le \int_0^1 p(\tau)\varphi_{(x,y)}(\tau)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 \varphi_{(x,y)}(\tau)\,d\tau \\
&\le \int_0^1 \left(\int_\tau^1 p(s)\,ds\right)\tau d\tau \nabla_- f_y(y-x) \\
&- \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \nabla_+ f_x(y-x),
\end{aligned}
\tag{2.11}
$$

namely (2.6).

If we change the variable $y = 1-\tau$, then we have

$$
\int_0^1 \left(\int_\tau^1 p(s)\,ds\right)\tau d\tau = \int_0^1 \left(\int_{1-y}^1 p(s)\,ds\right)(1-y)\,dy.
$$

Also by the change of variable $u = 1-s$, we get

$$
\int_{1-y}^1 p(s)\,ds = \int_0^y p(1-u)\,du,
$$

which implies that

$$
\int_0^1 \left(\int_\tau^1 p(s)\,ds\right)\tau d\tau = \int_0^1 \left(\int_0^\tau p(1-s)\,ds\right)(1-\tau)\,d\tau.
$$

Therefore

$$\int_0^1\left(\int_\tau^1 p(s)\,ds\right)\tau d\tau\nabla_- f_y(y-x)-\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau\nabla_+ f_x(y-x)$$
$$=\int_0^1\left(\int_0^\tau p(1-s)\,ds\right)(1-\tau)\,d\tau\nabla_- f_y(y-x)$$
$$-\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau\nabla_+ f_x(y-x)$$
$$=\int_0^1(1-\tau)\left(\int_0^\tau\left[p(1-s)\nabla_- f_y(y-x)-p(s)\nabla_+ f_x(y-x)\right]ds\right)d\tau$$

and

$$\int_0^1\left(\int_\tau^1 p(s)\,ds\right)\tau d\tau\nabla_+ f_x(y-x)-\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau\nabla_- f_y(y-x)$$
$$=\int_0^1\left(\int_0^\tau p(1-s)\,ds\right)(1-\tau)\,d\tau\nabla_+ f_x(y-x)$$
$$-\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau\nabla_- f_y(y-x)$$
$$=\int_0^1(1-\tau)\left(\int_0^\tau\left[p(1-s)\nabla_+ f_x(y-x)-p(s)\nabla_- f_y(y-x)\right]ds\right)d\tau,$$

and by (2.11) we get (2.7). □

We say that the function $p:[0,1]\to\mathbb{R}$ is symmetric on $[0,1]$ if

$$p(1-t)=p(t)\ \text{ for all } t\in[0,1].$$

Corollary 2.1. *Let f be a convex function on C and $x,\ y\in C$, with $x\neq y$. If $p:[0,1]\to\mathbb{R}$ is a Lebesgue integrable and symmetric function such that the condition (2.5) holds, then we have*

$$\left|\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1 p(\tau)\,f((1-\tau)x+\tau y)\,d\tau-\int_0^1 f((1-\tau)x+\tau y)\,d\tau\right| \tag{2.12}$$
$$\leq\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau\left[\nabla_- f_y(y-x)-\nabla_+ f_x(y-x)\right]$$
$$\leq\frac{1}{2}\left[\nabla_- f_y(y-x)-\nabla_+ f_x(y-x)\right].$$

Proof. Since p is symmetric, then $p(1-s)=p(s)$ for all $s\in[0,1]$ and by (2.7) we get

$$\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau\left[\nabla_+ f_x(y-x)-\nabla_- f_y(y-x)\right]$$
$$\leq\int_0^1 p(\tau)\,\varphi_{(x,y)}(\tau)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1\varphi_{(x,y)}(\tau)\,d\tau$$
$$\leq\left[\nabla_- f_y(y-x)-\nabla_+ f_x(y-x)\right]\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau,$$

which is equivalent to the first inequality in (2.12).

Since $0 \leq \int_0^\tau p(s)\,ds \leq \int_0^1 p(\tau)\,d\tau$, hence

$$\int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \leq \int_0^1 p(\tau)\,d\tau \int_0^1 (1-\tau)\,d\tau = \frac{1}{2}\int_0^1 p(\tau)\,d\tau$$

and the last part of (2.12) is proved. □

Remark 2.1. If the function p is nonnegative and symmetric then the inequality (2.12) holds true.

If we consider the weight $p:[0,1]\to[0,\infty)$, $p(s)=\left|s-\frac{1}{2}\right|$, then

$$\begin{aligned}
&\int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \\
&= \int_0^1 \left(\int_0^\tau \left|s-\frac{1}{2}\right| ds\right)(1-\tau)\,d\tau \\
&= \int_0^{\frac{1}{2}} \left(\int_0^\tau \left|s-\frac{1}{2}\right| ds\right)(1-\tau)\,d\tau \\
&+ \int_{\frac{1}{2}}^1 \left(\int_0^\tau \left|s-\frac{1}{2}\right| ds\right)(1-\tau)\,d\tau \\
&= \int_0^{\frac{1}{2}} \left(\int_0^\tau \left(\frac{1}{2}-s\right) ds\right)(1-\tau)\,d\tau \\
&+ \int_{\frac{1}{2}}^1 \left(\int_0^{\frac{1}{2}} \left(\frac{1}{2}-s\right) ds + \int_{\frac{1}{2}}^\tau \left(s-\frac{1}{2}\right)\right)(1-\tau)\,d\tau \\
&= \int_0^{\frac{1}{2}} \left(\frac{1}{2}\tau-\frac{\tau^2}{2}\right)(1-\tau)\,d\tau \\
&+ \int_{\frac{1}{2}}^1 \left(\int_0^{\frac{1}{2}} \left(\frac{1}{2}-s\right) ds + \int_{\frac{1}{2}}^\tau \left(s-\frac{1}{2}\right) ds\right)(1-\tau)\,d\tau.
\end{aligned}$$

We have

$$\begin{aligned}
&\int_0^{\frac{1}{2}} \left(\frac{1}{2}\tau-\frac{\tau^2}{2}\right)(1-\tau)\,d\tau = \frac{1}{2}\int_0^{\frac{1}{2}} (1-\tau)\,\tau\,(1-\tau)\,d\tau \\
&= \frac{1}{2}\int_0^{\frac{1}{2}} (1-\tau)^2\,\tau d\tau = \frac{11}{384}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\frac{1}{2}}^1 \left(\int_0^{\frac{1}{2}} \left(\frac{1}{2}-s\right) ds + \int_{\frac{1}{2}}^\tau \left(s-\frac{1}{2}\right) ds\right)(1-\tau)\,d\tau \\
&= \int_{\frac{1}{2}}^1 \left(\frac{1}{8}+\frac{1}{2}\left(\tau-\frac{1}{2}\right)^2\right)(1-\tau)\,d\tau \\
&= \frac{1}{8}\int_{\frac{1}{2}}^1 (1-\tau)\,d\tau + \frac{1}{2}\int_{\frac{1}{2}}^1 \left(\tau-\frac{1}{2}\right)^2 (1-\tau)\,d\tau = \frac{7}{384}.
\end{aligned}$$

Therefore

$$\int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau = \frac{3}{64}.$$

Since $\int_0^1 \left|\tau - \frac{1}{2}\right| d\tau = \frac{1}{4}$, hence

$$\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau = \frac{3}{16}.$$

Utilising (2.12) for symmetric weight $p : [0,1] \to [0,\infty)$, $p(s) = \left|s - \frac{1}{2}\right|$, we get

$$\left|4\int_0^1 \left|\tau - \frac{1}{2}\right| f((1-\tau)x + \tau y)\,d\tau - \int_0^1 f((1-\tau)x+\tau y)\,d\tau\right| \tag{2.13}$$
$$\leq \frac{3}{16}\left[\nabla_- f_y(y-x) - \nabla_+ f_x(y-x)\right]$$

where f is a convex function on C and x, $y \in C$, with $x \neq y$.

Consider now the symmetric function $p(s) = (1-s)s$, $x \in [0,1]$. Then

$$\int_0^\tau p(s)\,ds = \int_a^\tau (1-s)\,sds = -\frac{1}{6}\tau^2(2\tau - 3),\ \ \tau \in [0,1]$$

and

$$\int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau = -\frac{1}{6}\int_0^1 \tau^2(2\tau-3)(1-\tau)\,d\tau = \frac{1}{40}.$$

Also

$$\int_0^1 p(\tau)\,d\tau = \int_0^1 (1-\tau)\,\tau d\tau = \frac{1}{6}$$

and

$$\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau = \frac{3}{20}$$

and by (2.12) we obtain

$$\left|6\int_0^1 (1-\tau)\,\tau f((1-\tau)x+\tau y)\,d\tau - \int_0^1 f((1-\tau)x+\tau y)\,d\tau\right| \tag{2.14}$$
$$\leq \frac{3}{20}\left[\nabla_- f_y(y-x) - \nabla_+ f_x(y-x)\right],$$

where f is a convex function on C and x, $y \in C$, with $x \neq y$.

3. Examples for Norms

Now, assume that $(X, \|\cdot\|)$ is a normed linear space. The function $f_0(s) = \frac{1}{2}\|x\|^2$, $x \in X$ is convex and thus the following limits exist

(iv) $\langle x, y\rangle_s := \nabla_+ f_{0,y}(x) = \lim\limits_{t\to 0+} \frac{\|y+tx\|^2 - \|y\|^2}{2t}$;

(v) $\langle x, y\rangle_i := \nabla_- f_{0,y}(x) = \lim\limits_{s\to 0-} \frac{\|y+sx\|^2 - \|y\|^2}{2s}$;

for any $x,\ y\in X$. They are called the *lower* and *upper semi-inner* products associated to the norm $\|\cdot\|$.

For the sake of completeness we list here some of the main properties of these mappings that will be used in the sequel (see for example [2] or [6]), assuming that $p,\ q\in\{s,i\}$ and $p\neq q$:

(a) $\langle x,x\rangle_p=\|x\|^2$ for all $x\in X$;
(aa) $\langle\alpha x,\beta y\rangle_p=\alpha\beta\langle x,y\rangle_p$ if $\alpha,\ \beta\geq 0$ and $x,\ y\in X$;
(aaa) $\left|\langle x,y\rangle_p\right|\leq\|x\|\,\|y\|$ for all $x,\ y\in X$;
(av) $\langle\alpha x+y,x\rangle_p=\alpha\langle x,x\rangle_p+\langle y,x\rangle_p$ if $x,\ y\in X$ and $\alpha\in\mathbb{R}$;
(v) $\langle -x,y\rangle_p=-\langle x,y\rangle_q$ for all $x,\ y\in X$;
(va) $\langle x+y,z\rangle_p\leq\|x\|\,\|z\|+\langle y,z\rangle_p$ for all $x,\ y,\ z\in X$;
(vaa) The mapping $\langle\cdot,\cdot\rangle_p$ is continuous and subadditive (superadditive) in the first variable for $p=s$ (or $p=i$);
(vaaa) The normed linear space $(X,\|\cdot\|)$ is smooth at the point $x_0\in X\setminus\{0\}$ if and only if $\langle y,x_0\rangle_s=\langle y,x_0\rangle_i$ for all $y\in X$; in general $\langle y,x\rangle_i\leq\langle y,x\rangle_s$ for all $x,\ y\in X$;
(ax) If the norm $\|\cdot\|$ is induced by an inner product $\langle\cdot,\cdot\rangle$, then $\langle y,x\rangle_i=\langle y,x\rangle=\langle y,x\rangle_s$ for all $x,\ y\in X$.

The function $f_r(x)=\|x\|^r$ ($x\in X$ and $1\leq r<\infty$) is also convex. Therefore, the following limits, which are related to the superior (inferior) semi-inner products,

$$\begin{aligned}\nabla_{\pm}f_{r,y}(x)&:=\lim_{t\to0\pm}\frac{\|y+tx\|^r-\|y\|^r}{t}\\&=r\|y\|^{r-1}\lim_{t\to0\pm}\frac{\|y+tx\|-\|y\|}{t}=r\|y\|^{r-2}\langle x,y\rangle_{s(i)}\end{aligned}$$

exist for all $x,\ y\in X$ whenever $r\geq 2$; otherwise, they exist for any $x\in X$ and nonzero $y\in X$. In particular, if $r=1$, then the following limits

$$\nabla_{\pm}f_{1,y}(x):=\lim_{t\to0\pm}\frac{\|y+tx\|-\|y\|}{t}=\frac{\langle x,y\rangle_{s(i)}}{\|y\|}$$

exist for $x,\ y\in X$ and $y\neq 0$.

If $p:[0,1]\to\mathbb{R}$ is a Lebesgue integrable and symmetric function such that the condition

$$0\leq\int_0^\tau p(s)\,ds\leq\int_0^1 p(s)\,ds\text{ for all }\tau\in[0,1],$$

is valid, then by (2.12) we get

$$\left|\frac{1}{\int_0^1p(\tau)\,d\tau}\int_0^1p(\tau)\|(1-\tau)x+\tau y\|^r\,d\tau-\int_0^1\|(1-\tau)x+\tau y\|^r\,d\tau\right|\tag{3.1}$$
$$\leq\frac{r}{\int_0^1p(\tau)\,d\tau}\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau$$
$$\times\left[\|y\|^{r-2}\langle y-x,y\rangle_i-\|x\|^{r-2}\langle y-x,x\rangle_s\right].$$

If $r\geq 2$, then the inequality (3.1) holds for all $x,\ y\in X$. If $r\in[1,2)$, then the inequality (3.1) holds for all $x,\ y\in X$ with $x,\ y\neq 0$.

For $r=2$ we get

$$\left|\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1 p(\tau)\left\|(1-\tau)x+\tau y\right\|^2 d\tau-\int_0^1\left\|(1-\tau)x+\tau y\right\|^2 d\tau\right| \tag{3.2}$$
$$\leq \frac{2}{\int_0^1 p(\tau)\,d\tau}\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau\left[\langle y-x,y\rangle_i-\langle y-x,x\rangle_s\right]$$

for all $x,\ y\in X$.

If we take $p(\tau)=\left|\tau-\frac{1}{2}\right|,\ \tau\in[0,1]$ in (3.1), then we obtain

$$\left|4\int_0^1\left|\tau-\frac{1}{2}\right|\left\|(1-\tau)x+\tau y\right\|^r d\tau-\int_0^1\left\|(1-\tau)x+\tau y\right\|^r d\tau\right| \tag{3.3}$$
$$\leq \frac{3r}{16}\left[\|y\|^{r-2}\langle y-x,y\rangle_i-\|x\|^{r-2}\langle y-x,x\rangle_s\right].$$

If $X=H$ a real inner product space, then from (3.2) we get

$$\left|\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1 p(\tau)\left\|(1-\tau)x+\tau y\right\|^2 d\tau-\int_0^1\left\|(1-\tau)x+\tau y\right\|^2 d\tau\right| \tag{3.4}$$
$$\leq \frac{2}{\int_0^1 p(\tau)\,d\tau}\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau\,\|y-x\|^2$$

for all $x,\ y\in H$.

4. Examples for Functions of Several Variables

Now, let $\Omega\subset\mathbb{R}^n$ be an open convex set in $\mathbb{R}^n$. If $F:\Omega\to\mathbb{R}$ is a differentiable convex function on Ω, then, obviously, for any $\bar{c}\in\Omega$ we have

$$\nabla F_{\bar{c}}(\bar{y})=\sum_{i=1}^n\frac{\partial F(\bar{c})}{\partial x_i}\cdot y_i,\ \ \bar{y}=(y_1,...,y_n)\in\mathbb{R}^n,$$

where $\frac{\partial F}{\partial x_i}$ are the partial derivatives of F with respect to the variable x_i $(i=1,...,n)$.

If $p:[0,1]\to\mathbb{R}$ is a Lebesgue integrable and symmetric function such that the condition (2.5) holds, then we have for all $\bar{a},\ \bar{b}\in\Omega$ that

$$\left|\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1 p(\tau)\,F\left((1-\tau)\,\bar{a}+\tau\bar{b}\right)d\tau-\int_0^1 f\left((1-\tau)\,\bar{a}+\tau\bar{b}\right)d\tau\right| \tag{4.1}$$
$$\leq \frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau$$
$$\times\sum_{i=1}^n\left(\frac{\partial F(\bar{b})}{\partial x_i}-\frac{\partial F(\bar{a})}{\partial x_i}\right)(b_i-a_i)$$
$$\leq\frac{1}{2}\sum_{i=1}^n\left(\frac{\partial F(\bar{b})}{\partial x_i}-\frac{\partial F(\bar{a})}{\partial x_i}\right)(b_i-a_i).$$

If we take $p(\tau) = \left|\tau - \frac{1}{2}\right|$, $\tau \in [0,1]$ in (4.1), then we get

$$\left|4\int_0^1 \left|\tau - \frac{1}{2}\right| F\left((1-\tau)\bar{a} + \tau\bar{b}\right) d\tau - \int_0^1 f\left((1-\tau)\bar{a} + \tau\bar{b}\right) d\tau\right| \le \frac{3}{16}\sum_{i=1}^{n}\left(\frac{\partial F(\bar{b})}{\partial x_i} - \frac{\partial F(\bar{a})}{\partial x_i}\right)(b_i - a_i) \tag{4.2}$$

for all $\bar{a}$, $\bar{b} \in \Omega$.

References

[1] M. Alomari and M. Darus, Refinements of s-Orlicz convex functions in normed linear spaces. *Int. J. Contemp. Math. Sci.* **3** (2008), no. 29-32, 1569–1594.

[2] I. Ciorănescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Kluwer Academic Publishers, Dordrecht, 1990.

[3] X. Chen, New convex functions in linear spaces and Jensen's discrete inequality. *J. Inequal. Appl.* **2013**, 2013:472, 8 pp.

[4] S. S. Dragomir, An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products, *J. Inequal. Pure & Appl. Math.* **3** (2002), no. 2, Article 31, 8 pp.

[5] S. S. Dragomir, An inequality improving the second Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products, *J. Inequal. Pure & Appl. Math.* **3** (2002), no. 3, Article 35, 8 pp.

[6] S. S. Dragomir, *Semi-inner Products and Applications.* Nova Science Publishers, Inc., Hauppauge, NY, 2004. x+222 pp. ISBN: 1-59033-947-9

[7] S. S. Dragomir, Refinements and reverses of Féjer's inequalities for convex functions on linear spaces, Preprint *RGMIA Res. Rep. Coll.* **22** (2019), Art. 92, 14 pp. [Online http://rgmia.org/papers/v22/v22a92.pdf].

[8] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000. [Online https://rgmia.org/monographs/hermite_hadamard.html].

[9] M. Adil Khan, S. Khalid and J. Pečarić, Improvement of Jensen's inequality in terms of Gâteaux derivatives for convex functions in linear spaces with applications. *Kyungpook Math. J.* **52** (2012), no. 4, 495–511.

[10] E. Kikianty and S. S. Dragomir, Hermite-Hadamard's inequality and the *p-HH*-norm on the Cartesian product of two copies of a normed space. *Math. Inequal. Appl.* **13** (2010), no. 1, 1–32.

[11] E. Kikianty, S. S. Dragomir and P. Cerone, Ostrowski type inequality for absolutely continuous functions on segments in linear spaces. *Bull. Korean Math. Soc.* **45** (2008), no. 4, 763–780.

[12] E. Kikianty, S. S. Dragomir and P. Cerone, Sharp inequalities of Ostrowski type for convex functions defined on linear spaces and application. *Comput. Math. Appl.* **56** (2008), no. 9, 2235–2246.

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