Multi-Objective Robust H-infinity Control of Spacecraft Rendezvous

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Huijun Gao*, **Xuebo Yang**†, **Peng Shi**‡

Abstract

Based on the relative motion dynamic model illustrated by C-W equations, the problem of robust H_∞ control for a class of spacecraft rendezvous systems is investigated, which contain parametric uncertainties, external disturbances and input constraints. An H_∞ state-feedback controller is designed via a Lyapunov approach, which guarantees the closed-loop system to meet the multi-objective design requirements. The existence conditions for admissible controllers are formulated in the form of liner matrix inequalities (LMIs), and the controller design is cast into a convex optimization problem subject to LMI constraints. An illustrative example is provided to show the effectiveness of the proposed control design method.

Keywords: C-W equations; rendezvous; H_∞ control; linear matrix inequalities (LMIs); uncertainty; input constraint

1 Introduction

Spacecraft rendezvous has been well recognized as an important issue in aerospace engineering. Successful rendezvous is the precondition of many astronautic missions such as intercepting, repairing, saving, docking, large-scale structure assembling and satellite networking. During the last few decades, the spacecraft rendezvous control problem has attracted considerable attention and many design methods have been developed. C-W equations, derived by Clohessey and Wiltshire in 1960 [5], have been widely used to describe the linear relative motion between two neighboring spacecraft if the target orbit is approximately circular and the distance between them is much smaller than the orbit radius. For example, [13], [18] and [25] studied the optimal impulsive rendezvous problem based on C-W equations, and [17] utilized annealing algorithm to design orbital controllers for the rendezvous model depicted by C-W equations. In recent years, with the development of control theory, many advanced methods have been utilized to solve the rendezvous control problem. For example, adaptive control theory was applied to solve the rendezvous and docking problem in [24]; in [7], a new rendezvous guidance method was proposed based on sliding-mode control theory; and in [26], the problem of rendezvous was cast as a stabilization problem analyzed by Lyapunov theory. Although

*Department of Electrical and Computer Engineering, University of Alberta, Edmonton, T6G 2V4, Canada. Email: hjgao@hit.edu.cn

*Space Control and Inertial Technology Research Center, Harbin Institute of Technology, Harbin, 150001, China. Email: yangxuebo@gmail.com

†Faculty of Advanced Technology, University of Glamorgan, Pontypridd, CF371DL, United Kingdom. Email: pshi@glam.ac.uk

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there have been some results in this field, the orbital rendezvous problem has not been fully investigated and still remains challenging.

In the dynamic model depicted by C-W equations, the angle velocity of target spacecraft is an important parameter (the proper calculation of control force depends on its accurate real-time value). However, the accuracy of the velocity is affected by many factors, which can be illustrated separately according to different kinds of astronautic missions. Firstly, if the two spacecraft are cooperative during docking, assembling or satellite networking, the angle velocity of target should be a determined value before execution. But the actual velocity varies continuously and it is usually difficult to determine due to the complex perturbations in space. Secondly, the angle velocity of target must be measured real-time during repairing, saving or intercepting. The inevitable error of detection is another source of uncertainty. These uncertainties have much to do with the stability and accuracy of rendezvous. Therefore, in recent years, some results have been reported to deal with the uncertain parameters, see, for instance [10], [11], [22], [23] and [27]. Nevertheless, for the spacecraft rendezvous problem, the parameter uncertainty and other design requirements are, somehow surprisingly, studied separately, and it is practically necessary to take them into consideration simultaneously.

For astronautic missions, control input constraint is another important issue we must pay attention to. In practical engineering, the orbital control input force is limited due to the constraints of the thrust equipment and the limited quantity of fuel. So far, some researchers have studied the optimal- or minimum-fuel problem in spacecraft rendezvous. For instance, the problems of optimal-fuel rendezvous for spacecraft drived by normal power were studied in [15], [20] and [21]; for the spacecraft drived by solar energy, the studies of rendezvous or correlative problems can be found in [12], [16] and [19].

It is obvious that the rendezvous controller design is a multi-objective problem, and we must synthetically consider the requirements of the stability, input constraints and closed-loop poles placement. Meanwhile, the parameter uncertainties and the external perturbations must also be taken into consideration simultaneously. In [6], a novel hybrid optimization method was developed to deal with an optimal multi-objective problem for spacecraft trajectories. It is worth mentioning that, although there have been some results towards solving the rendezvous problem, most of these results just consider one or two aspects of the multi-objective design problem, and few attempts have been made towards solving the multi-objective design problems with the simultaneous consideration of parameter uncertainties and external perturbations.

In this paper, we propose a multi-objective robust H_∞ controller design method for the rendezvous problem of two neighboring spacecraft subject to parameter uncertainties, external perturbation, control input constraints and poles constraint. We first formulate the parameter uncertainty by norm-bounded approach, and a new rendezvous model is established. Based on this uncertain dynamic model, the system stability, H_∞ performance, input constraints and poles assignment are synthetically taken into consideration. For this multi-objective problem, a new robust H_∞ state-feedback controller design method is developed by a Lyapunov approach. The existence conditions for admissible controllers are formulated in the form of liner matrix inequalities (LMIs), and the controller design problem is cast into a convex optimization problem subject to LMI constraints. If the optimization problem is solvable, a desired controller can be readily constructed. An illustrative example is provided to show the effectiveness and advantage of the proposed control design method.

The rest of this paper is organized as follows. Section 2 presents the dynamic model of spacecraft rendezvous, and the multi-objective robust control design problem is formulated. In Section 3, the H_∞ state-feedback controller design method is proposed. Then, an example is given to illustrate the applicability of the presented approach in Section 4. Finally, Section 5 draws the conclusion.

Notations: The notation used throughout the paper is fairly standard. The superscript "T" stands for matrix transposition; $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $\mathbb{R}^{n\times m}$ denotes the set of all $n \times m$ real matrices; $\|\cdot\|$ refers to either the Euclidean vector norm or the induced matrix 2-norm. For a real symmetric matrix W, the notation $W > 0$ ($W < 0$) is used to denote its positive- (negative-) definiteness. diag$\{\ldots\}$ stands for a block-diagonal matrix. In symmetric block matrices or complex matrix expressions, we use an asterisk ($*$) to represent a term that is induced by symmetry. I and 0 denote the identity matrix and zero matrix with compatible dimensions, respectively. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

2 Dynamic Model and Problem Formulation

In this section, we will review the C-W equations and establish the uncertain rendezvous model. By considering the actual requirements, the problem under study in this paper will be formulated based on this model.

The spacecraft rendezvous system is illustrated in Fig. 1. We assume the two spacecraft (Target and Chaser) are adjacent, and the orbital coordinate frame in our study is a right-handed Cartesian coordinate, with origin attached to the target spacecraft center of mass, x-axis along the vector from earth center to the target's center of mass, y-axis along the target orbit circumference, and z-axis completing the right-handed frame.

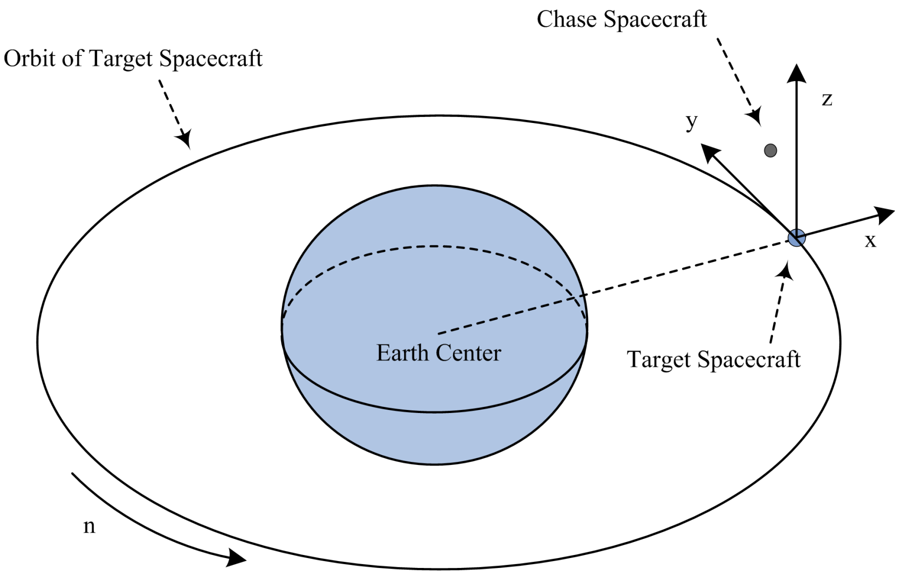

Fig. 1. Spacecraft rendezvous system

The relative dynamic model can be described by C-W's equations:

$$\begin{cases} \ddot{x} - 2n\dot{y} - 3n^2 x = \frac{1}{m}(T_x + \omega_x), \\ \ddot{y} + 2n\dot{x} = \frac{1}{m}(T_y + \omega_y), \\ \ddot{z} + n^2 z = \frac{1}{m}(T_z + \omega_z), \end{cases} \tag{1}$$

where x, y and z are the components of the relative position, n is the angle velocity of the target moving around the earth, m is the mass of the chaser, T_i ($i = x, y, z$) is the i^{th} component of the control input force acting on the relative motion dynamics, ω_i ($i = x, y, z$) is the i^{th} component of the external disturbance. By defining the state vector $q(t) = [x, y, z, \dot{x}, \dot{y}, \dot{z}]^T$, control input vector $u(t) = [T_x, T_y, T_z]^T$, external disturbance vector $\omega(t) = [\omega_x, \omega_y, \omega_z]^T$, and output vector $f(t) = [x,\ y,\ z]^T$, we have

$$\begin{cases} \dot{q}(t) = Aq(t) + Bu(t) + B_\omega \omega(t), \\ f(t) = Cq(t), \end{cases} \tag{2}$$

where

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3n^2 & 0 & 0 & 0 & 2n & 0 \\ 0 & 0 & 0 & -2n & 0 & 0 \\ 0 & 0 & -n^2 & 0 & 0 & 0 \end{bmatrix}, \quad B = B_\omega = \frac{1}{m} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^T.$$

The whole rendezvous process can be described by the transformation of state vector $q(t)$ from nonzero initial state $q(0)$ to the terminal state $q(t_m) = 0$, where t_m is the rendezvous time.

In order to meet the requirements of actual conditions, the following important aspects should be taken into consideration simultaneously:

(1) *Parameter uncertainty.* Due to the detection errors or the complex force among the objects in space, the angle velocity of target spacecraft n cannot be determined online accurately. It can be generally characterized as

$$n = n_0(1 + \delta), \tag{3}$$

where n_0 is the theoretical angle velocity, and δ denotes the magnitude of uncertainty.

(2) *Input constraint.* In view of the limited power of actuator, the actual control input force should be confined into a certain range, which means that

$$\|u(t)\|_2 \leq u_{\max}, \tag{4}$$

where $u_{\max}$ denotes the maximum input force.

(3) *Pole assignment.* In order to obtain a desired dynamic performance of the closed-loop system, usually the poles placement needs to be imposed. Here, we consider the disk regional poles constraint, and let $\mho(\eta, r)$ denote the disk region centered in η with radius r in the complex plane (η, $r \in \mathbb{R}$ and $r > 0$).

According to the parameter uncertainty, we have

$$\dot{q}(t) = \tilde{A}q(t) + Bu(t) + B_\omega \omega(t), \tag{5}$$

where

$$\tilde{A} = A_0 + \Delta_A,$$

$$A_0 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3n_0^2 & 0 & 0 & 0 & 2n_0 & 0 \\ 0 & 0 & 0 & -2n_0 & 0 & 0 \\ 0 & 0 & -n_0^2 & 0 & 0 & 0 \end{bmatrix},$$

$$\Delta_A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3n_0^2(2\delta+\delta^2) & 0 & 0 & 0 & 2\delta n_0 & 0 \\ 0 & 0 & 0 & -2\delta n_0 & 0 & 0 \\ 0 & 0 & -n_0^2(2\delta+\delta^2) & 0 & 0 & 0 \end{bmatrix}.$$

For the uncertain matrix Δ_A in (5), we assume the norm-bounded condition

$$\|\Delta_A\| \leq \alpha, \tag{6}$$

where α is positive which can be determined by δ.

Now, we consider the following controller structure:

$$u(t) = Kq(t), \tag{7}$$

where K is a constant feedback control gain to be determined. Then, the resulting closed-loop system (5) with (7) can be written as

$$\begin{cases} \dot{q}(t) = \overline{A}q(t) + B_\omega \omega(t), \\ f(t) = Cq(t), \end{cases} \tag{8}$$

where $\overline{A} = \tilde{A} + BK$. Our objective in this paper is to determine the controller gain K, such that the system in (8) is robustly stable and the performance $\|T_{f\omega}\|_\infty < \gamma$ is guaranteed subject to the parameter uncertainties, external disturbance and input constraints, and the poles of the closed-loop system lie inside the disk region $\mho(\eta, r)$, where $\|T_{f\omega}\|_\infty$ denotes the closed-loop transfer function from $\omega(t)$ to $f(t)$, and $\mho(\eta, r)$ denotes the disk region centered in η with radius r in the complex plane. It can be briefly summarized as the following minimization problem:

$$\min \gamma \quad \text{s.t.} \begin{cases} \text{the closed-loop system is stable and } \|T_{f\omega}\|_\infty < \gamma, \\ \|u(t)\|_2 \leq u_{\max}, \\ OC, \end{cases}$$

where $u_{\max}$ is a given constant, and OC represents other constraints, such as the poles constraint of the closed-loop system.

Remark 1 *In this paper, we only consider the state-feedback control problem, and it is assumed that the real-time state signals can be transmitted accurately. It is worth mentioning that the output-feedback control problem is more important in real application, and the possible data missing phenomena should also be taken into consideration [31]. For spacecraft rendezvous, the output-feedback control problem with possible missing measurements deserves to be further studied in the future work.*

Remark 2 *It is known that the transient response of a linear system is related to the location of its poles. By constraining the poles to lie in a prescribed region, specific bounds can be put on these quantities to ensure a satisfactory transient response. Till now, many different kinds of poles regions have been studied, such as vertical strip, elliptical and hyperbolic regions. The circular region has been proved to be more effective in both theory and practice, see, for instance [29], [30] and [32]. Readers are referred to [3] for more information about how to select the circular region.*

3 Controller Design

In this section, we will investigate the multi-objective robust H_∞ state-feedback controller design problem. The design requirements mentioned above will be analyzed separately, and the obtained results will be utilized for the controller design. First, we recall the following results which will be used in our later development, and their proofs and the applications can be found in [1], [2], [8], [9], [14], and [28].

Lemma 1 *Let L, E and F are real matrices of appropriate dimensions with $\|F\| \leq 1$. Then, for any scalar $\mu > 0$, we have*

$$LFE + E^T F^T L^T \leq \mu^{-1} LL^T + \mu E^T E.$$

Lemma 2 *Let M and N be real matrices of appropriate dimensions, for any scalar $\varepsilon > 0$,*

$$\begin{bmatrix} 0 & NM^T \\ MN^T & 0 \end{bmatrix} \leq \begin{bmatrix} \varepsilon NN^T & 0 \\ 0 & \varepsilon^{-1} MM^T \end{bmatrix}.$$

Lemma 3 *(Projection Lemma): Let Γ, Λ and Θ be given, there exists a matrix F satisfying*

$$\Theta + \Gamma F \Lambda^T + \Lambda F^T \Gamma^T < 0,$$

if and only if

$$\Gamma^\perp \Theta \Gamma^{\perp^T} < 0, \quad \Lambda^\perp \Theta \Lambda^{\perp^T} < 0.$$

In the multi-objective synthesis, in order to cast the controller design into a convex optimization problem, we usually need to set a common Lyapunov matrix for different performance objectives. This method is simple, but is inevitably conservative due to the fixed positive symmetric matrix. In the following, we will present a new approach which is potentially less conservative. Firstly, we will present three propositions, which convert the design requirements (H_∞ performance, poles assignment and input constraints) into LMI conditions respectively.

Proposition 1 *Consider the system in (8) and the state feedback control law in (7). The closed-loop system is stable and $\|T_{f\omega}\|_\infty < \gamma$ if and only if there exist positive symmetric matrix P_1, general matrices F and G satisfying*

$$\begin{bmatrix} \overline{A}^T G + G^T \overline{A} & \Phi & G^T B_\omega & C^T \\ * & -F - F^T & F^T B_\omega & 0 \\ * & * & -\gamma I & 0 \\ * & * & * & -\gamma I \end{bmatrix} < 0, \tag{9}$$

where $\overline{A} = \tilde{A} + BK$, $\Phi = P_1 - G^T + \overline{A}^T F$.

Proof. By defining $\Sigma = [G \;\; F]$, the inequality (9) can be written as

$$\Theta + \Gamma\Sigma\Lambda^T + \Lambda\Sigma^T\Gamma^T < 0, \tag{10}$$

where

$$\Theta = \begin{bmatrix} 0 & P_1 & 0 & C^T \\ * & 0 & 0 & 0 \\ * & * & -\gamma I & 0 \\ * & * & * & -\gamma I \end{bmatrix}, \; \Gamma = \begin{bmatrix} \overline{A}^T \\ -I \\ B_\omega^T \\ 0 \end{bmatrix}, \; \Lambda = \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The orthogonal complements of Γ and Λ^T are

$$\Gamma^\perp = \begin{bmatrix} I & \overline{A}^T & 0 & 0 \\ 0 & B_\omega^T & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \; \Lambda^\perp = \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}.$$

It is obvious that $\Lambda^\perp\Theta\Lambda^{\perp^T} < 0$. Then, by Lemma 3, it can be seen that (10) holds if and only if

$$\Gamma^\perp\Theta\Gamma^{\perp^T} = \begin{bmatrix} \overline{A}^T P_1 + P_1\overline{A} & P_1 B_\omega & C^T \\ * & -\gamma I & 0 \\ * & * & -\gamma I \end{bmatrix} < 0. \tag{11}$$

According to [3], the system in (8) is stable and $\|T_{z\omega}\|_\infty < \gamma$ if and only if (11) holds, which, together with the equivalence between (9) and (11), completes the proof. □

Proposition 2 *Consider the closed-loop system in (8) and the state feedback control law in (7), assuming that the initial state $q(0)$ is known, and given the positive symmetric matrix P_1 introduced in Proposition 1. Then, for all $t \geq 0$, the input constraint $\|u(t)\|_2 \leq u_{\max}$ can be ensured if there exist general matrix V and positive scalar ρ satisfying*

$$\begin{bmatrix} u_{\max}^2 I & K \\ K^T & \frac{1}{\rho}P_1 \end{bmatrix} \;\geq\; 0, \tag{12}$$

$$\begin{bmatrix} \rho I & q^T(0)V \\ V^T q(0) & V^T + V - P_1 \end{bmatrix} \;\geq\; 0. \tag{13}$$

Proof. Define a Lyapunov function $U(q(t)) = q^T(t)P_1q(t)$, which satisfies

$$U(q(t)) \leq \rho,$$

where ρ is a given positive scalar. Denote the ellipsoid $\Omega(P_1, \rho) = \{q(t) \mid q^T(t)P_1q(t) \leq \rho\}$. For $\|u(t)\|_2 \leq u_{\max}$, denote another ellipsoid $\Omega(K) = \{q(t) \mid |q^T(t)K^TKq(t)| \leq u_{\max}^2\}$. Thus, it can be seen that the input constraints can be ensured by

$$\Omega(P_1, \rho) \subset \Omega(K). \tag{14}$$

According to [4], we can see that (14) can be guaranteed if and only if

$$K(\frac{P_1}{\rho})^{-1}K^T \leq u_{\max}^2,$$

which can be readily obtained from (12) by Schur complement. At the same time, for P_1 is introduced by Proposition 1 which guarantees $\dot{U}(q(t)) < 0$, then we have $q(t)^T P_1 q(t) < q(0)^T P_1 q(0)$ for $t > 0$. Thus, the condition $q(t)^T P_1 q(t) \leq \rho$ can be ensured by $q(0)^T P_1 q(0) \leq \rho$, which is equivalent to

$$\begin{bmatrix} \rho I & q^T(0) \\ q(0) & P_1^{-1} \end{bmatrix} \geq 0. \tag{15}$$

Pre- and post-multiplying (15) by $\text{diag}\{I, V^T\}$ and its transpose respectively, we obtain

$$\begin{bmatrix} \rho I & q^T(0) \\ q(0) & V^T P_1^{-1} V \end{bmatrix} \geq 0. \tag{16}$$

For $(P_1 - V)^T P_1^{-1} (P_1 - V) \geq 0$, we have $V^T P_1^{-1} V \geq V^T + V - P_1$. Then, it is obvious that (16) can be ensured by (13). Thus, we can see that the conditions (12) and (13) can ensure $\Omega(P_1, \rho) \subset \Omega(K)$ and $q(t)^T P_1 q(t) \leq \rho$, which renders the input constraints to be respected. The proof is completed. □

Proposition 3 *Consider the system in (8). All the poles of the closed-loop system lie inside the disk region $\mho(\eta, r)$ (centered in η with radius r in the complex plane) if there exist positive symmetric matrix P_2 and general matrix H satisfying*

$$\begin{bmatrix} P_2 - H^T - H & H^T \left(\overline{A} - \eta I\right) \\ * & -r^2 P_2 \end{bmatrix} < 0. \tag{17}$$

Proof. For $(P_2 - H)^T P_2^{-1} (P_2 - H) \geq 0$, we have

$$-H P_2^{-1} H^T \leq P_2 - H^T - H.$$

Then, it can be seen that if (17) holds, then H is invertible and

$$\begin{bmatrix} -H^T P_2^{-1} H & H^T \left(\overline{A} - \eta I\right) \\ * & -r^2 P_2 \end{bmatrix} < 0. \tag{18}$$

Pre- and post-multiplying (18) by $\text{diag}\{P_2 H^{-T}, I\}$ and its transpose respectively, we obtain

$$\begin{bmatrix} -P_2 & P_2 \left(\overline{A} - \eta I\right) \\ * & -r^2 P_2 \end{bmatrix} < 0. \tag{19}$$

According to [3], the poles of the closed-loop system in (8) lie inside the disk region $\mho(\eta, r)$ (centered in η with radius r in the complex plane) if and only if (19) holds, which can be ensured by (17). The proof is completed. □

Propositions 1-3 formulate the conditions under which the closed-loop system meets the multi-objectives. Based on these propositions, the following theorem presents a controller design method via convex optimization.

Theorem 1 *For the uncertain rendezvous system in (8) and a given scalar $\gamma > 0$, under the constraint input in (4), the closed-loop system is robustly asymptotically stable with disturbance attenuation γ and the* *poles lie inside the disk region $\mho(\eta, r)$ (centered in η with radius r in the complex plane), if there exist scalars ϵ_i $(i = 1, 2, 3)$, and $\rho > 0$, $\varepsilon_j > 0$ $(j = 1, 2, 3)$, and matrices S, L, $\tilde{P}_k > 0$ $(k = 1, 2)$ satisfying*

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} & \epsilon_2 B_\omega & S^T C^T & \epsilon_1 S^T & \epsilon_2 S^T \\ * & \Xi_{22} & \epsilon_1 B_\omega & 0 & 0 & 0 \\ * & * & -\gamma I & 0 & 0 & 0 \\ * & * & * & -\gamma I & 0 & 0 \\ * & * & * & * & -\varepsilon_1 I & 0 \\ * & * & * & * & * & -\varepsilon_2 I \end{bmatrix} < 0, \tag{20}$$

$$\begin{bmatrix} u_{\max}^2 I & L \\ * & \frac{1}{\rho}\widetilde{P}_1 \end{bmatrix} \geq 0, \tag{21}$$

$$\begin{bmatrix} \rho & \epsilon_4 q^T(0) \\ * & \epsilon_4 S + \epsilon_4 S^T - \widetilde{P}_1 \end{bmatrix} \geq 0, \tag{22}$$

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} & 0 \\ * & -r^2 \widetilde{P}_2 & S^T \\ * & * & -\varepsilon_3 I \end{bmatrix} < 0, \tag{23}$$

where

$$\begin{aligned} \Xi_{11} &= \epsilon_2 S^T A_0^T + \epsilon_2 A_0 S + \epsilon_2 BL + \epsilon_2 L^T B^T + \varepsilon_2 \alpha^2 I, \\ \Xi_{12} &= \widetilde{P}_1 - \epsilon_2 S + \epsilon_1 S^T A_0^T + \epsilon_1 L^T B^T, \\ \Xi_{22} &= -\epsilon_1 S - \epsilon_1 S^T + \varepsilon_1 \alpha^2 I, \\ \Pi_{11} &= \widetilde{P}_2 - \epsilon_3 S - \epsilon_3 S^T + \varepsilon_3 \epsilon_3^2 \alpha^2 I, \\ \Pi_{12} &= \epsilon_3 A_0 S + \epsilon_3 BL - \epsilon_3 \eta S. \end{aligned}$$

Furthermore, the desired robust H_∞ state feedback control law is given by $u(t) = LS^{-1}q(t)$.

Proof. For the general matrices in Propositions 1–3, we select $F \triangleq \epsilon_1 V$, $G \triangleq \epsilon_2 V^T$, $H \triangleq \epsilon_3 V$ and $S \triangleq V^{-1}$. Then, condition (9) in Proposition 1 can be rewritten as

$$\begin{bmatrix} \epsilon_2 V^T A + \epsilon_2 A^T V & \Phi & \epsilon_2 V^T B_\omega & C^T \\ * & -\epsilon_1 V - \epsilon_1 V^T & \epsilon_1 V^T B_\omega & 0 \\ * & * & -\gamma^2 I & 0 \\ * & * & * & -I \end{bmatrix} < 0, \tag{24}$$

where

$$\Phi = P_1 - \epsilon_2 V^T + \epsilon_1 \overline{A}^T V.$$

Pre- and post-multiplying (24) by diag $\{S^T, S^T, I, I\}$ and its transpose respectively, we obtain

$$\begin{bmatrix} \epsilon_2 S^T \overline{A}^T + \epsilon_2 \overline{A} S & \Phi & \epsilon_2 B_\omega & S^T C^T \\ * & -\epsilon_1 S^T - \epsilon_1 S & \epsilon_1 B_\omega & 0 \\ * & * & -\gamma I & 0 \\ * & * & * & -\gamma I \end{bmatrix} < 0, \tag{25}$$

where

$$\Phi = S^T P_1 S - \epsilon_2 S + \epsilon_1 S^T \overline{A}^T.$$

Here, we define $\widetilde{P}_1 \triangleq S^T P_1 S$, $L \triangleq KS$, and then we have

$$\begin{bmatrix} \Psi_{11} & \Psi_{12} & \epsilon_2 B_\omega & S^T C^T \\ * & -\epsilon_1 S^T - \epsilon_1 S & \epsilon_1 B_\omega & 0 \\ * & * & -\gamma I & 0 \\ * & * & * & -\gamma I \end{bmatrix} < 0, \tag{26}$$

where

$$\begin{aligned} \Psi_{11} &= \epsilon_2 A_0 S + \epsilon_2 S^T A_0^T + \epsilon_2 BL + \epsilon_2 L^T B^T + \epsilon_2 \Delta_A S + \epsilon_2 S^T \Delta_A^T, \\ \Psi_{12} &= \widetilde{P}_1 - \epsilon_2 S + \epsilon_1 S^T A_0^T + \epsilon_1 S^T \Delta_A^T + \epsilon_1 L^T B^T. \end{aligned}$$

By Lemma 2 and (6), for any scalar $\varepsilon_1 > 0$,

$$\begin{bmatrix} \# & \epsilon_1 S^T \Delta_A^T \\ * & \& \end{bmatrix} \le \begin{bmatrix} \# + \varepsilon_1^{-1} \epsilon_1^2 S^T S & 0 \\ * & \& + \varepsilon_1 \alpha^2 I \end{bmatrix}, \tag{27}$$

where # and & represent the original corresponding matrix elements in (26). And by Lemma 1 and (6), with scalar $\varepsilon_2 > 0$,

$$\begin{aligned} \epsilon_2 \Delta_A S + \epsilon_2 S^T \Delta_A^T &\le \varepsilon_2 \Delta_A \Delta_A^T + \varepsilon_2^{-1} \epsilon_2^2 S^T S \\ &\le \varepsilon_2 \alpha^2 I + \varepsilon_2^{-1} \epsilon_2^2 S^T S. \end{aligned} \tag{28}$$

Thus, considering (26), (27) and (28), and by Schur complement, we can see that (24) can be ensured if (20) holds, which means that the stability and H_∞ performance can be guaranteed by (20).

Next, consider the conditions in Proposition 2. Pre- and post-multiplying (12) and (13) by diag $\{I, S^T\}$ and its transpose respectively, we can readily obtain the equivalent conditions (21) and (22), which can ensure the input constraints.

Finally, for $H = \epsilon_3 V$, (12) in Proposition 3 can be rewritten as

$$\begin{bmatrix} P_2 - \epsilon_3 V^T - \epsilon_3 V & \epsilon_3 V^T (\overline{A} - \eta I) \\ * & -r^2 P_2 \end{bmatrix} < 0. \tag{29}$$

Pre- and post-multiplying (29) by diag $\{S^T, S^T\}$ and its transpose respectively, and by defining $\widetilde{P}_2 = S^T P_2 S$, we obtain

$$\begin{bmatrix} \widetilde{P}_2 - \epsilon_3 S - \epsilon_3 S^T & \epsilon_3 A_0 S + \epsilon_3 \Delta_A S + \epsilon_3 BL - \epsilon_3 \eta S \\ * & -r^2 \widetilde{P}_2 \end{bmatrix} < 0. \tag{30}$$

By Lemma 2 and (6), there exists scalar $\varepsilon_3 > 0$ satisfying

$$\begin{bmatrix} \# & \epsilon_3 \Delta_A S \\ * & \& \end{bmatrix} \le \begin{bmatrix} \# + \varepsilon_3 \epsilon_3^2 \alpha^2 I & 0 \\ * & \& + \varepsilon_3^{-1} S^T S \end{bmatrix}, \tag{31}$$

where # and & represent the original corresponding matrix elements in (30). Thus, considering (30) and (31), and by Schur complement, we can obtain (29) if (21) holds, which means that the poles constraint can be guaranteed by (23).

Thus, we can see that all the conditions listed in Proposition 1–3 can be ensured by (20)–(23), which means that the controller design requirements can be guaranteed by these inequality constraints. And it is obvious that the desired controller can be calculated by $K = LS^{-1}$. The proof is completed. □

Remark 3 *The scalar γ can be included as an optimization variable to obtain a reduction of the H_∞ disturbance attention level bound. Then, the minimum H_∞ disturbance attention level bound in terms of the feasibility of admissible controllers can be readily found by solving the following convex optimization problem:*

$$\textit{Minimize } \gamma \textit{ subject to the LMIs in Theorem 1} \tag{32}$$

4 Illustrative Example

In this section, we provide an example to illustrate the usefulness and advantage of the controller design method proposed in the above sections. Here, we consider a pair of adjacent spacecraft, and make the following assumptions. The mass of the chaser is $300kg$, and the target is moving along a geosynchronous orbit of radius $r = 42241km$ with an orbital period of 24 hours. Thus, the angle velocity $n_0 = 7.2722 \times 10^{-5}$ rads/s. Assume that the initial relative position $(x_0, y_0, z_0) = (800, 600, 500)$ at time $t = 0$. Furthermore, for simplicity, we assume that the initial state $q(0) = [800, 600, 500, 0, 0, 0]$, which means that the spacecraft are relatively static before time $t = 0$. Assume that the maximum input control force is $3000N$. Then, our purpose is to design a state feedback controller K in the form of $u(t) = Kq(t)$, such that the closed-loop system satisfies

(1) stable and $\|T_{f\omega}\|_\infty < \gamma$,

(2) $\|u(t)\|_2 \leq 3000$,

(3) all the poles lie inside the disk region $\mho(-1, 1)$ (centered in -1 with radius 1 in the complex plane).

First, we consider the situation without external perturbations ($\omega(t) = 0$) and assume $\alpha = 0.01$. By solving the convex optimization problem in (32), we obtain the associated matrices as follows (for brevity, we only list the matrices necessary for the construction of the admissible controllers):

$$S = \begin{bmatrix} 0.5271 & -0.0310 & -0.0258 & -0.0308 & 0.0048 & 0.0041 \\ -0.0310 & 0.5448 & -0.0194 & 0.0049 & -0.0336 & 0.0030 \\ -0.0257 & -0.0194 & 0.5520 & 0.0040 & 0.0031 & -0.0347 \\ -0.0414 & 0.0032 & 0.0026 & 0.0074 & -0.0001 & -0.0001 \\ 0.0031 & -0.0432 & 0.0020 & -0.0001 & 0.0075 & -0.0001 \\ 0.0026 & 0.0020 & -0.0439 & -0.0001 & -0.0001 & 0.0075 \end{bmatrix},$$

$$L = \begin{bmatrix} -0.2482 & -0.1024 & -0.0853 & -0.0972 & 0.0094 & 0.0078 \\ -0.1028 & -0.1891 & -0.0643 & 0.0094 & -0.1025 & 0.0059 \\ -0.0854 & -0.0641 & -0.1653 & 0.0078 & 0.0059 & -0.1047 \end{bmatrix}.$$

Therefore, the gain matrix for the feedback controller is given by

$$K = L \times S^{-1} = \begin{bmatrix} -2.2541 & -0.0071 & -0.0072 & -22.3975 & 2.3256 & 1.9369 \\ -0.0104 & -2.2493 & -0.0055 & 2.3259 & -23.7456 & 1.4544 \\ -0.0072 & -0.0055 & -2.2471 & 1.9357 & 1.4552 & -24.2818 \end{bmatrix}.$$

With the controller K, we consider the poles assignment of the closed-loop system. Fig. 2 illustrates the poles placements of the open- and closed-loop system in the complex plane.

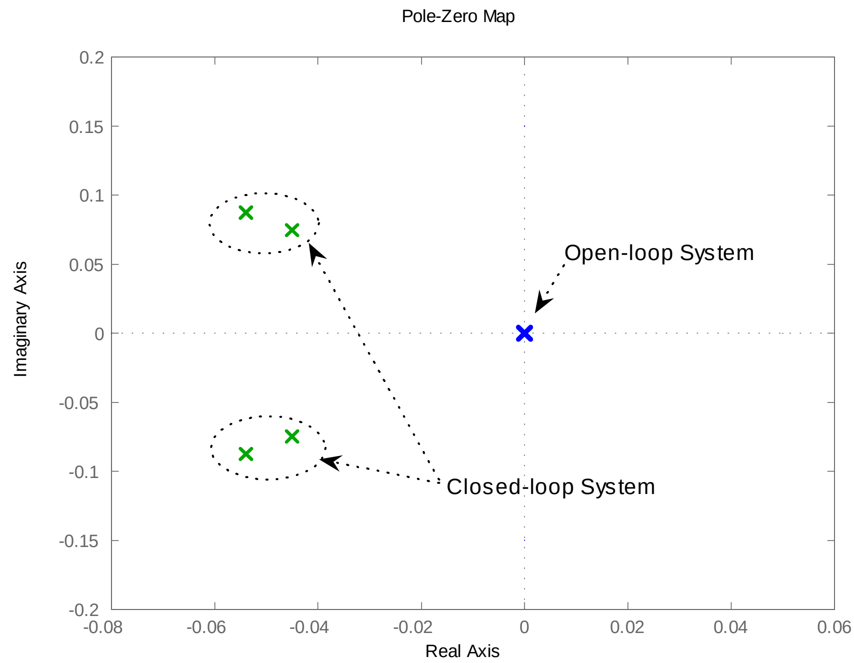

Fig. 2. Poles of open- and closed-loop systems.

We can see that all the poles of open-loop system are near by the origin, which means the weak stability of the system. And we can see that all the poles of the closed-loop system have been aparted from the imagine axis, and have been placed into the expected region $\mho(-1, 1)$. So the requirement of poles assignment can be satisfied by the designed controller.

Next, we assume $\alpha = 0.002$ and the following external disturbance signal:

$$\omega(t) = \begin{cases} 10 \sin 0.2t, & 0 < t < 60\text{s}, \\ 0, & \text{otherwise.} \end{cases}$$

By solving the convex optimization problem in (32), the gain matrix for the feedback controller is given by

$$K = \begin{bmatrix} -1.9770 & 0.0300 & 0.0237 & -23.4372 & 0.7655 & 0.6378 \\ 0.0270 & -1.9939 & 0.0177 & 0.7695 & -23.8812 & 0.4796 \\ 0.0237 & 0.0177 & -2.0004 & 0.6392 & 0.4804 & -24.0589 \end{bmatrix},$$

and $\gamma_{\min} = 4.9678$. The output of the closed-loop system (which means the relative position of the two spacecraft) is depicted in Fig. 3.

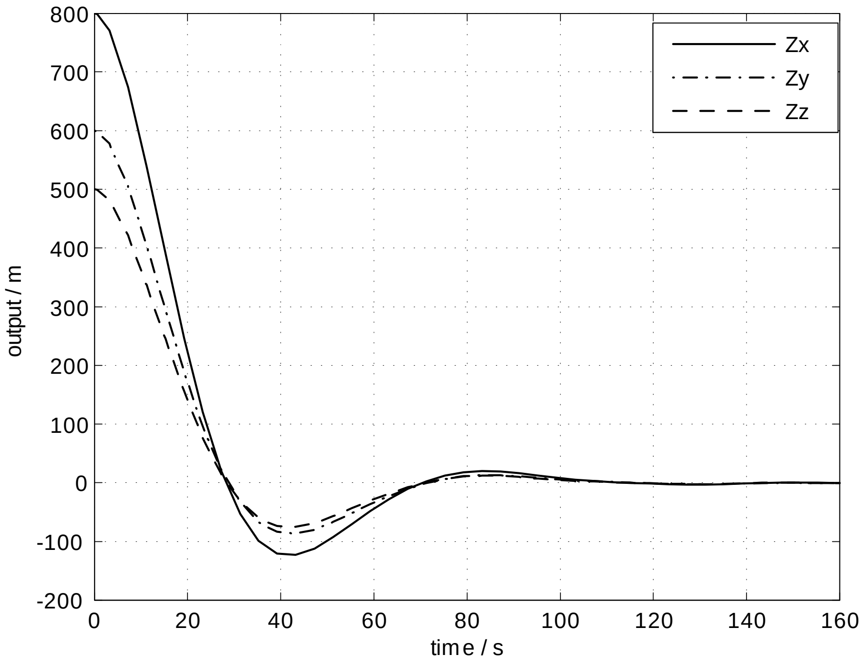

Fig. 3. The output of system (which also means the relative position of the two satellites)

Generally, the expected attitude and orbit of spacecraft are always heavily affected by low frequency disturbance force. Thus, it is necessary to investigate the frequency response of the system during the controller design. Fig. 4 shows the open- and closed-loop frequency responses from the disturbance $\omega(t)$ to the output $z(t)$. The zoomed area is depicted clearly in Fig. 5. It can be seen that the response of very low frequency disturbance is huge in open-loop system, which is unacceptable in practice. And we can see that the closed-loop system has significant reduction in amplitude compared with the open-loop system.

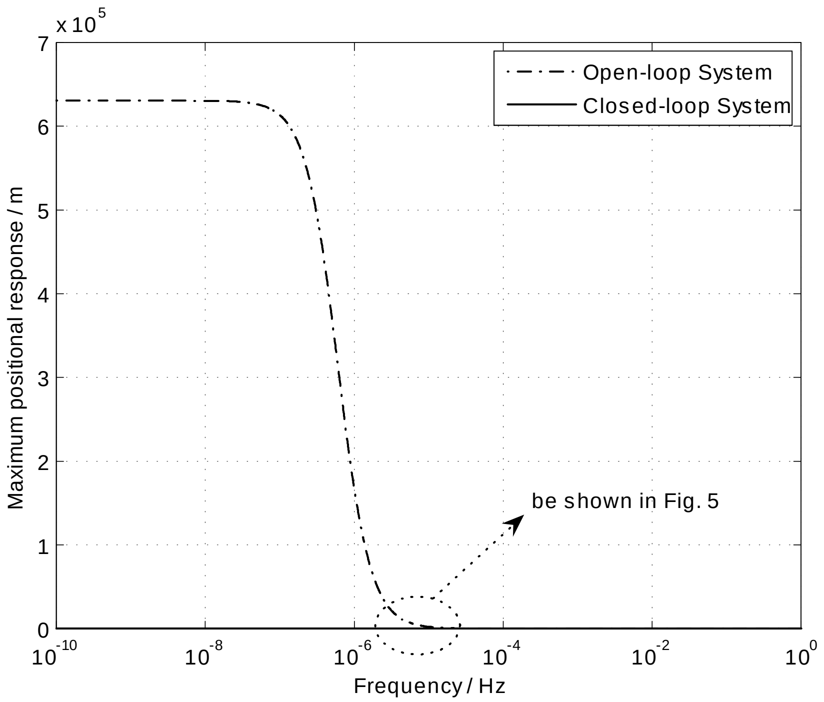

Fig. 4. Overview of the frequency responses of open- and closed-loop system from disturbance to the positional output

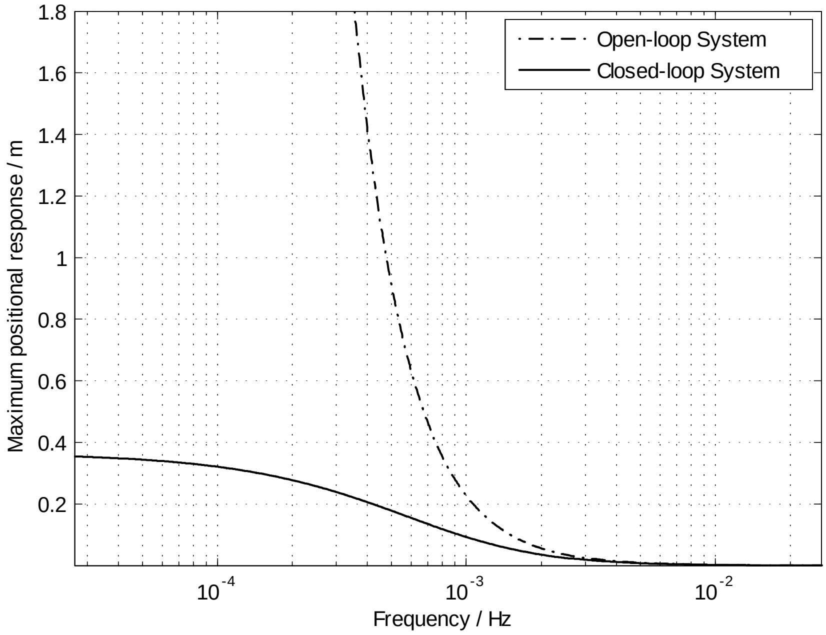

Fig. 5. Zoomed area of frequency responses of open- and closed-loop system from disturbance to the positional output

Furthermore, the maximal control input force must be investigated due to its constraint. The variations of control input force in three axes are depicted in Fig. 6. From the figure, we can see that the input force component in x-axis is the largest, which is obvious due to the initial state (the position component in x-axis is the largest when $t = 0$). And we can see that even the largest input force of three axes is bellow the maximum allowed force $3000N$, which means that the input constraints can be guaranteed by the designed controller.

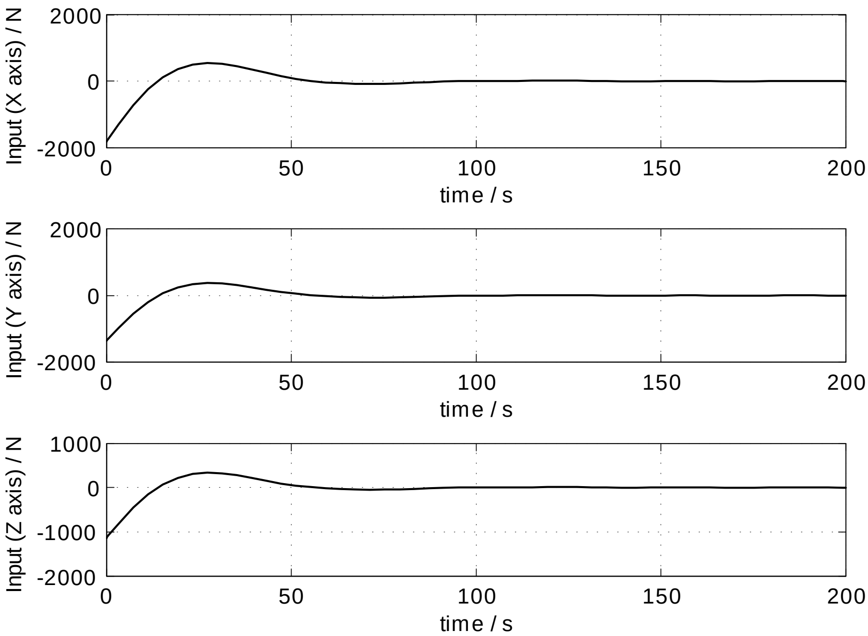

Fig. 6. Control input force in three axises

Finally, the approximated rendezvous orbit is given by Fig. 7. Here, we only show the terminal orbit of rendezvous process. It can be shown that the two spacecraft will eventually asymptotically rendezvous.

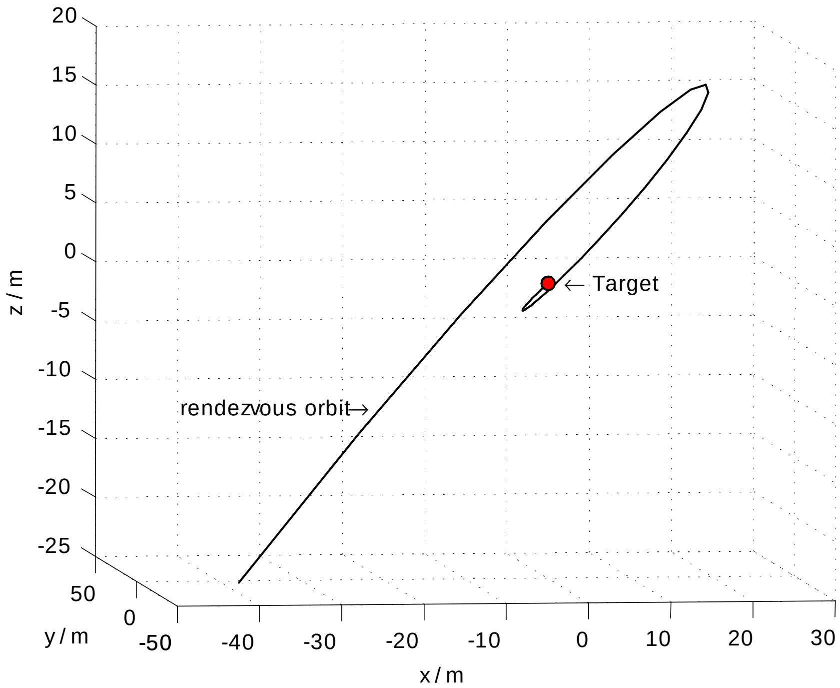

Fig. 7. Approximate rendevzous orbit in the final peroid

5 Conclusions

This paper has presented a new robust H_∞ state-feedback controller design method for spacecraft rendezvous subject to parameter uncertainty, external perturbation, input constraints and poles assignment. By using Lyapunov method and linear matrix inequality techniques, the multi-objective design problem has been transformed into a convex optimization problem with linear matrix inequality constraints. An illustrative example has shown the effectiveness of the proposed controller design methods.

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