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Inequalities for $D-$Synchronous Functions and Related Functionals

Silvestru Sever Dragomir ✉
Victoria University, College of Engineering and Science, Melbourne, Australia.
E-mail: sever.dragomir@vu.edu.au.

Abstract. We introduce in this paper the concept of quadruple $D-$synchronous functions which generalizes the concept of a pair of synchronous functions, we establish an inequality similar to Chebyshev inequality and we also provide some Cauchy-Bunyakovsky-Schwarz type inequalities for a functional associated with this quadruple. Some applications for univariate functions of real variable are given. Discrete inequalities are also stated.
Keywords: Synchronous Functions, Lipschitzian functions, Chebyshev inequality, Cauchy-Bunyakovsky-Schwarz inequality.
MSC2010: 26D15; 26D10.

Desigualdades para funciones $D-$sincrónicas y funciones relacionadas

Resumen. Introducimos en este artículo el concepto de funciones $D-$sincrónicas cuádruples, que generaliza el concepto de un par de funciones sincrónicas; estableceremos una desigualdad similar a la desigualdad de Chebyshev y también presentamos algunas desigualdades de tipo Cauchy-Bunyakovsky-Schwarz para un funcional asociado con este cuádruple. Se dan algunas aplicaciones para funciones univariadas de la variable real. También se indican desigualdades discretas.
Palabras clave: Funciones $D-$sincrónicas, funciones Lipschitzianas, desigualdad de Chebyshev, desigualdad de Cauchy-Bunyakovsky-Schwarz.

1. Introduction

Let $(\Omega, \mathcal{A}, \nu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of subsets of Ω and a countably additive and *positive measure* ν on $\mathcal{A}$ with values in $[0, +\infty]$. For

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a ν-measurable function $w:\Omega\to\mathbb{R}$, with $w(x)\geq 0$ for ν-a.e. (almost every) $x\in\Omega$, consider the *Lebesgue space*

$$L_w(\Omega,\nu):=\{f:\Omega\to\mathbb{R},\ f \text{ is } \nu\text{-measurable and } \int_\Omega w(x)\,|f(x)|\,d\nu(x)<\infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\nu$ instead of $\int_\Omega w(x)\,d\nu(x)$. Assume also that $\int_\Omega w d\nu=1$.

We say that the pair of measurable functions (f,g) are *synchronous* on Ω if

$$(f(x)-f(y))(g(x)-g(y))\geq 0 \tag{1}$$

for ν-a.e. $x,y\in\Omega$. If the inequality reverses in (1), the functions are called *asynchronous* on Ω.

If (f,g) are synchronous on Ω and $f,\ g,\ fg\in L_w(\Omega,\nu)$, then the following inequality, that is known in the literature as *Chebyshev's Inequality,* holds:

$$\int_\Omega wfgd\nu\geq\int_\Omega wfd\nu\int_\Omega wgd\nu, \tag{2}$$

where $w(x)\geq 0$ for ν-a.e. (almost every) $x\in\Omega$ and $\int_\Omega wd\nu=1$.

If $f,\ g:\Omega\to\mathbb{R}$ are ν-measurable functions and $f,\ g,\ fg\in L_w(\Omega,\nu)$, then we may consider the *Chebyshev functional*

$$T_w(f,g):=\int_\Omega wfgd\nu-\int_\Omega wfd\nu\int_\Omega wgd\nu.$$

The following result is known in the literature as the *Grüss inequality:*

$$|T_w(f,g)|\leq\frac{1}{4}(\Gamma-\gamma)(\Delta-\delta), \tag{3}$$

provided

$$-\infty<\gamma\leq f(x)\leq\Gamma<\infty,\qquad -\infty<\delta\leq g(x)\leq\Delta<\infty \tag{4}$$

for ν-a.e. $x\in\Omega$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

If $f\in L_w(\Omega,\nu)$, then we may define

$$D_w(f):=\int_\Omega w(x)\left|f(x)-\int_\Omega w(y)\,f(y)\,d\nu(y)\right|d\nu(x). \tag{5}$$

The following refinement of Grüss inequality in the general setting of measure spaces is due to Cerone & Dragomir [1]:

Theorem 1.1. *Let $w,\ f,\ g:\Omega\to\mathbb{R}$ be ν-measurable functions with $w\geq 0$ ν-a.e. on Ω and $\int_\Omega wd\nu=1$. If $f,\ g,\ fg\in L_w(\Omega,\nu)$ and there exist constants δ, Δ such that*

$$-\infty<\delta\leq g(x)\leq\Delta<\infty\quad \textit{for}\quad \nu\textit{-a.e. } x\in\Omega, \tag{6}$$

then we have the inequality

$$|T_w(f,g)| \leq \frac{1}{2}(\Delta - \delta) D_w(f). \tag{7}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

Motivated by the above results, we introduce in this paper the concept of quadruple $D-$synchronous functions that generalizes the concept of a pair of synchronous functions, we establish an inequality similar to Chebyshev inequality and also provide some Cauchy-Bunyakovsky-Schwarz type inequalities for a functional associated with this quadruple. Some applications for univariate functions of real variable are given. Discrete inequalities are also stated.

2. $D-$Synchronous functions

Let $(\Omega, \mathcal{A}, \nu)$ be a measurable space and $f, g, h, \ell : \Omega \to \mathbb{R}$ be four ν-measurable functions on Ω.

Definition 2.1. The quadruple (f,g,h,ℓ) is called $D-$Synchronous ($D-$Asynchronous) on Ω if

$$\det\begin{pmatrix} f(x) & f(y) \\ g(x) & g(y)\end{pmatrix} \det\begin{pmatrix} h(x) & h(y) \\ \ell(x) & \ell(y)\end{pmatrix} \geq (\leq)\, 0 \tag{8}$$

for ν-a.e. (almost every) $x, y \in \Omega$.

This concept is a generalization of synchronous functions, since for $g = 1$, $\ell = 1$ the quadruple (f,g,h,ℓ) is $D-$Synchronous if, and only if, (f,h) is synchronous on Ω.

If $g, \ell \neq 0$ ν-a.e on Ω, then

$$\begin{aligned}\det\begin{pmatrix} f(x) & f(y) \\ g(x) & g(y)\end{pmatrix} &\det\begin{pmatrix} h(x) & h(y) \\ \ell(x) & \ell(y)\end{pmatrix} \\ &= (f(x)g(y) - g(x)f(y))(h(x)\ell(y) - \ell(x)h(y)) \\ &= g(x)\ell(x)g(y)\ell(y)\left(\frac{f(x)}{g(x)} - \frac{f(y)}{g(y)}\right)\left(\frac{h(x)}{\ell(x)} - \frac{h(y)}{\ell(y)}\right)\end{aligned} \tag{9}$$

for ν-a.e. $x, y \in \Omega$. So, if $g\ell > 0$ ν-a.e on Ω the quadruple (f,g,h,ℓ) is $D-$Synchronous if, and only if, $\left(\frac{f}{g}, \frac{h}{\ell}\right)$ is synchronous on Ω.

Theorem 2.2. *Let $f, g, h, \ell : \Omega \to \mathbb{R}$ be ν-measurable functions on Ω and such that the quadruple (f,g,h,ℓ) is D-Synchronous ($D-$Asynchronous), $w \geq 0$ a.e. on Ω with $\int_\Omega w d\nu = 1$ and fh, $g\ell$, gh, $f\ell \in L_w(\Omega, \nu)$. Then,*

$$\det\begin{pmatrix} \int_\Omega wfh d\nu & \int_\Omega wgh d\nu \\ \int_\Omega wf\ell d\nu & \int_\Omega wg\ell d\nu\end{pmatrix} \geq (\leq)\, 0. \tag{10}$$

Proof. Since the quadruple (f,g,h,ℓ) is $D-$Synchronous, then

$$\begin{aligned} 0 \leq & \left(f(x)g(y)-g(x)f(y)\right)\left(h(x)\ell(y)-\ell(x)h(y)\right) \\ =& f(x)h(x)g(y)\ell(y)+g(x)\ell(x)f(y)h(y) \\ & -f(x)\ell(x)g(y)h(y)-g(x)h(x)f(y)\ell(y) \end{aligned} \tag{11}$$

for ν-a.e. $x,\ y\in\Omega$.

This is equivalent to

$$\begin{aligned} & f(x)h(x)g(y)\ell(y)+g(x)\ell(x)f(y)h(y) \\ & \qquad \geq f(x)\ell(x)g(y)h(y)+g(x)h(x)f(y)\ell(y) \end{aligned} \tag{12}$$

for ν-a.e. $x,\ y\in\Omega$.

Multiply (12) by $w(x)w(y)\geq 0$ to get

$$\begin{aligned} & w(x)f(x)h(x)w(y)g(y)\ell(y)+w(x)g(x)\ell(x)w(y)f(y)h(y) \\ & \qquad \geq w(x)f(x)\ell(x)w(y)g(y)h(y)+w(x)g(x)h(x)w(y)f(y)\ell(y) \end{aligned} \tag{13}$$

for ν-a.e. $x,\ y\in\Omega$.

If we integrate the inequality (13) over $x\in\Omega$, then we get

$$\begin{aligned} & w(y)g(y)\ell(y)\int_\Omega wfhd\nu+w(y)f(y)h(y)\int_\Omega wg\ell d\nu \\ & \qquad \geq w(y)g(y)h(y)\int_\Omega wf\ell d\nu+w(y)f(y)\ell(y)\int_\Omega wghd\nu \end{aligned} \tag{14}$$

for ν-a.e. $y\in\Omega$.

Finally, if we integrate the inequality (14) over $y\in\Omega$, then we get

$$\begin{aligned} & \int_\Omega wfhd\nu\int_\Omega wg\ell d\nu+\int_\Omega wg\ell d\nu\int_\Omega wfhd\nu \\ & \qquad \geq \int_\Omega wf\ell d\nu\int_\Omega wghd\nu+\int_\Omega wghd\nu\int_\Omega wf\ell d\nu, \end{aligned}$$

which is equivalent to the desired result (10). ☑

Corollary 2.3. *Let $f,\ g,\ h,\ \ell:\Omega\to\mathbb{R}$ be ν-measurable functions on Ω and such that $g\ell>0$ ν-a.e on Ω, $\left(\frac{f}{g},\frac{h}{\ell}\right)$ is synchronous (asynchronous) on Ω, $w\geq 0$ a.e. on Ω with $\int_\Omega wd\nu=1$ and $fh,\ g\ell,\ gh,\ f\ell\in L_w(\Omega,\nu)$; then the inequality (10) is valid.*

Let $f,\ g,\ h,\ \ell:\Omega\to\mathbb{R}$ be ν-measurable functions on Ω , $w\geq 0$ a.e. on Ω with $\int_\Omega wd\nu=1$ and $fh,\ g\ell,\ gh,\ f\ell\in L_w(\Omega,\nu)$; then we can consider the functionals

$$\begin{aligned} \mathcal{D}(f,g,h,\ell;w,\Omega) :=& \det\begin{pmatrix} \int_\Omega wfhd\nu & \int_\Omega wghd\nu \\ \int_\Omega wf\ell d\nu & \int_\Omega wg\ell d\nu \end{pmatrix} \\ =& \int_\Omega wfhd\nu\int_\Omega wg\ell d\nu-\int_\Omega wf\ell d\nu\int_\Omega wghd\nu, \end{aligned} \tag{15}$$

and, for $(f,g)=(h,\ell)$,

$$\begin{aligned}\mathcal{D}(f,g;w,\Omega) &:= \mathcal{D}(f,g,f,g;w,\Omega) \\ &= \det\begin{pmatrix} \int_\Omega wf^2 d\nu & \int_\Omega wfg d\nu \\ \int_\Omega wfg d\nu & \int_\Omega wg^2 d\nu \end{pmatrix} \\ &= \int_\Omega wf^2 d\nu \int_\Omega wg^2 d\nu - \left(\int_\Omega wfg d\nu\right)^2,\end{aligned} \tag{16}$$

provided $f^2,\ g^2 \in L_w(\Omega,\nu)$.

We can improve the inequality (10) as follows:

Theorem 2.4. *Let $f,\ g,\ h,\ \ell:\Omega\to\mathbb{R}$ be ν-measurable functions on Ω and such that the quadruple (f,g,h,ℓ) is $D-$Synchronous, $w\geq 0$ a.e. on Ω with $\int_\Omega wd\nu=1$ and $fh,\ g\ell,\ gh,\ f\ell\in L_w(\Omega,\nu)$; then,*

$$\begin{aligned}\mathcal{D}(f,g,h,\ell;w,\Omega) \geq\ & \max\{|\mathcal{D}(|f|,|g|,h,\ell;w,\Omega)|, \\ & |\mathcal{D}(f,g,|h|,|\ell|;w,\Omega)|, |\mathcal{D}(|f|,|g|,|h|,|\ell|;w,\Omega)|\} \\ \geq\ & 0.\end{aligned} \tag{17}$$

Proof. We use the continuity property of the modulus, namely

$$|a-b|\geq ||a|-|b||,\ a,b\in\mathbb{R}.$$

Since (f,g,h,ℓ) is $D-$Synchronous, then

$$\begin{aligned}&(f(x)g(y)-g(x)f(y))(h(x)\ell(y)-\ell(x)h(y)) \\ &\quad = |f(x)g(y)-g(x)f(y)||h(x)\ell(y)-\ell(x)h(y)| \\ &\quad \geq \begin{cases} |(|f(x)||g(y)|-|g(x)||f(y)|)(h(x)\ell(y)-\ell(x)h(y))| \\ \\ |(f(x)g(y)-g(x)f(y))(|h(x)||\ell(y)|-|\ell(x)||h(y)|)| \\ \\ |(|f(x)||g(y)|-|g(x)||f(y)|)(|h(x)||\ell(y)|-|\ell(x)||h(y)|)| \end{cases}\end{aligned} \tag{18}$$

for ν-a.e. $x,\ y\in\Omega$.

As in the proof of Theorem 2.2, we have the identity

$$\begin{aligned}\mathcal{D}(f,g,h,\ell;w,\Omega) = \frac{1}{2}\int_\Omega\int_\Omega & w(x)w(y)(f(x)g(y)-g(x)f(y)) \\ & \times(h(x)\ell(y)-\ell(x)h(y))d\nu(x)d\nu(y).\end{aligned} \tag{19}$$

By using the identity (19) and the first branch in (18) we have

$$\begin{aligned}\mathcal{D}(f,g,h,\ell;w,\Omega) &\geq \frac{1}{2}\int_\Omega\int_\Omega w(x)\,w(y)\,|(|f(x)|\,|g(y)|-|g(x)|\,|f(y)|)\\ &\qquad\times(h(x)\,\ell(y)-\ell(x)\,h(y))|\,d\nu(x)\,d\nu(y)\\ &\geq \frac{1}{2}\left|\int_\Omega\int_\Omega w(x)\,w(y)\,(|f(x)|\,|g(y)|-|g(x)|\,|f(y)|)\right.\\ &\qquad\times(h(x)\,\ell(y)-\ell(x)\,h(y))\,d\nu(x)\,d\nu(y)\Big|\\ &= |\mathcal{D}(|f|,|g|,h,\ell;w,\Omega)|,\end{aligned}$$

which proves the first part of (17).

The second and third part of (17) can be proved in a similar way and details are omitted. ☑

3. Further results for the functional $\mathcal{D}$

We have the following Schwarz's type inequality for the functional $\mathcal{D}$:

Theorem 3.1. *Let $f,\ g,\ h,\ \ell:\Omega\to\mathbb{R}$ be ν-measurable functions on Ω , $w\geq 0$ a.e. on Ω with $\int_\Omega w d\nu=1$ and $f^2,\ g^2,\ h^2,\ \ell^2\in L_w(\Omega,\nu)$. Then,*

$$\mathcal{D}^2(f,g,h,\ell;w,\Omega)\leq\mathcal{D}(f,g;w,\Omega)\,\mathcal{D}(h,\ell;w,\Omega). \tag{20}$$

Proof. As in the proof of Theorem 2.4, we have the identities

$$\begin{aligned}\mathcal{D}(f,g,h,\ell;w,\Omega) &= \frac{1}{2}\int_\Omega\int_\Omega w(x)\,w(y)\,(f(x)\,g(y)-g(x)\,f(y))\\ &\qquad\times(h(x)\,\ell(y)-\ell(x)\,h(y))\,d\nu(x)\,d\nu(y),\end{aligned}$$

$$\mathcal{D}(f,g;w,\Omega)=\frac{1}{2}\int_\Omega\int_\Omega w(x)\,w(y)\,(f(x)\,g(y)-g(x)\,f(y))^2\,d\nu(x)\,d\nu(y)$$

and

$$\mathcal{D}(h,\ell;w,\Omega)=\frac{1}{2}\int_\Omega\int_\Omega w(x)\,w(y)\,(h(x)\,\ell(y)-\ell(x)\,h(y))^2\,d\nu(x)\,d\nu(y).$$

By the Cauchy-Bunyakovsky-Schwarz double integral inequality we have

$$\begin{aligned}&\left(\int_\Omega\int_\Omega w(x)\,w(y)\,(f(x)\,g(y)-g(x)\,f(y))\,(h(x)\,\ell(y)-\ell(x)\,h(y))\,d\nu(x)\,d\nu(y)\right)^2\\ &\leq\int_\Omega\int_\Omega w(x)\,w(y)\,(h(x)\,g(y)-g(x)\,h(y))^2\,d\nu(x)\,d\nu(y)\\ &\qquad\times\int_\Omega\int_\Omega w(x)\,w(y)\,(h(x)\,\ell(y)-\ell(x)\,h(y))^2\,d\nu(x)\,d\nu(y),\end{aligned}$$

which produces the desired result (20). ☑

Corollary 3.2. *Let f, g, h, $\ell : \Omega \to \mathbb{R}$ be ν-measurable functions on Ω with g^2, $\ell^2 \in L_w(\Omega, \nu)$, $w \geq 0$ a.e. on Ω with $\int_\Omega w d\nu = 1$, and a, A, b, $B \in \mathbb{R}$ such that $A > a$, $B > b$,*

$$ag \leq f \leq Ag \quad and \quad b\ell \leq h \leq B\ell \tag{21}$$

ν-a.e. on Ω. Then,

$$|\mathcal{D}(f, g, h, \ell; w, \Omega)| \leq \frac{1}{4}(A-a)(B-b)\int_\Omega wg^2 d\nu \int_\Omega w\ell^2 d\nu. \tag{22}$$

Proof. In [2] (see also [4, p. 8]) we proved the following reverse of Cauchy-Bunyakovsky-Schwarz integral inequality

$$\int_\Omega wf^2 d\nu \int_\Omega wg^2 d\nu - \left(\int_\Omega wfg d\nu\right)^2 \leq \frac{1}{4}(A-a)^2 \left(\int_\Omega wg^2 d\nu\right)^2$$

provided that $ag \leq f \leq Ag$ ν-a.e. on Ω and $g^2 \in L_w(\Omega, \nu)$.

Since, we also have

$$\int_\Omega wh^2 d\nu \int_\Omega w\ell^2 d\nu - \left(\int_\Omega wh\ell d\nu\right)^2 \leq \frac{1}{4}(B-b)^2 \left(\int_\Omega w\ell^2 d\nu\right)^2,$$

provided that $b\ell \leq h \leq B\ell$ ν-a.e. on Ω and $\ell^2 \in L_w(\Omega, \nu)$. Then, by (20) we have

$$\mathcal{D}^2(f, g, h, \ell; w, \Omega) \leq \frac{1}{16}(A-a)^2 (B-b)^2 \left(\int_\Omega wg^2 d\nu\right)^2 \left(\int_\Omega w\ell^2 d\nu\right)^2$$

that is equivalent to the desired result (22). ☑

For positive margins we also have:

Corollary 3.3. *Let f, g, h, $\ell : \Omega \to \mathbb{R}$ be four ν-measurable functions on Ω with $g^2, \ell^2 \in L_w(\Omega, \nu)$, $w \geq 0$ a.e. on Ω with $\int_\Omega w d\nu = 1$, and $a, A, b, B > 0$ such that $A > a$, $B > b$,*

$$ag \leq f \leq Ag \ and \ b\ell \leq h \leq B\ell \tag{23}$$

ν-a.e. on Ω. Then we have

$$|\mathcal{D}(f, g, h, \ell; w, \Omega)| \leq \frac{1}{4}\frac{(A-a)(B-b)}{\sqrt{aAbB}}\int_\Omega wfg d\nu \int_\Omega wh\ell d\nu. \tag{24}$$

Proof. In [3] (see also [4, p. 16]) we proved the following reverse of Cauchy-Bunyakovsky-Schwarz integral inequality:

$$\int_\Omega wf^2 d\nu \int_\Omega wg^2 d\nu - \left(\int_\Omega wfg d\nu\right)^2 \leq \frac{(A-a)^2}{4aA} \left(\int_\Omega wfg d\nu\right)^2,$$

whenever $ag \leq f \leq Ag$ ν-a.e. on Ω.

Since

$$\int_\Omega wh^2 d\nu \int_\Omega w\ell^2 d\nu - \left(\int_\Omega wh\ell d\nu\right)^2 \leq \frac{(B-b)^2}{4bB} \left(\int_\Omega wh\ell d\nu\right)^2,$$

provided $b\ell \leq h \leq B\ell$ ν-a.e. on Ω, then by (20) we get the desired result (24). ☑

If bounds for the sum and difference are available, then we have:

Corollary 3.4. *Let $f,\ g,\ h,\ \ell : \Omega \to \mathbb{R}$ be ν-measurable functions on Ω with $g^2,\ \ell^2 \in L_w(\Omega,\nu)$, $w \geq 0$ a.e. on Ω with $\int_\Omega w d\nu = 1$. Assume that there exists the constants P_1, Q_1, P_2, Q_2 such that*

$$|g-f| \leq P_1, \quad |g+f| \leq Q_1, \quad |h-\ell| \leq P_2, \quad |h+\ell| \leq Q_2 \tag{25}$$

a.e. on Ω; then,

$$|\mathcal{D}(f,g,h,\ell;w,\Omega)| \leq \frac{1}{4}P_1Q_1P_2Q_2. \tag{26}$$

Proof. In the recent paper [5] we obtained amongst other the following reverse of Cauchy-Bunyakovsky-Schwarz integral inequality:

$$\int_\Omega wf^2 d\nu \int_\Omega wg^2 d\nu - \left(\int_\Omega wfg d\nu\right)^2 \leq \frac{1}{4}P_1^2Q_1^2,$$

provided $|g-f| \leq P_1$, $|g+f| \leq Q_1$ a.e. on Ω.

Since

$$\int_\Omega wh^2 d\nu \int_\Omega w\ell^2 d\nu - \left(\int_\Omega wh\ell d\nu\right)^2 \leq \frac{1}{4}P_2^2Q_2^2,$$

if $|h-\ell| \leq P_2$, $|h+\ell| \leq Q_2$ a.e. on Ω, then by (20) we get the desired result (26). ☑

If bounds for each function are available, then we have:

Corollary 3.5. *Let $f,\ g,\ h,\ \ell : \Omega \to \mathbb{R}$ be ν-measurable functions on Ω and $w \geq 0$ a.e. on Ω with $\int_\Omega w d\nu = 1$. Assume that there exists the constants a_i, A_i, b_i and B_i with $i \in \{1,2\}$ such that*

$$0 < a_1 \leq f \leq A_1 < \infty, \qquad 0 < a_2 \leq g \leq A_2 < \infty, \tag{27}$$

and

$$0 < b_1 \leq h \leq B_1 < \infty, \qquad 0 < b_2 \leq \ell \leq B_2 < \infty, \tag{28}$$

a.e. on Ω; then,

$$|\mathcal{D}(f,g,h,\ell;w,\Omega)| \leq \frac{1}{3}(A_1A_2 - a_1a_2)(B_1B_2 - b_1b_2). \tag{29}$$

Proof. We use the following Ozeki's type inequality obtained in [6]:

$$\int_\Omega wf^2 d\nu \int_\Omega wg^2 d\nu - \left(\int_\Omega wfg d\nu\right)^2 \leq \frac{1}{3}(A_1A_2 - a_1a_2)^2,$$

provided $0 < a_1 \leq f \leq A_1 < \infty$, $0 < a_2 \leq g \leq A_2 < \infty$ a.e. on Ω.

Since

$$\int_\Omega wh^2 d\nu \int_\Omega w\ell^2 d\nu - \left(\int_\Omega wh\ell d\nu\right)^2 \leq \frac{1}{3}(B_1B_2 - b_1b_2)^2,$$

when $0 < b_1 \leq h \leq B_1 < \infty$, $0 < b_2 \leq \ell \leq B_2 < \infty$ a.e. on Ω, then by (20) we get the desired result (29). ☑

4. Results for univariate functions

Let $\Omega = [a, b]$ be an interval of real numbers, and assume that $f,\ g,\ h,\ \ell : [a, b] \to \mathbb{R}$ are measurable $D-$Synchronous ($D-$Aynchronous), $w \geq 0$ a.e. on $[a, b]$ with $\int_a^b w(t)\, dt = 1$ and $fh,\ g\ell,\ gh,\ f\ell \in L_w([a, b])$; then,

$$\int_a^b w(t) f(t) h(t)\, dt \int_a^b w(t) g(t) \ell(t)\, dt \tag{30}$$
$$\geq (\leq) \int_a^b w(t) g(t) h(t)\, dt \int_a^b w(t) f(t) \ell(t)\, dt.$$

Now, assume that $[a, b] \subset (0, \infty)$ and take $f(t) = t^p,\ g(t) = t^q,\ h(t) = t^r$ and $\ell(t) = t^s$ with $p,\ q,\ r,\ s \in \mathbb{R}$. Then,

$$\frac{f(t)}{g(t)} = t^{p-q} \quad \text{and} \quad \frac{h(t)}{\ell(t)} = t^{r-s}.$$

If $(p-q)(r-s) > 0$, then the functions $\left(\frac{f}{g}, \frac{h}{\ell}\right)$ have the same monotonicity on $[a, b]$ while if $(p-q)(r-s) < 0$ then $\left(\frac{f}{g}, \frac{h}{\ell}\right)$ have opposite monotonicity on $[a, b]$. Therefore, by (30) we have for any nonnegative integrable function w with $\int_a^b w(t)\, dt = 1$ that

$$\int_a^b w(t) t^{p+r} dt \int_a^b w(t) t^{q+s} dt \geq (\leq) \int_a^b w(t) t^{q+r} dt \int_a^b w(t) t^{p+s} dt, \tag{31}$$

provided $(p-q)(r-s) > (<)\, 0$.

Assume that $[a, b] \subset (0, \infty)$ and take $f(t) = \exp(\alpha t)$, $g(t) = \exp(\beta t)$, $h(t) = \exp(\gamma t)$ and $\ell(t) = \exp(\delta t)$, with $\alpha,\ \beta,\ \gamma,\ \delta \in \mathbb{R}$. Then,

$$\frac{f(t)}{g(t)} = \exp[(\alpha - \beta) t] \quad \text{and} \quad \frac{h(t)}{\ell(t)} = \exp[(\gamma - \delta) t].$$

If $(\alpha - \beta)(\gamma - \delta) > 0$, then the functions $\left(\frac{f}{g}, \frac{h}{\ell}\right)$ have the same monotonicity on $[a, b]$, while if $(\alpha - \beta)(\gamma - \delta) < 0$ then $\left(\frac{f}{g}, \frac{h}{\ell}\right)$ have opposite monotonicity on $[a, b]$. Therefore, by (30) we have for any nonnegative integrable function w with $\int_a^b w(t)\, dt = 1$ that

$$\int_a^b w(t) \exp[(\alpha + \gamma) t]\, dt \int_a^b w(t) \exp[(\beta + \delta) t]\, dt \tag{32}$$
$$\geq (\leq) \int_a^b w(t) \exp[(\beta + \gamma) t]\, dt \int_a^b w(t) \exp[(\alpha + \delta) t]\, dt,$$

provided $(\alpha - \beta)(\gamma - \delta) > (<)\, 0$.

Consider the functional

$$\mathcal{D}_{p,q,r,s}(w) := \int_a^b w(t) t^{p+r} dt \int_a^b w(t) t^{q+s} dt \tag{33}$$
$$- \int_a^b w(t) t^{q+r} dt \int_a^b w(t) t^{p+s} dt,$$

for any nonnegative integrable function w with $\int_a^b w(t)\,dt = 1$, and $p, q, r, s \in \mathbb{R}$.

We observe that for $t \in [a,b] \subset (0,\infty)$ we have

$$k_{p,q}(a,b) := \begin{cases} a^{p-q}, & \text{if } p \geq q, \\ \\ b^{p-q}, & \text{if } p < q, \end{cases} \leq \frac{f(t)}{g(t)} = t^{p-q} \tag{34}$$
$$\leq K_{p,q}(a,b) := \begin{cases} b^{p-q}, & \text{if } p \geq q, \\ \\ a^{p-q} & \text{if } p < q, \end{cases}$$

and, similarly,

$$k_{r,s}(a,b) \leq \frac{h(t)}{\ell(t)} = t^{r-s} \leq K_{r,s}(a,b).$$

Using the inequality (22) we have

$$|\mathcal{D}_{p,q,r,s}(w)| \leq \frac{1}{4}\left[K_{p,q}(a,b) - k_{p,q}(a,b)\right]\left[K_{r,s}(a,b) - k_{r,s}(a,b)\right] \tag{35}$$
$$\times \int_a^b w(t)\,t^{2q}dt \int_a^b w(t)\,t^{2s}dt,$$

while from (24) we have

$$|\mathcal{D}_{p,q,r,s}(w)| \leq \frac{1}{4}\frac{\left[K_{p,q}(a,b) - k_{p,q}(a,b)\right]\left[K_{r,s}(a,b) - k_{r,s}(a,b)\right]}{\sqrt{k_{p,q}(a,b)\,k_{r,s}(a,b)\,K_{p,q}(a,b)\,K_{r,s}(a,b)}} \tag{36}$$
$$\times \int_a^b w(t)\,t^{p+q}dt \int_a^b w(t)\,t^{r+s}dt.$$

We also have for $t \in [a,b] \subset (0,\infty)$ that

$$u_p(a,b) := \begin{cases} a^p, & \text{if } p \geq 0, \\ \\ b^p, & \text{if } p < 0, \end{cases} \leq f(t) = t^p$$
$$\leq U_p(a,b) := \begin{cases} b^p, & \text{if } p \geq 0, \\ \\ a^p, & \text{if } p < 0, \end{cases}$$

and the corresponding bounds for $g(t) = t^q$, $h(t) = t^r$ and $\ell(t) = t^s$, with $p, q, r, s \in \mathbb{R}$.

Making use of the inequality (29) we get

$$|\mathcal{D}_{p,q,r,s}(w)| \leq \frac{1}{3}\left(U_p(a,b)\,U_q(a,b) - u_p(a,b)\,u_q(a,b)\right) \tag{37}$$
$$\times \left(U_r(a,b)\,U_s(a,b) - u_r(a,b)\,u_s(a,b)\right).$$

Similar results may be stated for the functional

$$\mathcal{D}_{\alpha,\beta,\gamma,\delta}(w) := \int_a^b w(t)\exp\left[(\alpha+\gamma)t\right]dt \int_a^b w(t)\exp\left[(\beta+\delta)t\right]dt$$
$$- \int_a^b w(t)\exp\left[(\beta+\gamma)t\right]dt \int_a^b w(t)\exp\left[(\alpha+\delta)t\right]dt$$

for any nonnegative integrable function w with $\int_a^b w(t)\,dt = 1$, for $\alpha,\ \beta,\ \gamma,\ \delta \in \mathbb{R}$ and $[a,b] \subset (0,\infty)$. Details are omitted.

We say that the function $\varphi : [a,b] \to \mathbb{R}$ is Lipschitzian with the constant $L > 0$ if

$$|\varphi(t) - \varphi(s)| \leq L\,|t-s|$$

for any $t,\ s \in [a,b]$.

Define the functional

$$\begin{aligned}\mathcal{D}(f,g,h,\ell;w,[a,b]) :=& \int_a^b w(t) f(t) h(t)\,dt \int_a^b w(t) g(t) \ell(t)\,dt\\ &- \int_a^b w(t) g(t) h(t)\,dt \int_a^b w(t) f(t) \ell(t)\,dt.\end{aligned}$$

In the next result we provided two upper bounds in terms of Lipschitzian constants:

Theorem 4.1. *Let $f,\ g,\ h,\ \ell : [a,b] \to \mathbb{R}$ be measurable functions and $w \geq 0$ a.e. on $[a,b]$ with $\int_a^b w(t)\,dt = 1$.*

(i) If $g(t), \ell(t) \neq 0$ for any $t \in [a,b]$, and $\frac{f}{g}$ is Lipschitzian with the constant $L > 0$, and $\frac{h}{\ell}$ is Lipschitzian with the constant $K > 0$, and $g\ell,\ g\ell e^2 \in L_w([a,b])$ with $e(t) = t,\ t \in [a,b]$, then

$$\begin{aligned}&|\mathcal{D}(f,g,h,\ell;w,[a,b])|\\ &\leq LK\left[\int_a^b w(s)|g(s)||\ell(s)|\,ds \int_a^b w(t)|\ell(t)||g(t)|\,t^2 dt\right.\\ &\qquad\left.- \left(\int_a^b w(t)|g(t)||\ell(t)|\,t dt\right)^2\right]. \quad (38)\end{aligned}$$

(ii) If, in addition, we have $wg\ell \in L_\infty[a,b]$ and

$$\|wg\ell\|_\infty = \operatorname{esssup}_{t\in[a,b]} |w(t) g(t) \ell(t)| < \infty,$$

then

$$|\mathcal{D}(f,g,h,\ell;w,[a,b])| \leq \frac{1}{12}(b-a)^4 LK \|wg\ell\|_\infty^2. \quad (39)$$

Proof. We have

$$\begin{aligned}\mathcal{D}(f,g,h,\ell;w,[a,b]) =& \frac{1}{2}\int_a^b \int_a^b w(t) w(s) (f(t) g(s) - g(t) f(s))\\ &\times (h(t) \ell(s) - \ell(t) h(s))\,dtds\\ =& \frac{1}{2}\int_a^b \int_a^b w(t) w(s) g(t) g(s) \ell(t) \ell(s)\\ &\times \left(\frac{f(t)}{g(t)} - \frac{f(s)}{g(s)}\right)\left(\frac{h(t)}{\ell(t)} - \frac{h(s)}{\ell(s)}\right) dtds.\end{aligned}$$

By taking modulus in this equality, we get

$$
\begin{aligned}
&\left|\mathcal{D}\left(f,g,h,\ell;w,[a,b]\right)\right| &(40)\\
&\quad\le \frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\left|g(t)\right|\left|g(s)\right|\left|\ell(t)\right|\left|\ell(s)\right| \times \left|\frac{f(t)}{g(t)}-\frac{f(s)}{g(s)}\right|\left|\frac{h(t)}{\ell(t)}-\frac{h(s)}{\ell(s)}\right| dt\,ds\\
&\quad\le \frac{1}{2}LK\int_a^b\int_a^b w(t)\,w(s)\left|g(t)\right|\left|g(s)\right|\left|\ell(t)\right|\left|\ell(s)\right|(t-s)^2\,dt\,ds.
\end{aligned}
$$

Now, observe that

$$
\begin{aligned}
&\int_a^b\int_a^b w(t)\,w(s)\left|g(t)\right|\left|g(s)\right|\left|\ell(t)\right|\left|\ell(s)\right|(t-s)^2\,dtds &(41)\\
&\quad=\int_a^b\int_a^b w(t)\,w(s)\left|g(t)\right|\left|g(s)\right|\left|\ell(t)\right|\left|\ell(s)\right|\left(t^2-2ts+s^2\right)dtds\\
&\quad=2\left(\int_a^b\int_a^b w(t)\,w(s)\left|g(t)\right|\left|g(s)\right|\left|\ell(t)\right|\left|\ell(s)\right|t^2dtds\right.\\
&\qquad\left.-\int_a^b\int_a^b w(t)\,w(s)\left|g(t)\right|\left|g(s)\right|\left|\ell(t)\right|\left|\ell(s)\right|tsdtds\right)\\
&\quad=2\left[\int_a^b w(s)\left|g(s)\right|\left|\ell(s)\right|ds\int_a^b w(t)\left|g(t)\right|\left|\ell(t)\right|t^2dt\right.\\
&\qquad\left.-\left(\int_a^b w(t)\left|g(t)\right|\left|\ell(t)\right|tdt\right)^2\right].
\end{aligned}
$$

On making use of (40) and (41) we get the desired result (38).

If $wg\ell \in L_\infty[a,b]$, then

$$
\begin{aligned}
&\int_a^b\int_a^b w(t)\,w(s)\left|g(t)\right|\left|g(s)\right|\left|\ell(t)\right|\left|\ell(s)\right|(t-s)^2\,dtds\\
&\qquad\le \left\|wg\ell\right\|_\infty^2\int_a^b\int_a^b(t-s)^2\,dtds=\frac{1}{6}(b-a)^4\left\|wg\ell\right\|_\infty^2. &(42)
\end{aligned}
$$

Therefore, by inequalities (40) and (42) we obtain the desired result (39). ☑

5. Discrete inequalities

Consider the n-tuples of real numbers $x=(x_1,...,x_n)$, $y=(y_1,...,y_n)$, $z=(z_1,...,z_n)$ and $u=(u_1,...,u_n)$. We say that the quadruple (x,y,z,u) is $D-$*Synchronous* if

$$
\begin{aligned}
0&\le \det\begin{pmatrix} x_i & x_j\\ y_i & y_j\end{pmatrix}\det\begin{pmatrix} z_i & z_j\\ u_i & u_j\end{pmatrix} &(43)\\
&=\left(x_iy_j-x_jy_i\right)\left(z_iu_j-z_ju_i\right)
\end{aligned}
$$

for any $i,\ j\in\{1,...,n\}$.

If $p=(p_1,...,p_n)$ is a probability distribution, namely, $p_i\geq 0$ for any $i\in\{1,...,n\}$ and $\sum_{i=1}^n p_i=1$, and the quadruple (x,y,z,u) is $D-$Synchronous, then by (10) we have:

$$\begin{aligned}\mathcal{D}_n(x,y,z,u;p)&:=\det\begin{pmatrix}\sum_{i=1}^n p_ix_iz_i & \sum_{i=1}^n p_iy_iz_i\\ \\ \sum_{i=1}^n p_ix_iu_i & \sum_{i=1}^n p_iy_iu_i\end{pmatrix} \\ &=\sum_{i=1}^n p_ix_iz_i\sum_{i=1}^n p_iy_iu_i-\sum_{i=1}^n p_ix_iu_i\sum_{i=1}^n p_iy_iz_i\geq 0.\end{aligned} \tag{44}$$

For an n-tuples of real numbers $x=(x_1,...,x_n)$, we denote by $|x|:=(|x_1|,...,|x_n|)$. On making use of the inequality (17), then for any $D-$Synchronous quadruple (x,y,z,u) and for any probability distribution $p=(p_1,...,p_n)$ we have

$$\begin{aligned}&\mathcal{D}_n(x,y,z,u;p)\\ &\quad\geq\max\{|\mathcal{D}_n(|x|,y,z,u;p)|,|\mathcal{D}_n(x,|y|,z,u;p)|,|\mathcal{D}_n(|x|,|y|,z,u;p)|\}\geq 0.\end{aligned} \tag{45}$$

Observe that if we consider

$$\mathcal{D}_n(x,y;p):=\mathcal{D}_n(x,y,x,y;p)=\sum_{i=1}^n p_ix_i^2\sum_{i=1}^n p_iy_i^2-\left(\sum_{i=1}^n p_ix_iy_i\right)^2,$$

then by (20) we have

$$|\mathcal{D}_n(x,y,z,u;p)|^2\leq\mathcal{D}_n(x,y;p)\,\mathcal{D}_n(z,u;p) \tag{46}$$

for any quadruple (x,y,z,u) and any probability distribution $p=(p_1,...,p_n)$.

If $a,\ A,\ b,\ B\in\mathbb{R}$ and (x,y,z,u) are such that $A>a,\ B>b$,

$$ay_i\leq x_i\leq Ay_i\text{ and }bu_i\leq z_i\leq Bu_i \tag{47}$$

for any $i\in\{1,...,n\}$, then by (22) we have

$$|\mathcal{D}_n(x,y,z,u;p)|\leq\frac{1}{4}(A-a)(B-b)\sum_{i=1}^n p_iy_i^2\sum_{i=1}^n p_iu_i^2. \tag{48}$$

If $a,\ A,\ b,\ B>0$ and condition (47) is valid, then by (24) we have

$$|\mathcal{D}_n(x,y,z,u;p)|\leq\frac{1}{4}\frac{(A-a)(B-b)}{\sqrt{aAbB}}\sum_{i=1}^n p_ix_iy_i\sum_{i=1}^n p_iz_iu_i. \tag{49}$$

Now, if we use the *Klamkin-McLenaghan's inequality*

$$\sum_{i=1}^n p_ix_i^2\sum_{i=1}^n p_iy_i^2-\left(\sum_{i=1}^n p_ix_iy_i\right)^2\leq\left(\sqrt{A}-\sqrt{a}\right)^2\sum_{i=1}^n p_ix_iy_i\sum_{i=1}^n p_ix_i^2$$

that holds for x, y satisfying the condition (47) with A, $a > 0$, then by (46) we get

$$\begin{aligned} & |\mathcal{D}_n(x, y, z, u; p)| \\ & \quad \leq \left(\sqrt{A} - \sqrt{a}\right)\left(\sqrt{B} - \sqrt{b}\right) \\ & \qquad \times \left(\sum_{i=1}^{n} p_i x_i y_i\right)^{1/2} \left(\sum_{i=1}^{n} p_i x_i^2\right)^{1/2} \left(\sum_{i=1}^{n} p_i z_i u_i\right)^{1/2} \left(\sum_{i=1}^{n} p_i z_i^2\right)^{1/2}, \end{aligned} \tag{50}$$

provided (x, y, z, u) satisfy (47) with a, A, b, $B > 0$.

Now, assume that

$$0 < a_1 \leq x_i \leq A_1 < \infty, \qquad 0 < a_2 \leq y_i \leq A_2 < \infty, \tag{51}$$

and

$$0 < b_1 \leq x_i \leq B_1 < \infty, \qquad 0 < b_2 \leq u_i \leq B_2 < \infty, \tag{52}$$

for any $i \in \{1, ..., n\}$; then by (29) we get

$$|\mathcal{D}_n(x, y, z, u; p)| \leq \frac{1}{3}(A_1 A_2 - a_1 a_2)(B_1 B_2 - b_1 b_2), \tag{53}$$

for any probability distribution $p = (p_1, ..., p_n)$.

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