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REFINEMENTS AND REVERSES OF HöLDER-MCCARTHY OPERATOR INEQUALITY VIA A CARTWRIGHT-FIELD RESULT

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Abstract. By the use of a classical result of Cartwright and Field, in this paper we have obtained new refinements and reverses of Hölder-McCarthy operator inequality in the case of $p \in (0, 1)$. A comparison for the two upper bounds obtained showing that neither of them is better in general, has also been performed.

Keywords: Hölder-McCarthy operator inequality; selfadjoint operator; Hilbert space; nonnegative operator.

1. Introduction

Let A be a nonnegative operator on the complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$, namely $\langle Ax, x\rangle \geq 0$ for any $x \in H$. We write this as $A \geq 0$.

By the use of the spectral resolution of A and the Hölder inequality, C. A. McCarthy [16] proved that

$$\langle Ax, x\rangle^p \leq \langle A^p x, x\rangle\,,\ p \in (1, \infty) \tag{1.1}$$

and

$$\langle A^p x, x\rangle \leq \langle Ax, x\rangle^p\,,\ p \in (0, 1) \tag{1.2}$$

for any $x \in H$ with $\|x\| = 1$.

Let A be a selfadjoint operator on H with

$$mI \leq A \leq MI, \tag{1.3}$$

where I is the *identity operator* and m, M are real numbers with $m < M$.

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In [7, Theorem 3] Fujii et al. obtained the following interesting ratio inequality that provides a reverse of the *Hölder-McCarthy inequality* (1.1) for an operator A that satisfies the condition (1.3) with $m > 0$

$$\langle A^p x, x\rangle \leq \left\{ \frac{1}{p^{1/p} q^{1/q}} \frac{M^p - m^p}{(M-m)^{1/p} (mM^p - Mm^p)^{1/q}} \right\}^p \langle Ax, x\rangle^p, \tag{1.4}$$

for any $x \in H$ with $\|x\| = 1$, where $q = p/(p-1)$, $p > 1$.

If A satisfies the condition (1.3) with $m \geq 0$, then we also have the additive reverse of (1.1) that has been obtained by the author in 2008, see [4]

$$\begin{aligned}\langle A^p x, x\rangle - \langle Ax, x\rangle^p &\leq \tfrac{1}{2} p (M-m) \left[\left\| A^{p-1} x \right\|^2 - \left\langle A^{p-1} x, x\right\rangle^2 \right]^{1/2} \\ &\leq \tfrac{1}{4} p (M-m) \left(M^{p-1} - m^{p-1} \right)\end{aligned}$$

and

$$\begin{aligned}\langle A^p x, x\rangle - \langle Ax, x\rangle^p &\leq \tfrac{1}{2} p \left(M^{p-1} - m^{p-1} \right) \left[\|Ax\|^2 - \langle Ax, x\rangle^2 \right]^{1/2} \\ &\leq \tfrac{1}{4} p (M-m) \left(M^{p-1} - m^{p-1} \right)\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$, where $p > 1$.

We also have the alternative upper bounds [4]

$$\begin{aligned}\langle A^p x, x\rangle - \langle Ax, x\rangle^p &\leq \tfrac{1}{4} p \tfrac{(M-m)\left(M^{p-1} - m^{p-1}\right)}{M^{p/2} m^{p/2}} \langle Ax, x\rangle \left\langle A^{p-1} x, x\right\rangle, \text{ (for } m > 0), \\ &\leq p \tfrac{1}{4} (M-m) \left(M^{p-1} - m^{p-1} \right) \left(\tfrac{M}{m} \right)^{p/2}, \text{ (for } m > 0)\end{aligned}$$

and

$$\begin{aligned}\langle A^p x, x\rangle - \langle Ax, x\rangle^p &\leq p \left(\sqrt{M} - \sqrt{m} \right) \left(M^{(p-1)/2} - m^{(p-1)/2} \right) \left[\langle Ax, x\rangle \left\langle A^{p-1} x, x\right\rangle \right]^{\frac{1}{2}} \\ &\leq p \left(\sqrt{M} - \sqrt{m} \right) \left(M^{(p-1)/2} - m^{(p-1)/2} \right) M^{p/2}\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$, where $p > 1$.

For various related inequalities, see [6]-[10] and [14]-[15].

We have the following inequality that provides a refinement and a reverse for the celebrated *Young's scalar inequality*

$$\begin{aligned}\frac{1}{2} \nu (1-\nu) \frac{(b-a)^2}{\max\{a,b\}} &\leq (1-\nu) a + \nu b - a^{1-\nu} b^\nu \\ &\leq \frac{1}{2} \nu (1-\nu) \frac{(b-a)^2}{\min\{a,b\}}\end{aligned} \tag{1.5}$$

for any a, $b > 0$ and $\nu \in [0,1]$.

This result was obtained in 1978 by Cartwright and Field [1] who established a more general result for n variables and gave an application for a probability measure supported on a finite interval.

For new recent reverses and refinements of Young's inequality see [2]-[3], [11]-[12], [13] and [19].

By the use of (1.5). we have obtained new refinements and reverses of Hölder-McCarthy operator inequality in the case when $p \in (0,1)$. A comparison for the two upper bounds obtained showing that neither of them is better in general, has also been performed.

2. Some Refinements and Reverse Results

We have:

Theorem 2.1. *Let m, M be real numbers so that $M > m > 0$. If A is a selfadjoint operator satisfying the condition (1.3) above, then for any $p \in (0,1)$ we have*

$$\begin{aligned}(2.1)\qquad \frac{p(1-p)}{2}\frac{m}{M}\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}-1\right) &\le \tfrac{p(1-p)}{2M}\langle Ax,x\rangle\left(\tfrac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}-1\right)\\ &\le 1-\tfrac{\langle A^px,x\rangle}{\langle Ax,x\rangle^p}\\ &\le \tfrac{p(1-p)}{2m}\langle Ax,x\rangle\left(\tfrac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}-1\right)\\ &\le \tfrac{p(1-p)}{2}\tfrac{M}{m}\left(\tfrac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}-1\right)\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

In particular,

$$\begin{aligned}(2.2)\qquad \frac{1}{8}\frac{m}{M}\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}-1\right) &\le \tfrac{\langle Ax,x\rangle}{8M}\left(\tfrac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}-1\right)\\ \le 1-\tfrac{\langle A^{1/2}x,x\rangle}{\langle Ax,x\rangle^{1/2}} &\le \tfrac{\langle Ax,x\rangle}{8m}\left(\tfrac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}-1\right)\\ &\le \tfrac{1}{8}\tfrac{M}{m}\left(\tfrac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}-1\right),\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

Proof. If $a,\ b \in [m, M]$, then by Cartwright-Field inequality (1.5) we have

$$\frac{1}{2M}p(1-p)(b-a)^2 \le (1-p)a + pb - a^{1-p}b^p \le \frac{1}{2m}p(1-p)(b-a)^2$$

or, equivalently

$$\begin{aligned}(2.3)\qquad \frac{1}{2M}p\,(1-p)\,(b^2-2ab+a^2) &\le (1-p)\,a+pb-a^{1-p}b^p \\ &\le \frac{1}{2m}p\,(1-p)\,(b^2-2ab+a^2),\end{aligned}$$

for any $p\in(0,1)$.

Fix $a\in[m,M]$ and by using the operator functional calculus for A with $mI\le A\le MI$ we have

$$\begin{aligned}(2.4)\qquad \frac{1}{2M}p\,(1-p)\,(A^2-2aA+a^2I) &\le (1-p)\,aI+pA-a^{1-p}A^p \\ &\le \frac{1}{2m}p\,(1-p)\,(A^2-2aA+a^2I).\end{aligned}$$

Then for any $x\in H$ with $\|x\|=1$ we have from (2.4) that

$$\begin{aligned}(2.5)\qquad &\tfrac{1}{2M}p\,(1-p)\,(\langle A^2x,x\rangle-2a\,\langle Ax,x\rangle+a^2) \\ &\le (1-p)\,a+p\,\langle Ax,x\rangle-a^{1-p}\,\langle A^px,x\rangle \\ &\le \tfrac{1}{2m}p\,(1-p)\,(\langle A^2x,x\rangle-2a\,\langle Ax,x\rangle+a^2),\end{aligned}$$

for any $a\in[m,M]$.

If we choose in (2.5) $a=\langle Ax,x\rangle\in[m,M]$, then we get for any $x\in H$ with $\|x\|=1$ that

$$\begin{aligned}\frac{1}{2M}p\,(1-p)\,(\langle A^2x,x\rangle-\langle Ax,x\rangle^2) &\le \langle Ax,x\rangle-\langle Ax,x\rangle^{1-p}\,\langle A^px,x\rangle \\ &\le \tfrac{1}{2m}p\,(1-p)\,(\langle A^2x,x\rangle-\langle Ax,x\rangle^2),\end{aligned}$$

and by division with $\langle Ax,x\rangle>0$ we obtain the second and third inequalities in (2.1).

The rest is obvious. ∎

Remark 2.1. It is well known that, if $mI\le A\le MI$ with $M>0$, then, see for instance [17, p. 27], we have

$$(1\le)\,\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}\le\frac{(m+M)^2}{4mM}$$

for any $x\in H$ with $\|x\|=1$, which implies that

$$(0\le)\,\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}-1\le\frac{(M-m)^2}{4mM}.$$

Using (2.1) and by denoting $h=\frac{M}{m}$ we get

$$(2.6)\qquad (0\le)\,1-\frac{\langle A^px,x\rangle}{\langle Ax,x\rangle^p}\le\frac{p\,(1-p)}{8}\,(h-1)^2$$

and, in particular,

$$(0 \leq) \, 1 - \frac{\left\langle A^{1/2} x, x \right\rangle}{\left\langle Ax, x \right\rangle^{1/2}} \leq \frac{1}{32} \left(h - 1 \right)^2, \tag{2.7}$$

for any $x \in H$ with $\|x\| = 1$.

We consider the *Kantorovich's constant* defined by

$$K\left(h \right) := \frac{\left(h + 1 \right)^2}{4h}, \ h > 0. \tag{2.8}$$

The function K is decreasing on $(0, 1)$ and increasing on $[1, \infty)$, $K(h) \geq 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

Observe that for any $h > 0$

$$K\left(h \right) - 1 = \frac{\left(h - 1 \right)^2}{4h} = K\left(\frac{1}{h} \right) - 1.$$

From (2.6) we then have

$$(0 \leq) \, 1 - \frac{\left\langle A^p x, x \right\rangle}{\left\langle Ax, x \right\rangle^p} \leq \frac{p\left(1 - p\right)}{2} h \left[K\left(h \right) - 1 \right] \tag{2.9}$$

and, in particular,

$$(0 \leq) \, 1 - \frac{\left\langle A^{1/2} x, x \right\rangle}{\left\langle Ax, x \right\rangle^{1/2}} \leq \frac{1}{8} h \left[K\left(h \right) - 1 \right], \tag{2.10}$$

for any $x \in H$ with $\|x\| = 1$.

Also, if a, $b > 0$ then

$$K\left(\frac{b}{a} \right) - 1 = \frac{\left(b - a \right)^2}{4ab}.$$

Since $\min\{a, b\} \max\{a, b\} = ab$ if $a, b > 0$, then

$$\frac{\left(b - a \right)^2}{\max\{a, b\}} = \frac{\min\{a, b\} \left(b - a \right)^2}{ab} = 4 \min\{a, b\} \left[K\left(\frac{b}{a} \right) - 1 \right]$$

and

$$\frac{\left(b - a \right)^2}{\min\{a, b\}} = \frac{\max\{a, b\} \left(b - a \right)^2}{ab} = 4 \max\{a, b\} \left[K\left(\frac{b}{a} \right) - 1 \right]$$

and the inequality (1.5) can be written as

$$\begin{aligned} 2\nu \left(1 - \nu\right) \min\{a, b\} \left[K\left(\frac{b}{a} \right) - 1 \right] &\leq \left(1 - \nu\right) a + \nu b - a^{1-\nu} b^{\nu} \\ &\leq 2\nu \left(1 - \nu\right) \max\{a, b\} \left[K\left(\tfrac{b}{a} \right) - 1 \right] \end{aligned}$$

for any a, $b > 0$ and $\nu \in [0, 1]$.

Theorem 2.2. *Let m, M be real numbers so that $M > m > 0$. If A is a selfadjoint operator satisfying the condition (1.3) above, then for any $p \in (0,1)$ we have*

$$
\begin{aligned}
(0 \le)\, 1 - \tfrac{\langle A^p x,x\rangle}{\langle Ax,x\rangle^p} &\le p(1-p)[K(h)-1]\left(2 + \tfrac{\langle |A-\langle Ax,x\rangle I|x,x\rangle}{\langle Ax,x\rangle}\right) \\
&\le p(1-p)[K(h)-1]\left[2 + \left(\tfrac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2} - 1\right)^{1/2}\right] \\
&\le p(1-p)[K(h)-1]\left[2 + (K(h)-1)^{1/2}\right]
\end{aligned} \tag{2.11}
$$

for any $x \in H$ with $\|x\| = 1$.

In particular, we have

$$
\begin{aligned}
(0 \le)\, 1 - \tfrac{\langle A^{1/2} x,x\rangle}{\langle Ax,x\rangle^{1/2}} &\le \tfrac{1}{4}[K(h)-1]\left(2 + \tfrac{\langle |A-\langle Ax,x\rangle I|x,x\rangle}{\langle Ax,x\rangle}\right) \\
&\le \tfrac{1}{4}[K(h)-1]\left[2 + \left(\tfrac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2} - 1\right)^{1/2}\right] \\
&\le \tfrac{1}{4}[K(h)-1]\left[2 + (K(h)-1)^{1/2}\right]
\end{aligned} \tag{2.12}
$$

for any $x \in H$ with $\|x\| = 1$.

Proof. From (2.11) we have for any a, $b > 0$ and $p \in [0,1]$ that

$$
(1-p)\,a + pb - a^{1-p}b^p \le p(1-p)(a+b+|b-a|)\left[K\left(\frac{b}{a}\right) - 1\right] \tag{2.13}
$$

since

$$
\max\{a,b\} = \frac{1}{2}(a+b+|b-a|).
$$

If a, $b \in [m,M]$, then $\frac{b}{a} \in \left[\frac{m}{M}, \frac{M}{m}\right]$ and by the properties of Kantorovich's constant K, we have

$$
1 \le K\left(\frac{b}{a}\right) \le K\left(\frac{M}{m}\right) = K(h) \text{ for any } a, b \in [m,M].
$$

Therefore, by (2.13) we have

$$
(1-p)\,a + pb - a^{1-p}b^p \le p(1-p)(a+b+|b-a|)[K(h)-1]
$$

for any a, $b \in [m,M]$ and $p \in [0,1]$.

Fix $a \in [m,M]$ and by using the operator functional calculus for A with $mI \le A \le MI$, we have

$$
(1-p)\,aI + pA - a^{1-p}A^p \le p(1-p)[K(h)-1](aI + A + |A - aI|). \tag{2.14}
$$

Then for any $x \in H$ with $\|x\| = 1$ we get from (2.14) that

$$
\begin{aligned}
(2.15) \qquad & (1-p)\,a + p\,\langle Ax, x\rangle - a^{1-p}\,\langle A^p x, x\rangle \\
& \le p\,(1-p)\,[K\,(h)-1]\,(a + \langle Ax, x\rangle + \langle |A - aI|\,x, x\rangle)\,,
\end{aligned}
$$

for any $a \in [m, M]$ and $p \in [0, 1]$.

Now, if we take $a = \langle Ax, x\rangle \in [m, M]$, where $x \in H$ with $\|x\| = 1$ in (2.15), then we obtain

$$
\begin{aligned}
& \langle Ax, x\rangle - \langle Ax, x\rangle^{1-p}\,\langle A^p x, x\rangle \\
& \le p\,(1-p)\,[K\,(h)-1]\,(2\,\langle Ax, x\rangle + \langle |A - \langle Ax, x\rangle\, I|\,x, x\rangle)\,,
\end{aligned}
$$

which, by division with $\langle Ax, x\rangle > 0$ provides the first inequality in (2.11).

By Schwarz inequality, we have for $x \in H$ with $\|x\| = 1$ that

$$
\begin{aligned}
\langle |A - \langle Ax, x\rangle\, I|\,x, x\rangle &\le \left\langle (A - \langle Ax, x\rangle\, I)^2\, x, x\right\rangle^{1/2} \\
&= \left\langle \left(A^2 - 2\,\langle Ax, x\rangle\, A + \langle Ax, x\rangle^2\, I\right) x, x\right\rangle^{1/2} \\
&= \left(\left\langle A^2 x, x\right\rangle - \langle Ax, x\rangle^2\right)^{1/2},
\end{aligned}
$$

which proves the second part of (2.11).

Since

$$
\frac{\left\langle A^2 x, x\right\rangle}{\langle Ax, x\rangle^2} - 1 \le \frac{(M-m)^2}{4mM} = K\,(h) - 1
$$

for $x \in H$ with $\|x\| = 1$, then the last part of (2.11) is thus proved. ∎

3. A Comparison for Upper Bounds

We observe that the inequality (2.9) provides for the quantity

$$
(0 \le)\, 1 - \frac{\langle A^p x, x\rangle}{\langle Ax, x\rangle^p},\ x \in H \text{ with } \|x\| = 1,
$$

the following upper bound

$$
B_1\,(p, h) := \frac{p\,(1-p)}{2} h\,[K\,(h) - 1]\,, \tag{3.1}
$$

while the inequality (2.11) gives the upper bound

$$
B_2\,(p, h) := p\,(1-p)\,[K\,(h) - 1]\left[2 + (K\,(h) - 1)^{1/2}\right], \tag{3.2}
$$

where $p \in (0, 1)$ and $h > 1$.

Now, if we depict the 3D plot for the difference of the bounds B_1 and B_2, namely

$$D(x,y) := B_1(y,x) - B_2(y,x)$$

on the box $[1,8] \times [0,1]$, then we observe that it takes both positive and negative values, showing that the bounds $B_1(p,h)$ and $B_2(p,h)$ can not be compared in general, namely neither of them is better for any $p \in (0,1)$ and $h > 1$.

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