Two points Taylor's type representations for analytic complex functions with integral remainders

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Two Points Taylor's Type Representations for Analytic Complex Functions with Integral Remainders

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Abstract

In this paper we establish some two point weighted Taylor's expansions for analytic functions $f : D \subseteq \mathbb{C} \to \mathbb{C}$ defined on a convex domain D. Some error bounds for these expansions are also provided. Examples for the complex logarithm and the complex exponential are also given.

1 Introduction

Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the convex domain D and z, $v \in D$, then we have the following Taylor expansion with integral remainder

$$f(z) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(v)(z-v)^k + \frac{1}{n!}(z-v)^{n+1} \int_0^1 f^{(n+1)}[(1-s)v + sz](1-s)^n ds \quad (1)$$

for $n \geq 0$, see for instance [13].

In this paper we establish some two point weighted Taylor expansions for analytic functions $f : D \subseteq \mathbb{C} \to \mathbb{C}$ defined on a convex domain D. Some error bounds for these expansions are also provided. Examples for the complex logarithm and the complex exponential are given as well.

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Consider the function $f(z) = \mathrm{Log}(z)$ where $\mathrm{Log}(z) = \ln|z| + i\,\mathrm{Arg}(z)$ and $\mathrm{Arg}(z)$ is such that $-\pi < \mathrm{Arg}(z) \le \pi$. Log is called the *"principal branch"* of the complex logarithmic function. The function f is analytic on all of $\mathbb{C}_\ell := \mathbb{C}\backslash\{x+iy : x \le 0,\ y = 0\}$ and

$$f^{(k)}(z) = \frac{(-1)^{k-1}(k-1)!}{z^k},\ k \ge 1,\ z \in \mathbb{C}_\ell.$$

Using the representation (1) we then have

$$\mathrm{Log}(z) = \mathrm{Log}(v) + \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}\left(\frac{z-v}{v}\right)^k + (-1)^n (z-v)^{n+1} \int_0^1 \frac{(1-s)^n\, ds}{[(1-s)v + sz]^{n+1}} \tag{2}$$

for all $z, v \in \mathbb{C}_\ell$ with $(1-s)v + sz \in \mathbb{C}_\ell$ for $s \in [0,1]$.

Consider the complex exponential function $f(z) = \exp(z)$, then by (1) we get

$$\exp(z) = \sum_{k=0}^{n} \frac{1}{k!}(z-v)^k \exp(v) + \frac{1}{n!}(z-v)^{n+1} \int_0^1 (1-s)^n \exp[(1-s)v + sz]\, ds \tag{3}$$

for all $z, v \in \mathbb{C}$.

For various inequalities related to Taylor expansions for real functions see [1]-[12].

2 Two Points Taylor Expansions

We have the following two points Taylor expansion with integral remainder:

Theorem 1. *Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the convex domain D and $z, v, w \in D$, then for all $\lambda \in \mathbb{C}$ we have*

$$\begin{aligned} f(z) &= (1-\lambda) f(v) + \lambda f(w) \\ &+ \sum_{k=1}^{n} \frac{1}{k!}\left[(1-\lambda) f^{(k)}(v)(z-v)^k + (-1)^k \lambda f^{(k)}(w)(w-z)^k\right] \\ &+ S_{n,\lambda}(z,v,w), \end{aligned} \tag{4}$$

where the remainder $S_{n,\lambda}(z,v,w)$ *is given by*

$$\begin{aligned} & S_{n,\lambda}(z,v,w) \\ & := \frac{1}{n!}\left[(1-\lambda)(z-v)^{n+1}\int_0^1 f^{(n+1)}[(1-s)v+sz](1-s)^n\,ds \right. \\ & \left. +(-1)^{n+1}\lambda(w-z)^{n+1}\int_0^1 f^{(n+1)}[(1-s)z+sw]s^n ds\right]. \end{aligned} \tag{5}$$

Proof. If we replace in (1) v by w, then we get

$$\begin{aligned} f(z) &= \sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(w)(z-w)^k \\ & \quad + \frac{1}{n!}(z-w)^{n+1}\int_0^1 f^{(n+1)}[(1-s)w+sz](1-s)^n\,ds \end{aligned} \tag{6}$$

$$\begin{aligned} &= \sum_{k=0}^{n}\frac{(-1)^k}{k!}f^{(k)}(w)(w-z)^k \\ & \quad + \frac{(-1)^{n+1}}{n!}(w-z)^{n+1}\int_0^1 f^{(n+1)}[(1-s)w+sz](1-s)^n\,ds \\ &= \sum_{k=0}^{n}\frac{(-1)^k}{k!}f^{(k)}(w)(w-z)^k \\ & \quad + \frac{(-1)^{n+1}}{n!}(w-z)^{n+1}\int_0^1 f^{(n+1)}[(1-s)z+sw]s^n ds. \end{aligned}$$

Assume that $\lambda \neq 1, 0$. If we multiply (1) by $1-\lambda$ and (6) by λ we get the desired representation (4) with the remainder $S_{n,\lambda}(z,v,w)$ given by (5).

If either $\lambda = 1$ or $\lambda = 0$, then the theorem also holds by the use of Taylor usual expansion. □

Remark 1. *We observe that for* $n = 0$ *the representation from Theorem 1 becomes*

$$f(z) = (1-\lambda)f(v) + \lambda f(w) + S_\lambda(z,v,w), \tag{7}$$

where the remainder $S_\lambda(z,v,w)$ *is given by*

$$\begin{aligned} S_\lambda(z,v,w) &:= (1-\lambda)(z-v)\int_0^1 f'((1-s)v+sz)\,ds \\ & \quad - \lambda(w-z)\int_0^1 f'((1-s)z+sw)\,ds. \end{aligned} \tag{8}$$

Remark 2. *If we take in (6) $z = \frac{v+w}{2}$, with $v, w \in D$, then we have for any $\lambda \in \mathbb{C}$ that*

$$f\left(\frac{v+w}{2}\right) = (1-\lambda) f(v) + \lambda f(w) + \sum_{k=1}^{n} \frac{1}{2^k k!} \left[(1-\lambda) f^{(k)}(v) + (-1)^k \lambda f^{(k)}(w)\right] (w-v)^k + \tilde{S}_{n,\lambda}(v,w), \tag{9}$$

where the remainder $\tilde{S}_{n,\lambda}(v,w)$ is given by

$$\tilde{S}_{n,\lambda}(v,w) := \frac{1}{2^{n+1} n!} (w-v)^{n+1} \left[(1-\lambda) \int_0^1 f^{(n+1)}\left((1-s) v + s\frac{v+w}{2}\right) (1-s)^n ds + (-1)^{n+1} \lambda \int_0^1 f^{(n+1)}\left((1-s) \frac{v+w}{2} + sw\right) s^n ds\right]. \tag{10}$$

In particular, for $\lambda = \frac{1}{2}$ in (9) we have

$$f\left(\frac{v+w}{2}\right) = \frac{f(v) + f(w)}{2} + \sum_{k=1}^{n} \frac{1}{2^{k+1} k!} \left[f^{(k)}(v) + (-1)^k f^{(k)}(w)\right] (w-v)^k + \tilde{S}_n(v,w), \tag{11}$$

where the remainder $\tilde{S}_n(v,w)$ is given by

$$\tilde{S}_n(v,w) := \frac{1}{2^{n+2} n!} (w-v)^{n+1} \left[\int_0^1 f^{(n+1)}\left((1-s) v + s\frac{v+w}{2}\right) (1-s)^n ds + (-1)^{n+1} \int_0^1 f^{(n+1)}\left((1-s) \frac{v+w}{2} + sw\right) s^n ds\right]. \tag{12}$$

Now, by the change of variable in (12) we also get the following representation for the remainder $\tilde{S}_n(v,w)$ as a single integral

$$\tilde{S}_n(v,w) := \frac{1}{2^{n+2} n!} (w-v)^{n+1} \times \int_0^1 \left[f^{(n+1)}\left(sv + (1-s) \frac{v+w}{2}\right) + (-1)^{n+1} f^{(n+1)}\left((1-s) \frac{v+w}{2} + sw\right)\right] s^n ds, \tag{13}$$

for $n \geq 0$.

Corollary 2. *With the assumptions in Theorem 1 we have for each distinct* $z,\ v,\ w \in D$ *with* $w \neq v$

$$f(z) = \frac{1}{w-v}\left[(w-z)f(v)+(z-v)f(w)\right]+\frac{(w-z)(z-v)}{w-v} \times \sum_{k=1}^{n}\frac{1}{k!}\left\{(z-v)^{k-1}f^{(k)}(v)+(-1)^{k}(w-z)^{k-1}f^{(k)}(w)\right\} + L_n(z,v,w), \tag{14}$$

where

$$L_n(z,v,w) := \frac{(w-z)(z-v)}{n!(w-v)}\left[(z-v)^{n}\int_0^1 f^{(n+1)}((1-s)v+sz)(1-s)^{n}ds + (-1)^{n+1}(w-z)^{n}\int_0^1 f^{(n+1)}((1-s)z+sw)s^{n}ds\right]$$

and

$$f(z) = \frac{1}{w-v}\left[(z-v)f(v)+(w-z)f(w)\right] + \frac{1}{w-v}\sum_{k=1}^{n}\frac{1}{k!}\left\{(z-v)^{k+1}f^{(k)}(v)+(-1)^{k}(w-z)^{k+1}f^{(k)}(w)\right\} + P_n(z,v,w), \tag{15}$$

where

$$P_n(z,v,w) := \frac{1}{n!(w-v)}\left[(z-v)^{n+2}\int_0^1 f^{(n+1)}((1-s)v+sz)(1-s)^{n}ds + (-1)^{n+1}(w-z)^{n+2}\int_0^1 f^{(n+1)}((1-s)z+sw)s^{n}ds\right],$$

respectively.

The proof is obvious, by choosing $\lambda = (z-v)/(w-v)$ and $\lambda = (w-z)/(w-v)$, respectively, in Theorem 1. The details are omitted.

Corollary 3. *With the assumption in Theorem 1 we have for each* $\lambda \in [0,1]$ *and any distinct* $v,\ w \in D$ *that*

$$f((1-\lambda)v+\lambda w) = (1-\lambda)f(v)+\lambda f(w)+\lambda(1-\lambda) \times \sum_{k=1}^{n}\frac{1}{k!}\left[\lambda^{k-1}f^{(k)}(v)+(-1)^{k}(1-\lambda)^{k-1}f^{(k)}(w)\right](w-v)^{k}+S_{n,\lambda}(v,w), \tag{16}$$

where the remainder $S_{n,\lambda}(v,w)$ *is given by*

$$\begin{aligned} & S_{n,\lambda}(v,w) \\ & := \frac{1}{n!}(1-\lambda)\lambda(w-v)^{n+1}\left[\lambda^n \int_0^1 f^{(n+1)}((1-s\lambda)v+s\lambda w)(1-s)^n ds\right. \\ & \left. +(-1)^{n+1}(1-\lambda)^n \int_0^1 f^{(n+1)}((1-s-\lambda+s\lambda)v+(\lambda+s-s\lambda)w)s^n ds\right]. \end{aligned} \tag{17}$$

We also have

$$\begin{aligned} & f((1-\lambda)w+\lambda v) = (1-\lambda)f(v)+\lambda f(w) \\ & +\sum_{k=1}^{n}\frac{1}{k!}\left[(1-\lambda)^{k+1}f^{(k)}(v)+(-1)^k\lambda^{k+1}f^{(k)}(w)\right](w-v)^k + P_{n,\lambda}(v,w), \end{aligned} \tag{18}$$

where the remainder $P_{n,\lambda}(v,w)$ *is given by*

$$\begin{aligned} & P_{n,\lambda}(v,w) \\ & := \frac{1}{n!}(w-v)^{n+1}\left[(1-\lambda)^{n+2}\int_0^1 f^{(n+1)}((1-s+\lambda s)v+(1-\lambda)sw)(1-s)^n ds\right. \\ & \left. +(-1)^{n+1}\lambda^{n+2}\int_0^1 f^{(n+1)}((1-s)\lambda v+(1-\lambda+\lambda s)w)s^n ds\right]. \end{aligned} \tag{19}$$

The case $n=0$ produces the following simple identities for each distinct $z,\ v,\ w \in D$ and $\lambda \in \mathbb{C}$

$$f(z) = (1-\lambda)f(v)+\lambda f(w)+S_\lambda(z,v,w), \tag{20}$$

where the remainder $S_\lambda(z,v,w)$ is given by (8).

We then have for each distinct $z,\ v,\ w \in D$

$$f(z) = \frac{1}{w-v}\left[(w-z)f(v)+(z-v)f(w)\right]+L(z,v,w), \tag{21}$$

where

$$\begin{aligned} & L(z,v,w) \\ & := \frac{(w-z)(z-v)}{w-v}\left[\int_0^1 f'((1-s)v+sz)ds-\int_0^1 f'((1-s)z+sw)ds\right] \end{aligned} \tag{22}$$

and

$$f(z)=\frac{1}{w-v}\left[(z-v)f(v)+(w-z)f(w)\right]+P(z,v,w), \tag{23}$$

where

$$P(z,v,w) := \frac{1}{w-v}\left[(z-v)^2\int_0^1 f'((1-s)v+sz)\,ds-(w-z)^2\int_0^1 f'((1-s)z+sw)\,ds\right]. \tag{24}$$

We also have for $\lambda \in [0,1]$

$$f((1-\lambda)v+\lambda w)=(1-\lambda)f(v)+\lambda f(w)+S_\lambda(v,w), \tag{25}$$

where the remainder $S_\lambda(v,w)$ is given by

$$S_\lambda(v,w) := (1-\lambda)\lambda(w-v)\left[\int_0^1 f'((1-s\lambda)v+s\lambda w)\,ds - \int_0^1 f'((1-s-\lambda+s\lambda)v+(\lambda+s-s\lambda)w)\,ds\right] \tag{26}$$

and

$$f((1-\lambda)w+\lambda v)=(1-\lambda)f(v)+\lambda f(w)+P_\lambda(v,w), \tag{27}$$

where the remainder $P_\lambda(v,w)$ is given by

$$P_\lambda(v,w) := (w-v)\left[(1-\lambda)^2\int_0^1 f'((1-s+\lambda s)v+(1-\lambda)sw)\,ds - \lambda^2\int_0^1 f'((1-s)\lambda v+(1-\lambda+\lambda s)w)\,ds\right]. \tag{28}$$

Moreover, if we take in (20) $z=\frac{v+w}{2}$ for each distinct $v, w \in D$ and $\lambda \in \mathbb{C}$, then we have

$$f\left(\frac{v+w}{2}\right)=(1-\lambda)f(v)+\lambda f(w)+S_\lambda(v,w), \tag{29}$$

where the remainder $S_\lambda(v,w)$ is given by

$$S_\lambda(v,w) := \frac{1}{2}(w-v) \times\left[(1-\lambda)\int_0^1 f'\left((1-s)v+s\frac{v+w}{2}\right)ds-\lambda\int_0^1 f'\left((1-s)\frac{v+w}{2}+sw\right)ds\right]. \tag{30}$$

In particular, for $\lambda = \frac{1}{2}$ we have

$$f\left(\frac{v+w}{2}\right) = \frac{f(v)+f(w)}{2} + S(v,w), \tag{31}$$

where

$$S(v,w) := \frac{1}{4}(w-v) \times \left[\int_0^1 f'\left((1-s)v + s\frac{v+w}{2}\right)ds - \int_0^1 f'\left((1-s)\frac{v+w}{2} + sw\right)ds\right]. \tag{32}$$

Now, assume that $z,\ v,\ w \in D \subset \mathbb{C}_\ell$, with D a convex set, then for all $\lambda \in \mathbb{C}$ we have by Theorem 1 for the function $f(z) = \operatorname{Log}(z)$ that

$$\begin{aligned} \operatorname{Log}(z) &= (1-\lambda)\operatorname{Log}(v) + \lambda \operatorname{Log}(w) \\ &\quad + \sum_{k=1}^{n} \frac{1}{k}\left[(1-\lambda)(-1)^{k-1}\frac{(z-v)^k}{v^k} - \lambda\frac{(w-z)^k}{w^k}\right] \\ &\quad + S_{n,\lambda}(z,v,w), \end{aligned} \tag{33}$$

where the remainder $\Lambda_{n,\lambda}(z,v,w)$ is given by

$$\begin{aligned} \Lambda_{n,\lambda}(z,v,w) &:= (1-\lambda)(z-v)^{n+1}(-1)^n \int_0^1 \frac{(1-s)^n}{((1-s)v+sz)^{n+1}}ds \\ &\quad - \lambda(w-z)^{n+1}\int_0^1 \frac{s^n}{((1-s)z+sw)^{n+1}}ds \end{aligned} \tag{34}$$

for $n \geq 0$.

Consider the function $f : \mathbb{C} \longrightarrow \mathbb{C}$, $f(z) = \exp z$, then for $z,\ v,\ w \in \mathbb{C}$ we have by Theorem 1 that

$$\begin{aligned} \exp z &= (1-\lambda)\exp v + \lambda \exp w \\ &\quad + \sum_{k=1}^{n} \frac{1}{k!}\left[(1-\lambda)(z-v)^k \exp v + (-1)^k \lambda (w-z)^k \exp w\right] \\ &\quad + \Theta_{n,\lambda}(z,v,w), \end{aligned} \tag{35}$$

where the remainder $\Theta_{n,\lambda}(z,v,w)$ is given by

$$\begin{aligned} &\Theta_{n,\lambda}(z,v,w) \\ &:= \frac{1}{n!}\left[(1-\lambda)(z-v)^{n+1}\int_0^1 \exp((1-s)v+sz)(1-s)^n ds \right. \\ &\left. + (-1)^{n+1}\lambda(w-z)^{n+1}\int_0^1 \exp((1-s)z+sw)s^n ds\right]. \end{aligned} \tag{36}$$

for $n \geq 0$ and for all $\lambda \in \mathbb{C}$.

3 Some Inequalities

We can state now some results concerning error bounds in approximating an analytic function by two points Taylor expansion:

Theorem 4. *Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the convex domain D and z, v, $w \in D$, then for all $\lambda \in \mathbb{C}$ we have*

$$\begin{aligned} f(z) &= (1-\lambda) f(v) + \lambda f(w) \\ &+ \sum_{k=1}^{n} \frac{1}{k!} \left[(1-\lambda) f^{(k)}(v)(z-v)^k + (-1)^k \lambda f^{(k)}(w)(w-z)^k \right] \\ &+ S_{n,\lambda}(z,v,w), \end{aligned} \tag{37}$$

and the remainder $S_{n,\lambda}(z,v,w)$ satisfies the inequalities

$$\begin{aligned} &|S_{n,\lambda}(z,v,w)| \\ &\quad \leq \frac{1}{n!} |1-\lambda| |z-v|^{n+1} \int_0^1 \left| f^{(n+1)}((1-s)v+sz) \right| (1-s)^n ds \\ &\qquad + \frac{1}{n!} |\lambda| |w-z|^{n+1} \int_0^1 \left| f^{(n+1)}((1-s)z+sw) \right| s^n ds \\ &\leq \frac{1}{n!} |1-\lambda| |z-v|^{n+1} \begin{cases} \frac{1}{n+1} \sup_{s\in[0,1]} \left| f^{(n+1)}((1-s)v+sz) \right| \\ \\ \frac{1}{(qn+1)^{1/q}} \left(\int_0^1 \left| f^{(n+1)}((1-s)v+sz) \right|^p ds \right)^{1/p} \\ \textit{where } p, q > 1 \textit{ with } \frac{1}{p} + \frac{1}{q} = 1 \\ \\ \int_0^1 \left| f^{(n+1)}((1-s)v+sz) \right| ds \end{cases} \\ &+ \frac{1}{n!} |\lambda| |w-z|^{n+1} \begin{cases} \frac{1}{n+1} \sup_{s\in[0,1]} \left| f^{(n+1)}((1-s)z+sw) \right| \\ \\ \frac{1}{(qn+1)^{1/q}} \left(\int_0^1 \left| f^{(n+1)}((1-s)z+sw) \right|^p ds \right)^{1/p} \\ \textit{where } p, q > 1 \textit{ with } \frac{1}{p} + \frac{1}{q} = 1 \\ \\ \int_0^1 \left| f^{(n+1)}((1-s)z+sw) \right| ds \end{cases} \end{aligned} \tag{38}$$

for $n \geq 0$.

Proof. Taking the modulus in the representation (5), we get

$$
\begin{aligned}
&\left|S_{n,\lambda}(z,v,w)\right| \qquad (39)\\
&\le \frac{1}{n!}\left[\left|(1-\lambda)(z-v)^{n+1}\int_0^1 f^{(n+1)}((1-s)v+sz)(1-s)^n\,ds\right|\right.\\
&\left.+\left|\lambda(w-z)^{n+1}\int_0^1 f^{(n+1)}((1-s)z+sw)s^n ds\right|\right]\\
&\le |1-\lambda|\,|z-v|^{n+1}\int_0^1\left|f^{(n+1)}((1-s)v+sz)\right|(1-s)^n\,ds\\
&+|\lambda|\,|w-z|^{n+1}\int_0^1\left|f^{(n+1)}((1-s)z+sw)\right|s^n ds.
\end{aligned}
$$

By Hölder's integral inequality we have

$$
\int_0^1\left|f^{(n+1)}((1-s)v+sz)\right|(1-s)^n\,ds
$$

$$
\le\begin{cases}
\sup_{s\in[0,1]}\left|f^{(n+1)}((1-s)v+sz)\right|\int_0^1(1-s)^n\,ds\\[2ex]
\left(\int_0^1\left|f^{(n+1)}((1-s)v+sz)\right|^p ds\right)^{1/p}\left(\int_0^1(1-s)^{qn}\,ds\right)^{1/q}\\
\text{where } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1\\[2ex]
\sup_{s\in[0,1]}\left\{(1-s)^n\right\}\int_0^1\left|f^{(n+1)}((1-s)v+sz)\right|ds
\end{cases}
$$

$$
=\begin{cases}
\frac{1}{n+1}\sup_{s\in[0,1]}\left|f^{(n+1)}((1-s)v+sz)\right|\\[2ex]
\frac{1}{(qn+1)^{1/q}}\left(\int_0^1\left|f^{(n+1)}((1-s)v+sz)\right|^p ds\right)^{1/p}\\
\text{where } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1\\[2ex]
\int_0^1\left|f^{(n+1)}((1-s)v+sz)\right|ds
\end{cases}
$$

and

$$\int_0^1 \left| f^{(n+1)}\left((1-s)z+sw\right) \right| s^n ds$$
$$\leq \begin{cases} \sup_{s\in[0,1]} \left| f^{(n+1)}\left((1-s)z+sw\right) \right| \int_0^1 s^n ds \\ \\ \left(\int_0^1 \left| f^{(n+1)}\left((1-s)z+sw\right) \right|^p ds\right)^{1/p} \left(\int_0^1 s^{qn} ds\right)^{1/q} \\ \text{where } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1 \\ \\ \sup_{s\in[0,1]} \{s^n\} \int_0^1 \left| f^{(n+1)}\left((1-s)z+sw\right) \right| ds \end{cases}$$
$$= \begin{cases} \frac{1}{n+1} \sup_{s\in[0,1]} \left| f^{(n+1)}\left((1-s)z+sw\right) \right| \\ \\ \frac{1}{(qn+1)^{1/q}} \left(\int_0^1 \left| f^{(n+1)}\left((1-s)z+sw\right) \right|^p ds\right)^{1/p} \\ \text{where } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1 \\ \\ \int_0^1 \left| f^{(n+1)}\left((1-s)z+sw\right) \right| ds, \end{cases}$$

which proves the second inequality in (38). □

Corollary 5. *With the assumptions of Theorem 4 and if*

$$\left\| f^{(n+1)} \right\|_{D,\infty} := \sup_{y\in D} \left| f^{(n+1)}(y) \right| < \infty,$$

then we have the simple bound

$$|S_{n,\lambda}(z,v,w)| \leq \frac{1}{(n+1)!} \left\| f^{(n+1)} \right\|_{D,\infty} \left(|1-\lambda| \, |z-v|^{n+1} + |\lambda| \, |w-z|^{n+1} \right) \tag{40}$$

for $n \geq 0$.

Remark 3. *If we take $z = \frac{v+w}{2}$, with v, $w \in D$, then we have for any $\lambda \in \mathbb{C}$ that*

$$f\left(\frac{v+w}{2}\right) = (1-\lambda) f(v) + \lambda f(w) \tag{41}$$
$$+ \sum_{k=1}^{n} \frac{1}{2^k k!} \left[(1-\lambda) f^{(k)}(v) + (-1)^k \lambda f^{(k)}(w) \right] (w-v)^k$$
$$+ \tilde{S}_{n,\lambda}(v,w),$$

and if $\left\|f^{(n+1)}\right\|_{D,\infty} := \sup_{y\in D}\left|f^{(n+1)}(y)\right| < \infty$, *then by (40) we get*

$$\left|\tilde{S}_{n,\lambda}(v,w)\right| \leq \frac{1}{2^{n+1}(n+1)!}\left\|f^{(n+1)}\right\|_{D,\infty}(|1-\lambda|+|\lambda|)|w-v|^{n+1}$$

or any $\lambda \in \mathbb{C}$.

In particular, if $\lambda = \frac{1}{2}$, *then we have*

$$\begin{aligned} f\left(\frac{v+w}{2}\right) &= \frac{f(v)+f(w)}{2} \\ &+ \sum_{k=1}^{n}\frac{1}{2^{k+1}k!}\left[f^{(k)}(v)+(-1)^k f^{(k)}(w)\right](w-v)^k \\ &+ \tilde{S}_n(v,w), \end{aligned} \tag{42}$$

and the remainder $\tilde{S}_n(v,w)$ *satisfies the bound*

$$\left|\tilde{S}_n(v,w)\right| \leq \frac{1}{2^{n+1}(n+1)!}\left\|f^{(n+1)}\right\|_{D,\infty}|w-v|^{n+1}$$

for $n \geq 0$.

Remark 4. *The case* $n = 0$ *provides some simple inequalities as follows*

$$\begin{aligned} &|f(z)-(1-\lambda)f(v)-\lambda f(w)| \\ &\leq |1-\lambda||z-v|\int_0^1|f'((1-s)v+sz)|ds+|\lambda||w-z|\int_0^1|f'((1-s)z+sw)|ds \\ &\leq |1-\lambda||z-v|\begin{cases}\sup_{s\in[0,1]}|f'((1-s)v+sz)| \\ \left(\int_0^1|f'((1-s)v+sz)|^p ds\right)^{1/p} \\ \text{where } p>1 \\ \int_0^1|f'((1-s)v+sz)|ds\end{cases} \\ &+ |\lambda||w-z|\begin{cases}\sup_{s\in[0,1]}|f'((1-s)z+sw)| \\ \left(\int_0^1|f'((1-s)z+sw)|^p ds\right)^{1/p} \\ \text{where } p>1 \\ \int_0^1|f'((1-s)z+sw)|ds\end{cases} \end{aligned} \tag{43}$$

where $f : D \subseteq \mathbb{C} \to \mathbb{C}$ *is an analytic function on the convex domain* D, z, v, $w \in D$ *and* $\lambda \in \mathbb{C}$.

If

$$\|f'\|_{D,\infty} := \sup_{y \in D} |f'(y)| < \infty,$$

then we have the simple bound

$$|f(z) - (1-\lambda) f(v) - \lambda f(w)| \le \|f'\|_{D,\infty} (|1-\lambda| |z-v| + |\lambda| |w-z|) \quad (44)$$

for z, v, $w \in D$ *and* $\lambda \in \mathbb{C}$.

If we take $z = \frac{v+w}{2}$, *with* v, $w \in D$, *then we have for any* $\lambda \in \mathbb{C}$ *that*

$$\left| f\left(\frac{v+w}{2}\right) - (1-\lambda) f(v) - \lambda f(w) \right| \le \frac{1}{2} \|f'\|_{D,\infty} (|1-\lambda| + |\lambda|) |w-v|, \quad (45)$$

which for $\lambda = \frac{1}{2}$ *gives the simple inequality*

$$\left| f\left(\frac{v+w}{2}\right) - \frac{f(v)+f(w)}{2} \right| \le \frac{1}{2} \|f'\|_{D,\infty} |w-v|. \quad (46)$$

If n is even, namely $n = 2m$, $m \ge 0$, then by (11) we have the representation

$$f\left(\frac{v+w}{2}\right) = \frac{f(v)+f(w)}{2} + \sum_{k=1}^{2m} \frac{1}{2^{k+1} k!} \left[f^{(k)}(v) + (-1)^k f^{(k)}(w)\right] (w-v)^k + \tilde{S}_{2m}(v,w), \quad (47)$$

where the remainder $\tilde{S}_{2m}(v,w)$ is given by (13) as

$$\tilde{S}_{2m}(v,w) := \frac{1}{2^{2m+2}(2m)!} (w-v)^{2m+1} \times \int_0^1 \left[f^{(2m+1)}\left(sv + (1-s)\frac{v+w}{2}\right) - f^{(2m+1)}\left((1-s)\frac{v+w}{2} + sw\right)\right] s^{2m} ds, \quad (48)$$

We also have the following result:

Theorem 6. *Let* $f : D \subseteq \mathbb{C} \to \mathbb{C}$ *be an analytic function on the convex domain* D *and* v, $w \in D$. *Let* $m \ge 0$ *and assume that*

$$\left|f^{(2m+1)}(z) - f^{(2m+1)}(y)\right| \le L_{2m+1} |z-y| \text{ for all } z,\ y \in D \quad (49)$$

for some $L_{2m+1} > 0$, namely that $f^{(2m+1)}$ is Lipschitzian on D. Then we have the representation (47) and the remainder $\tilde{S}_{2m}(v,w)$ satisfies the bound

$$\left|\tilde{S}_{2m}(v,w)\right| \leq \frac{1}{2^{2m+2}(2m+2)(2m)!}|w-v|^{2m+2} L_{2m+1} \tag{50}$$

Proof. By taking the modulus in (48), we have

$$\begin{aligned}
&\left|\tilde{S}_{2m}(v,w)\right| \leq \frac{1}{2^{2m+2}(2m)!}|w-v|^{2m+1} \\
&\times \int_0^1 \left|f^{(2m+1)}\left(sv+(1-s)\frac{v+w}{2}\right) - f^{(2m+1)}\left((1-s)\frac{v+w}{2}+sw\right)\right| s^{2m}ds \\
&\qquad \leq \frac{1}{2^{2m+2}(2m)!}|w-v|^{2m+1} \\
&\qquad \times L_{2m+1}\int_0^1 \left|sv+(1-s)\frac{v+w}{2}-(1-s)\frac{v+w}{2}-sw\right| s^{2m}ds \\
&\qquad = \frac{1}{2^{2m+2}(2m)!}|w-v|^{2m+2} L_{2m+1}\int_0^1 s^{2m+1}ds \\
&\qquad = \frac{1}{2^{2m+2}(2m+2)(2m)!}|w-v|^{2m+2} L_{2m+1},
\end{aligned}$$

which proves the desired result (50). □

Corollary 7. *Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the convex domain D and v, $w \in D$. Assume that*

$$|f'(z) - f'(y)| \leq L|z-y| \text{ for all } z,\ y \in D \tag{51}$$

for some $L > 0$. Then we have the inequality

$$\left|f\left(\frac{v+w}{2}\right) - \frac{f(v)+f(w)}{2}\right| \leq \frac{1}{8}L|w-v|^2. \tag{52}$$

4 Inequalities for Convex Derivatives in Absolute Value

We have:

Theorem 8. *Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be an analytic function on the convex domain D and such that for a given $n \geq 0$, $\left|f^{(n+1)}\right|$ is convex on D. If z, v,*

$w \in D$, then for all $\lambda \in \mathbb{C}$ we have

$$\begin{aligned} f(z) &= (1-\lambda) f(v) + \lambda f(w) \\ &+ \sum_{k=1}^{n} \frac{1}{k!} \left[(1-\lambda) f^{(k)}(v)(z-v)^k + (-1)^k \lambda f^{(k)}(w)(w-z)^k \right] \\ &+ S_{n,\lambda}(z,v,w), \end{aligned} \tag{53}$$

and the remainder $S_{n,\lambda}(z,v,w)$ satisfies the inequality

$$\begin{aligned} |S_{n,\lambda}(z,v,w)| &\le \frac{1}{n!(n+2)} \left[|1-\lambda| |z-v|^{n+1} \left| f^{(n+1)}(v) \right| \right. \\ &+ \frac{1}{(n+1)} \left[|1-\lambda| |z-v|^{n+1} + |\lambda| |w-z|^{n+1} \right] \left| f^{(n+1)}(z) \right| \\ &\left. + |\lambda| |w-z|^{n+1} \left| f^{(n+1)}(w) \right| \right]. \end{aligned} \tag{54}$$

Proof. Using the representation (5), we get

$$\begin{aligned} & |S_{n,\lambda}(z,v,w)| \\ & \le \frac{1}{n!} \left[|1-\lambda| |z-v|^{n+1} \left| \int_0^1 f^{(n+1)}((1-s)v + sz)(1-s)^n ds \right| \right. \\ & \left. + |\lambda| |w-z|^{n+1} \left| \int_0^1 f^{(n+1)}((1-s)z + sw) s^n ds \right| \right] \\ & \le \frac{1}{n!} \left[|1-\lambda| |z-v|^{n+1} \int_0^1 \left| f^{(n+1)}((1-s)v + sz) \right| (1-s)^n ds \right. \\ & \left. + |\lambda| |w-z|^{n+1} \int_0^1 \left| f^{(n+1)}((1-s)z + sw) \right| s^n ds \right] \\ & =: A_n(\lambda, w) \end{aligned} \tag{55}$$

By the convexity of $\left|f^{(n+1)}\right|$ we have

$$\begin{aligned}
&\int_0^1 \left|f^{(n+1)}\left((1-s)v+sz\right)\right|(1-s)^n\,ds \\
&\leq \int_0^1 \left[(1-s)\left|f^{(n+1)}(v)\right|+s\left|f^{(n+1)}(z)\right|\right](1-s)^n\,ds \\
&= \left|f^{(n+1)}(v)\right|\int_0^1 (1-s)^{n+1}\,ds+\left|f^{(n+1)}(z)\right|\int_0^1 s(1-s)^n\,ds \\
&= \frac{1}{n+2}\left|f^{(n+1)}(v)\right|+\left|f^{(n+1)}(z)\right|\int_0^1 (1-s)s^n ds \\
&= \frac{1}{n+2}\left|f^{(n+1)}(v)\right|+\left|f^{(n+1)}(z)\right|\left(\frac{1}{n+1}-\frac{1}{n+2}\right) \\
&= \frac{1}{n+2}\left|f^{(n+1)}(v)\right|+\frac{1}{(n+1)(n+2)}\left|f^{(n+1)}(z)\right|
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \left|f^{(n+1)}\left((1-s)z+sw\right)\right|s^n ds \\
&\leq \int_0^1 \left[(1-s)\left|f^{(n+1)}(z)\right|+s\left|f^{(n+1)}(w)\right|\right]s^n ds \\
&= \left|f^{(n+1)}(z)\right|\int_0^1 (1-s)s^n ds+\left|f^{(n+1)}(w)\right|\int_0^1 s^{n+1}ds \\
&= \frac{1}{(n+1)(n+2)}\left|f^{(n+1)}(z)\right|+\frac{1}{n+2}\left|f^{(n+1)}(w)\right|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&A_n(\lambda,w) \\
&\leq \frac{1}{n!}\left[|1-\lambda|\,|z-v|^{n+1}\left[\frac{1}{n+2}\left|f^{(n+1)}(v)\right|+\frac{1}{(n+1)(n+2)}\left|f^{(n+1)}(z)\right|\right]\right. \\
&\left.+|\lambda|\,|w-z|^{n+1}\left[\frac{1}{(n+1)(n+2)}\left|f^{(n+1)}(z)\right|+\frac{1}{n+2}\left|f^{(n+1)}(w)\right|\right]\right]
\end{aligned}$$

$$= \frac{1}{n!\,(n+2)}\left[|1-\lambda|\,|z-v|^{n+1}\left[\left|f^{(n+1)}(v)\right| + \frac{1}{(n+1)}\left|f^{(n+1)}(z)\right|\right]\right.$$
$$\left. + |\lambda|\,|w-z|^{n+1}\left[\frac{1}{(n+1)}\left|f^{(n+1)}(z)\right| + \left|f^{(n+1)}(w)\right|\right]\right]$$
$$= \frac{1}{n!\,(n+2)}\left[|1-\lambda|\,|z-v|^{n+1}\left|f^{(n+1)}(v)\right|\right.$$
$$+ \frac{1}{(n+1)}\left[|1-\lambda|\,|z-v|^{n+1} + |\lambda|\,|w-z|^{n+1}\right]\left|f^{(n+1)}(z)\right|$$
$$\left. + |\lambda|\,|w-z|^{n+1}\left|f^{(n+1)}(w)\right|\right],$$

which together with (55) produce the desired result (55). □

Remark 5. *Assume that for a given $n \geq 0$, $\left|f^{(n+1)}\right|$ is convex on D. If we take in (53) $z = \frac{v+w}{2}$, with v, $w \in D$, then we have for any $\lambda \in \mathbb{C}$ that*

$$f\left(\frac{v+w}{2}\right) = (1-\lambda)\,f(v) + \lambda f(w) \tag{56}$$
$$+ \sum_{k=1}^{n} \frac{1}{2^k k!}\left[(1-\lambda)\,f^{(k)}(v) + (-1)^k \lambda f^{(k)}(w)\right](w-v)^k$$
$$+ \tilde{S}_{n,\lambda}(v,w),$$

where the remainder $\tilde{S}_{n,\lambda}(v,w)$ satisfies the bound

$$\left|\tilde{S}_{n,\lambda}(v,w)\right| \leq \frac{1}{2^{n+1} n!\,(n+2)}\,|w-v|^{n+1}\left[|1-\lambda|\left|f^{(n+1)}(v)\right|\right.$$
$$\left. + \frac{1}{(n+1)}\left[|1-\lambda| + |\lambda|\right]\left|f^{(n+1)}\left(\frac{v+w}{2}\right)\right| + |\lambda|\left|f^{(n+1)}(w)\right|\right]. \tag{57}$$

In particular, for $\lambda = \frac{1}{2}$ in (57) we have

$$f\left(\frac{v+w}{2}\right) = \frac{f(v) + f(w)}{2} \tag{58}$$
$$+ \sum_{k=1}^{n} \frac{1}{2^{k+1} k!}\left[f^{(k)}(v) + (-1)^k f^{(k)}(w)\right](w-v)^k$$
$$+ \tilde{S}_n(v,w),$$

where the remainder $\tilde{S}_n(v,w)$ *satisfies the bound*

$$\left|\tilde{S}_n(v,w)\right| \le \frac{1}{2^{n+1}n!(n+2)}|w-v|^{n+1}\left[\frac{1}{2}\left|f^{(n+1)}(v)\right|\right.$$
$$\left.+\frac{1}{(n+1)}\left|f^{(n+1)}\left(\frac{v+w}{2}\right)\right|+\frac{1}{2}\left|f^{(n+1)}(w)\right|\right]. \quad (59)$$

Corollary 9. *With the assumption in Theorem 8 we have for each* $\lambda \in [0,1]$ *and any distinct* v, $w \in D$ *that*

$$f((1-\lambda)v+\lambda w) = (1-\lambda)f(v)+\lambda f(w)+\lambda(1-\lambda)$$
$$\times\sum_{k=1}^{n}\frac{1}{k!}\left[\lambda^{k-1}f^{(k)}(v)+(-1)^k(1-\lambda)^{k-1}f^{(k)}(w)\right](w-v)^k+S_{n,\lambda}(v,w), \quad (60)$$

where the remainder $S_{n,\lambda}(v,w)$ *satisfies the bound*

$$|S_{n,\lambda}(v,w)| \le \frac{1}{n!(n+2)}(1-\lambda)\lambda|w-v|^{n+1}\left[\lambda^n\left|f^{(n+1)}(v)\right|\right.$$
$$\left.+\frac{1}{(n+1)}[\lambda^n+(1-\lambda)^n]\left|f^{(n+1)}((1-\lambda)v+\lambda w)\right|+(1-\lambda)^n\left|f^{(n+1)}(w)\right|\right]. \quad (61)$$

We also have

$$f((1-\lambda)w+\lambda v) = (1-\lambda)f(v)+\lambda f(w)$$
$$+\sum_{k=1}^{n}\frac{1}{k!}\left[(1-\lambda)^{k+1}f^{(k)}(v)+(-1)^k\lambda^{k+1}f^{(k)}(w)\right](w-v)^k+P_{n,\lambda}(v,w), \quad (62)$$

where the remainder $P_{n,\lambda}(v,w)$ *satisfies the bound*

$$|P_{n,\lambda}(v,w)| \le \frac{1}{n!(n+2)}|w-v|^{n+1}\left[(1-\lambda)^{n+2}\left|f^{(n+1)}(v)\right|\right.$$
$$\left.+\frac{1}{(n+1)}\left[(1-\lambda)^{n+2}+\lambda^{n+2}\right]\left|f^{(n+1)}((1-\lambda)w+\lambda v)\right|+\lambda^{n+2}\left|f^{(n+1)}(w)\right|\right]. \quad (63)$$

For $n=0$, namely if $|f'|$ is convex on D, then by (54) we get

$$|f(z)-(1-\lambda)f(v)-\lambda f(w)| \le \frac{1}{2}[|1-\lambda||z-v||f'(v)|$$
$$+[|1-\lambda||z-v|+|\lambda||w-z|]|f'(z)|+|\lambda||w-z||f'(w)|], \quad (64)$$

for $z, v, w \in D$ and for all $\lambda \in \mathbb{C}$.

From (57) we get

$$\left|f\left(\frac{v+w}{2}\right)-(1-\lambda) f(v)-\lambda f(w)\right| \leq \frac{1}{4}|w-v|\left[|1-\lambda|\left|f'(v)\right|\right.$$
$$\left.+[|1-\lambda|+|\lambda|]\left|f'\left(\frac{v+w}{2}\right)\right|+|\lambda|\left|f'(w)\right|\right] \quad (65)$$

for $v, w \in D$ and for all $\lambda \in \mathbb{C}$.

In particular, for $\lambda=\frac{1}{2}$ we get

$$\left|f\left(\frac{v+w}{2}\right)-\frac{f(v)+f(w)}{2}\right|$$
$$\leq \frac{1}{8}|w-v|\left[\left|f'(v)\right|+2\left|f'\left(\frac{v+w}{2}\right)\right|+\left|f'(w)\right|\right] \quad (66)$$

for $v, w \in D$.

5 Examples for Logarithm and Exponential

Consider the function $f(z)=\operatorname{Log}(z)$ where $\operatorname{Log}(z)=\ln|z|+i \operatorname{Arg}(z)$ and $\operatorname{Arg}(z)$ is such that $-\pi<\operatorname{Arg}(z) \leq \pi$. Log is called the *"principal branch"* of the complex logarithmic function. The function f is analytic on all of $\mathbb{C}_{\ell}:=\mathbb{C} \backslash\{x+i y: x \leq 0,\ y=0\}$ and

$$f^{(k)}(z)=\frac{(-1)^{k-1}(k-1)!}{z^{k}},\ k \geq 1,\ z \in \mathbb{C}_{\ell}.$$

Let D be a convex domain in $\mathbb{C}_{\ell}$ an assume that $d_{D}:=\inf_{z \in D}|z|$ is a positive and finite number. If $z, v, w \in D \subset \mathbb{C}_{\ell}$, then by the representation (17) and the inequality (40)

$$\left|\operatorname{Log}(z)-(1-\lambda) \operatorname{Log}(v)-\lambda \operatorname{Log}(w)\right.$$
$$\left.-\sum_{k=1}^{n} \frac{1}{k}\left[(1-\lambda)(-1)^{k-1} \frac{(z-v)^{k}}{v^{k}}-\lambda \frac{(w-z)^{k}}{w^{k}}\right]\right|$$
$$\leq \frac{1}{(n+1) d_{D}^{n+1}}\left(|1-\lambda||z-v|^{n+1}+|\lambda||w-z|^{n+1}\right) \quad (67)$$

for $n \geq 1$ and for $n=0$ we have

$$|\operatorname{Log}(z)-(1-\lambda) \operatorname{Log}(v)-\lambda \operatorname{Log}(w)| \leq \frac{1}{d_{D}}(|1-\lambda||z-v|+|\lambda||w-z|), \quad (68)$$

for all $\lambda \in \mathbb{C}$.

If $\lambda = \tau \in [0,1]$, then by (67) we get

$$\begin{aligned} &\left|\operatorname{Log}(z) - (1-\tau)\operatorname{Log}(v) - \tau \operatorname{Log}(w)\right. \\ &\qquad \left. - \sum_{k=1}^{n} \frac{1}{k}\left[(1-\tau)(-1)^{k-1}\frac{(z-v)^k}{v^k} - \tau\frac{(w-z)^k}{w^k}\right]\right| \\ &\qquad \le \frac{1}{(n+1)\, d_D^{n+1}}\left((1-\tau)|z-v|^{n+1} + \tau|w-z|^{n+1}\right) \\ &\qquad\qquad \le \frac{1}{(n+1)\, d_D^{n+1}} \max\left\{|z-v|^{n+1}, |w-z|^{n+1}\right\} \quad (69) \end{aligned}$$

and for $\tau = \frac{1}{2}$ we get

$$\begin{aligned} &\left|\operatorname{Log}(z) - \frac{\operatorname{Log}(v)+\operatorname{Log}(w)}{2}\right. \\ &\qquad \left. -\frac{1}{2}\sum_{k=1}^{n}\frac{1}{k}\left[(-1)^{k-1}\frac{(z-v)^k}{v^k} - \frac{(w-z)^k}{w^k}\right]\right| \\ &\qquad\qquad \le \frac{1}{2(n+1)\, d_D^{n+1}}\left(|z-v|^{n+1} + |w-z|^{n+1}\right) \quad (70) \end{aligned}$$

for $z, v, w \in D \subset \mathbb{C}_\ell$.

Moreover, if we take $z = \frac{v+w}{2}$ in (70), then we get

$$\begin{aligned} &\left|\operatorname{Log}\left(\frac{v+w}{2}\right) - \frac{\operatorname{Log}(v)+\operatorname{Log}(w)}{2}\right. \\ &\qquad \left. -\frac{1}{2}\sum_{k=1}^{n}\frac{1}{2^k k}\left[\frac{(-1)^{k-1}}{v^k} - \frac{1}{w^k}\right](w-v)^k\right| \\ &\qquad\qquad \le \frac{1}{2^{n+1}(n+1)\, d_D^{n+1}}|w-v|^{n+1} \quad (71) \end{aligned}$$

for $v, w \in D \subset \mathbb{C}_\ell$.

The case $n = 0$ gives that

$$\left|\operatorname{Log}\left(\frac{v+w}{2}\right) - \frac{\operatorname{Log}(v)+\operatorname{Log}(w)}{2}\right| \le \frac{1}{2d_D}|w-v| \quad (72)$$

for $v, w \in D \subset \mathbb{C}_\ell$.

For $f(z) = \operatorname{Log}(z)$, $z \in D \subset \mathbb{C}_\ell$, we have

$$|f'(z) - f'(w)| = \left|\frac{1}{z} - \frac{1}{w}\right| = \frac{|w-v|}{|z||w|} \le \frac{1}{d_D^2}|w-v|$$

showing that f' is Lipschitzian on D with the constant $L = \frac{1}{d_D^2}$.

By the inequality (52) we then get

$$\left|\operatorname{Log}\left(\frac{v+w}{2}\right) - \frac{\operatorname{Log}(v) + \operatorname{Log}(w)}{2}\right| \leq \frac{1}{8d_D^2}|w-v|^2, \tag{73}$$

for $v,\ w \in D \subset \mathbb{C}_\ell$.

Now consider the exponential function $f(z) = \exp z$. Then

$$|\exp z| = \exp(\operatorname{Re} z)$$

and

$$|\exp((1-t)z + tw)| \leq (1-t)|\exp z| + t|\exp w|$$

for any $z,\ w \in \mathbb{C}$ and $t \in [0,1]$, showing that f is convex in absolute value.

Now let D be a convex domain in $\mathbb{C}$ and assume that $E_D := \sup_{z\in D}[\exp(\operatorname{Re} z)] < \infty$. If we use the representation (35) and the inequality (40), we have

$$\begin{aligned} &\Big|\exp z - (1-\lambda)\exp v - \lambda \exp w \\ &\quad - \sum_{k=1}^{n} \frac{1}{k!}\left[(1-\lambda)(z-v)^k \exp v + (-1)^k \lambda (w-z)^k \exp w\right]\Big| \\ &\qquad \leq \frac{1}{(n+1)!} E_D \left(|1-\lambda|\,|z-v|^{n+1} + |\lambda|\,|w-z|^{n+1}\right) \end{aligned} \tag{74}$$

for all $z,\ v,\ w \in D \subset \mathbb{C}$ and $n \geq 1$.

For $n = 0$ we have the simpler inequality

$$|\exp z - (1-\lambda)\exp v - \lambda \exp w| \leq E_D\left(|1-\lambda|\,|z-v| + |\lambda|\,|w-z|\right) \tag{75}$$

for all $z,\ v,\ w \in D \subset \mathbb{C}$.

If $\lambda = \tau \in [0,1]$, then by (74) we get

$$\begin{aligned} &\Big|\exp z - (1-\tau)\exp v - \tau \exp w \\ &\quad - \sum_{k=1}^{n} \frac{1}{k!}\left[(1-\tau)(z-v)^k \exp v + (-1)^k \tau (w-z)^k \exp w\right]\Big| \\ &\qquad \leq \frac{1}{(n+1)!} E_D \left((1-\tau)\,|z-v|^{n+1} + \tau\,|w-z|^{n+1}\right) \\ &\qquad\qquad \leq \frac{1}{(n+1)!} E_D \max\left\{|z-v|^{n+1}, |w-z|^{n+1}\right\} \end{aligned} \tag{76}$$

for all z, v, $w \in D \subset \mathbb{C}$ and for $\tau = \frac{1}{2}$ we get

$$\left| \exp z - \frac{\exp v + \exp w}{2} - \frac{1}{2} \sum_{k=1}^{n} \frac{1}{k!} \left[(z-v)^k \exp v + (-1)^k (w-z)^k \exp w \right] \right| \leq \frac{1}{2(n+1)!} E_D \left(|z-v|^{n+1} + |w-z|^{n+1} \right) \quad (77)$$

for all z, v, $w \in D \subset \mathbb{C}$.

Moreover, if we take $z = \frac{v+w}{2}$ in (77), then we get

$$\left| \exp\left(\frac{v+w}{2} \right) - \frac{\exp v + \exp w}{2} - \frac{1}{2} \sum_{k=1}^{n} \frac{1}{2^k k!} \left[\exp v + (-1)^k \exp w \right] (w-v)^k \right| \leq \frac{1}{2^{n+1}(n+1)!} E_D |w-v|^{n+1} \quad (78)$$

for all v, $w \in D \subset \mathbb{C}$.

The case $n = 0$ gives that

$$\left| \exp\left(\frac{v+w}{2} \right) - \frac{\exp v + \exp w}{2} \right| \leq \frac{1}{2} E_D |w-v| \quad (79)$$

for all v, $w \in D \subset \mathbb{C}$.

The function $f(z) = \exp z$ is Lipschitzian on D with the constant $L = E_D$, then by (52) we get

$$\left| \exp\left(\frac{v+w}{2} \right) - \frac{\exp v + \exp w}{2} \right| \leq \frac{1}{8} E_D |w-v|^2 . \quad (80)$$

for all v, $w \in D \subset \mathbb{C}$.

By the convexity in modulus of the complex function and by (66) we also have

$$\left| \exp\left(\frac{v+w}{2} \right) - \frac{\exp v + \exp w}{2} \right| \leq \frac{1}{8} |w-v| \left[\exp(\operatorname{Re} v) + 2 \exp\left(\operatorname{Re}\left(\frac{v+w}{2} \right) \right) + \exp(\operatorname{Re} w) \right] \quad (81)$$

for all v, $w \in \mathbb{C}$.

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