Zeta Identities in Parameter Form

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Abstract

We develop new parameterized series representations of zeta functions.

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1. Introduction and Preliminaries

In this paper the author considers the integral

$$I(m) = \int_0^1 \frac{\log^m(1-t)\ \log t}{t(1-t)} dt = \int_0^1 \frac{\log^m t\ \log(1-t)}{t(1-t)} dt \tag{1.1}$$

for $m \in \mathbb{N}$, and develops new parameterized series identities for zeta functions. In particular, for $m = 2$, the author obtains a parameterized series representation for $\zeta(4)$ in the form

$$\zeta(4) = 4 \sum_{n=1}^{\infty} \frac{1}{(1-a)^{n+1}} \sum_{j=1}^{n} \binom{n}{j} (-a)^{n-j} \left(\frac{H_j}{j^3} - \frac{\psi^{(\prime)}(j+1)}{j^2} + \frac{\psi^{(\prime\prime)}(j+1)}{2j} \right), \tag{1.2}$$

where the parameter $a \leq \frac{1}{2}$. As usual we let

$$H_n = \sum_{r=1}^{n} \frac{1}{r} = \int_0^1 \frac{1-t^n}{1-t}\, dt = \gamma + \psi(n+1) = \sum_{j=1}^{\infty} \frac{n}{j(j+n)},\ H_0 := 0$$

be the nth harmonic number, where γ denotes the Euler-Mascheroni constant, and $\psi(z)$ is the digamma (or psi) function defined, for $z \in \mathbb{C} \backslash \{0, -1, -2, -3, ...\}$, by

$$\psi(z) := \frac{d}{dz}\{\log \Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)} \text{ and } \psi(1+z) = \psi(z) + \frac{1}{z},$$

moreover

$$\psi(z) = -\gamma + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+z} \right).$$

The polygamma function, for $k \in \mathbb{N}$

$$\psi^{(k)}(z) = \frac{d^k}{dz^k}\{\psi(z)\} = (-1)^{k+1}\, k! \sum_{r=0}^{\infty} \frac{1}{(r+z)^{k+1}}$$

and satisfies the recurrence

$$\psi^{(k)}(z+1)=\psi^{(k)}(z)+\frac{(-1)^k\,k!}{z^{k+1}}.$$

Also, $H_n^{(m)}=\sum_{r=1}^{n}\frac{1}{r^m}$ is the m^{th} order harmonic number and is intrinsically connected to the polygamma function via the relation

$$H_\rho^{(m+1)} = \zeta(m+1)+\frac{(-1)^m}{m!}\,\psi^{(m)}(\rho+1)$$

$$(\rho\in\mathbb{R}\setminus\{-1,-2,-3,\cdots\};\ m\in\mathbb{N}),$$

where $\zeta(z)$ is the Riemann zeta function. In 1735 Euler announced his celebrated result for the $\zeta(2)$ function, that is

$$\zeta(2)=\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}. \tag{1.3}$$

He went further and reported the generalized result

$$\zeta(2k)=\sum_{n=1}^{\infty}\frac{1}{n^{2k}}=\frac{(-1)^{k+1}\,2^{2k-1}B_{2k}}{(2k)!}\pi^{2k}$$

where B_j is the j^{th} Bernoulli number and k is a positive integer. In particular for $k=2$ we have

$$\zeta(4)=\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}.$$

There are many representations of $\zeta(4)$, both in integral form and in series representation, some notable identities, that may be seen in [4], [9], [16], are:

$$\begin{aligned}\zeta(4) &= \frac{4}{5}\sum_{n=1}^{\infty}\frac{H_n}{n^3}=\frac{4}{17}\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^2=\frac{4}{7}\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^2}=\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n(n+1)}\\ &= \frac{8}{7}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(2n+1)H_n^{(3)}}{n(n+1)}=\frac{1}{3}\sum_{n=1}^{\infty}\frac{H_nH_{n+1}}{(n+1)^2}=2\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{ij(i+j)^2}\\ &= \frac{7}{40}\sum_{n=1}^{\infty}\left(\frac{H_{2n-1}}{n}\right)^2=\frac{36}{17}\sum_{n=1}^{\infty}\frac{1}{n^4\binom{2n}{n}},\\ &= 2\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{ijk(i+j+k)}.\end{aligned}$$

Also, from [4], [9], [16],

$$\begin{aligned}\zeta(4) &= -\frac{1}{6}\int_0^1 \frac{\log^3(1-x)\,dx}{x} = -\frac{8}{3}\int_0^1 \frac{\log(1-x)\log^2(1+x)\,dx}{x} \\ &= \frac{1}{6}\int_0^1\int_0^1 \frac{\log^2(xy)\,dxdy}{1-xy} = \int_0^1 \frac{\mathrm{Li}_3(x)}{x}dx,\end{aligned}$$

where the Polylogarithm, or de Jonquière's function,

$$\mathrm{Li}_t(z) := \sum_{m=1}^{\infty} \frac{z^m}{m^t},\ t \in \mathbb{C} \text{ when } |z| < 1;\ \Re(t) > 1 \text{ when } |z| = 1.$$

The representation for general even zeta constants, see [1] and [10], is also given as,

$$\zeta(2n) = \frac{(-1)^{n+1}\, 2^{2n-3}\pi^{2n}}{(2^{2n}-1)(2n-2)!}\int_0^1 E_{2(n-1)}(x)\,dx$$

where $E_n(x)$ is an Euler polynomial defined by

$$\frac{2e^{xt}}{e^t+1} = \sum_{n\geq 0} E_n(x)\frac{t^n}{n!}.$$

The work in this paper extends the results of Alzer and Sondow [2], in which they gave a parameterized series representation of Apery's constant, which can be obtained from the integral $I(m)$ for the case $m = 1$. The following Lemma will be useful in the development of the main Theorem.

Lemma 1. *For* $m \in \mathbb{N}\setminus\{1\}$

$$I(m) = \int_0^1 \frac{\log^m(1-t)\ \log t}{t(1-t)}dt = \int_0^1 \frac{\log^m t\ \log(1-t)}{t(1-t)}dt \tag{1.4}$$

$$= \frac{(-1)^{m+1}\, m!}{2}\left((m+3)\,\zeta(m+2) - \sum_{r=1}^{m-1}\zeta(r+1)\,\zeta(m+1-r)\right). \tag{1.5}$$

For $m = 1$,

$$I(1) = \int_0^1 \frac{\log(1-t)\ \log t}{t(1-t)}dt = \int_0^1 \frac{\log t\ \log(1-t)}{t(1-t)}dt = 2\zeta(3).$$

Proof. We have that for $-1 < t < 1$

$$\frac{\log(1-t)}{1-t} = -\sum_{r=1}^{\infty} H_r t^r \tag{1.6}$$

so that, substituting (1.6) and integrating by parts yields,

$$\begin{aligned}\int_0^1 \frac{\log^m t \ \log(1-t)}{t(1-t)} dt &= -\sum_{r=1}^{\infty} H_r \int_0^1 t^{r-1} \log^m t \ dt \\ &= (-1)^{m+1} m! \sum_{r=1}^{\infty} \frac{H_r}{r^{m+1}}.\end{aligned}$$

We know from Euler's identity [5], [6] that for $q \in \mathbb{N} \setminus \{1\}$

$$2\sum_{n=1}^{\infty} \frac{H_n}{n^q} = (q+2)\zeta(q+1) - \sum_{r=1}^{q-2} \zeta(r+1)\zeta(q-r)$$

hence (1.5) follows. For the case $m = 1$, the details of the evaluation of the integral $I(1) = 2\zeta(3)$ are clearly and sufficiently explained in Janous [8]. □

2. Main Results

We now prove the following Theorem.

Theorem 1. *Suppose that the parameter $a \le \frac{1}{2}$ then*

$$(m+1)\zeta(m+2) - \sum_{r=1}^{m-1} \zeta(r+1)\zeta(m+1-r) \tag{2.1}$$

$$= \frac{2(-1)^{m+1}}{m!} \sum_{n=1}^{\infty} \frac{1}{(1-a)^{n+1}} \sum_{j=1}^{n} \binom{n}{j} (-a)^{n-j} \delta(j,m)$$

where

$$\delta(j,m) = (-1)^{m+1} m! \left(\frac{H_j}{j^{m+1}} - \sum_{s=1}^{m} \frac{(-1)^{s+1}}{s! \ j^{m+1-s}} \psi^{(s)}(j+1) \right). \tag{2.2}$$

Proof. Let the parameter $a \le \frac{1}{2}$ and $0 < t < 1$ then

$$-1 < \frac{t-a}{1-a} < 1.$$

Expanding in a geometric series gives

$$\frac{1}{1-t}=\frac{1}{1-a}\left(\frac{1}{1-\frac{t-a}{1-a}}\right)=\frac{1}{1-a}\sum_{n=0}^{\infty}\left(\frac{t-a}{1-a}\right)^{n}$$

since

$$(t-a)^{n}=\sum_{j=0}^{n}\binom{n}{j}t^{j}(-a)^{n-j}$$

then

$$\frac{1}{1-t}=\sum_{n=0}^{\infty}\frac{1}{(1-a)^{n+1}}\sum_{j=0}^{n}\binom{n}{j}t^{j}(-a)^{n-j}.$$

From the integral

$$\begin{aligned}I(m) &= \int_0^1\frac{\log^{m}t\log(1-t)}{t(1-t)}dt\\ &= \int_0^1\log^{m}t\log(1-t)\sum_{n=0}^{\infty}\frac{1}{(1-a)^{n+1}}\sum_{j=0}^{n}\binom{n}{j}t^{j-1}(-a)^{n-j}\,dt\\ &= \sum_{n=0}^{\infty}\frac{1}{(1-a)^{n+1}}\sum_{j=0}^{n}\binom{n}{j}(-a)^{n-j}\,\delta(j,m),\end{aligned}$$

where

$$\delta(j,m)=\int_0^1 t^{j-1}\log^{m}t\log(1-t)\,dt.$$

To simplify the integral we note

$$\log(1-t)=-\sum_{r=1}^{\infty}\frac{t^{r}}{r}$$

and simple substitution leads to,

$$\delta(j,m) = -\int_0^1\sum_{r=1}^{\infty}\frac{t^{r+j-1}}{r}\log^{m}t\;dt,$$

hence, reordering and integrating by part we have,

$$\delta(j,m)=-\sum_{r=1}^{\infty}\frac{(-1)^{m}m!}{r(r+j)^{m+1}}.$$

For $j = 0$,

$$\delta(0,m) = (-1)^{m+1} m! \sum_{r=1}^{\infty} \frac{1}{r^{m+2}} = (-1)^{m+1} m! \zeta(m+2).$$

For $j \geq 1$

$$\delta(j,m) = (-1)^{m+1} m! \sum_{r=1}^{\infty} \frac{1}{r(r+j)^{m+1}},$$

we apply partial fraction decomposition so that,

$$\begin{aligned}
\delta(j,m) &= (-1)^{m+1} m! \sum_{r=1}^{\infty} \left(\frac{1}{j^{m+1}} \left(\frac{1}{r} - \frac{1}{r+j} \right) - \sum_{s=1}^{m} \frac{1}{j^{m+1-s}(j+r)^{1+s}} \right) \\
&= (-1)^{m+1} m! \left(\frac{H_j}{j^{m+1}} - \sum_{s=1}^{m} \frac{(-1)^{s+1}}{s! j^{m+1-s}} \psi^{(s)}(j+1) \right),
\end{aligned}$$

which confirms (2.2). Now we have

$$\begin{aligned}
I(m) &= \frac{(-1)^{m+1} m!}{2} \left((m+3)\zeta(m+2) - \sum_{r=1}^{m-1} \zeta(r+1)\zeta(m+1-r) \right) \\
&= \sum_{n=0}^{\infty} \frac{1}{(1-a)^{n+1}} \sum_{j=0}^{n} \binom{n}{j} (-a)^{n-j} \delta(j,m) \\
&= \frac{\delta(0,m)}{(1-a)} \sum_{n=0}^{\infty} \left(\frac{-a}{1-a} \right)^n + \sum_{n=1}^{\infty} \frac{1}{(1-a)^{n+1}} \sum_{j=1}^{n} \binom{n}{j} (-a)^{n-j} \delta(j,m) \\
&= (-1)^{m+1} m! \zeta(m+2) + \sum_{n=1}^{\infty} \frac{1}{(1-a)^{n+1}} \sum_{j=1}^{n} \binom{n}{j} (-a)^{n-j} \delta(j,m).
\end{aligned}$$

Since the integral

$$\begin{aligned}
I(m) &= \int_0^1 \frac{\log^m t \log(1-t)}{t(1-t)} dt \\
&= \frac{(-1)^{m+1} m!}{2} \left((m+3)\zeta(m+2) - \sum_{r=1}^{m-1} \zeta(r+1)\zeta(m+1-r) \right)
\end{aligned}$$

then a little simplification leads to (2.1). □

Remark 1. *The case $m = 1$, leads to the representation of Apery's $\zeta(3)$ constant, and the details of the representation have been clearly given by*

[2] *and will not be detailed here. For completeness we list the following as obtained in* [2]

$$\zeta(3)=\sum_{n=1}^{\infty}\frac{1}{(1-a)^{n+1}}\sum_{j=1}^{n}\binom{n}{j}(-a)^{n-j}\left(\frac{H_j}{j^2}-\frac{\psi^{(\prime)}(j+1)}{j}\right),$$

and when $a=-1$, *we have*

$$\zeta(3)=\sum_{n=1}^{\infty}\frac{1}{2^{n+1}}\sum_{j=1}^{n}\binom{n}{j}\left(\frac{H_j}{j^2}-\frac{\psi^{(\prime)}(j+1)}{j}\right).$$

The case $m=2$, *leads to the new representation of the* $\zeta(4)$ *constant. From (2.1) we have*

$$3\zeta(4)-\zeta^2(2)=-\sum_{n=1}^{\infty}\frac{1}{(1-a)^{n+1}}\sum_{j=1}^{n}\binom{n}{j}(-a)^{n-j}\,\delta(j,2)$$

and from (2.2)

$$\delta(j,2)=-2\left(\frac{H_j}{j^3}-\frac{\psi^{(\prime)}(j+1)}{j^2}+\frac{\psi^{(\prime\prime)}(j+1)}{2j}\right) \tag{2.3}$$

Since we know that $\zeta^2(2)=\frac{5}{2}\zeta(4)$ *then,*

$$\zeta(4) = 4\sum_{n=1}^{\infty}\frac{1}{(1-a)^{n+1}}\sum_{j=1}^{n}\binom{n}{j}(-a)^{n-j}\left(\frac{H_j}{j^3}-\frac{\psi^{(\prime)}(j+1)}{j^2}+\frac{\psi^{(\prime\prime)}(j+1)}{2j}\right), \tag{2.4}$$

which confirms (1.2). For $a=-1$

$$\zeta(4)=\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}\sum_{j=1}^{n}\binom{n}{j}\left(\frac{H_j}{j^3}-\frac{\psi^{(\prime)}(j+1)}{j^2}+\frac{\psi^{(\prime\prime)}(j+1)}{2j}\right),$$

which may be compared to the "global" Hasse [7] *identity, and rediscovered by Sondow* [15], *see also* [17]

$$\zeta(4)=\frac{4}{7}\sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{j=0}^{n}\binom{n}{j}\frac{(-1)^j}{(j+1)^4},$$

or its "cousin" identity

$$\zeta(4)=\sum_{n=1}^{\infty}\frac{1}{2^{n+1}}\sum_{j=1}^{n}\binom{n}{j}\frac{1}{j^4}.$$

For $a=-\frac{1}{2}$

$$\zeta(4)=8\sum_{n=1}^{\infty}\frac{1}{3^{n+1}}\sum_{j=1}^{n}2^j\binom{n}{j}\left(\frac{H_j}{j^3}-\frac{\psi^{(\prime)}(j+1)}{j^2}+\frac{\psi^{(\prime\prime)}(j+1)}{2j}\right), \tag{2.5}$$

and for $a = \frac{1}{4}$

$$\zeta(4) = 16 \sum_{n=1}^{\infty} \frac{(-1)^n}{3^{n+1}} \sum_{j=1}^{n} (-1)^j \, 2^{2j} \begin{pmatrix} n \\ j \end{pmatrix} \left(\frac{H_j}{j^3} - \frac{\psi^{(\prime)}(j+1)}{j^2} + \frac{\psi^{(\prime\prime)}(j+1)}{2j} \right).$$

The case $m = 3$*, leads to the new representation of the product of zeta constants. From (2.1) we have*

$$4\zeta(5) - 2\zeta(2)\zeta(3) = \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{(1-a)^{n+1}} \sum_{j=1}^{n} \begin{pmatrix} n \\ j \end{pmatrix} (-a)^{n-j} \delta(j,3)$$

and from (2.2)

$$\delta(j,3) = 6 \left(\frac{H_j}{j^4} - \frac{\psi^{(\prime)}(j+1)}{j^3} + \frac{\psi^{(\prime\prime)}(j+1)}{2j^2} - \frac{\psi^{(\prime\prime\prime)}(j+1)}{6j} \right)$$

then,

$$2\zeta(5) - \zeta(2)\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{(1-a)^{n+1}} \sum_{j=1}^{n} \begin{pmatrix} n \\ j \end{pmatrix} (-a)^{n-j} \begin{pmatrix} \frac{H_j}{j^4} - \frac{\psi^{(\prime)}(j+1)}{j^3} \\ + \frac{\psi^{(\prime\prime)}(j+1)}{2j^2} - \frac{\psi^{(\prime\prime\prime)}(j+1)}{6j} \end{pmatrix},$$

and when $a = -1$,

$$\zeta(5) = \frac{1}{2}\zeta(2)\zeta(3) + \sum_{n=1}^{\infty} \frac{1}{2^{n+2}} \sum_{j=1}^{n} \begin{pmatrix} n \\ j \end{pmatrix} \begin{pmatrix} \frac{H_j}{j^4} - \frac{\psi^{(\prime)}(j+1)}{j^3} \\ + \frac{\psi^{(\prime\prime)}(j+1)}{2j^2} - \frac{\psi^{(\prime\prime\prime)}(j+1)}{6j} \end{pmatrix}.$$

Remark 2. *If we take the representation (2.4) and multiply both sides by* $(1-a)^{b+1}$ *with* $b \in \mathbb{R}$*, we have*

$$(1-a)^{b+1} \zeta(4) = 4 \sum_{n=1}^{\infty} (1-a)^{b-n} \sum_{j=1}^{n} \begin{pmatrix} n \\ j \end{pmatrix} (-a)^{n-j} \delta(j,2),$$

where $\delta(j,2)$ *is given by (2.3). Now, differentiating with respect to a results in*

$$(b+1)\zeta(4) = 4 \sum_{n=1}^{\infty} \frac{1}{(1-a)^{n+1}} \sum_{j=1}^{n} \begin{pmatrix} n \\ j \end{pmatrix} (-a)^{n-j-1} (n - ab + (a-1)j) \, \delta(j,2)$$

and with $b = 1, a = -\frac{1}{2}$*, upon using (2.5) gives us a new representation*

$$\begin{aligned}\zeta(4) &= \sum_{n=1}^{\infty} \frac{1}{3^{n+1}} \sum_{j=1}^{n} \binom{n}{j} 2^{j+3} \left(n + \frac{1}{2} - \frac{3}{2}j\right) \delta(j,2) \\ &= \sum_{n=1}^{\infty} \frac{1}{3^{n+1}} \sum_{j=1}^{n} \binom{n}{j} 2^{j+3} \left(n + \frac{1}{2} - \frac{3}{2}j\right) \\ &\quad \times \left(\frac{H_j}{j^3} - \frac{\psi^{(\prime)}(j+1)}{j^2} + \frac{\psi^{(\prime\prime)}(j+1)}{2j}\right).\end{aligned}$$

Summary 1. *In this paper, we have provided a method of obtaining zeta identities from the consideration of the integral (1.1). Another approach in obtaining identities for values of the Riemann zeta function, Bernoulli and Euler numbers is based on the theory of polynomials related to the Kontorovich-Lebedev transform. Yakubovich* [18]*, considers the sequence of polynomials* $(p_n(x))_{n\geq 0}$ *defined in terms of a certain second-order differential operator, belonging to the Sheffer's sequences and related to the Kontorovich-Lebedev transform. Some manipulation of the sequence* $(p_n(x))_{n\geq 0}$*, then exhibit several interesting identities. Other representations of harmonic number sums and zeta functions may be seen in the works* [3]*,* [11]*,* [12]*,* [13]*,* [14] *and references therein.*

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References

[1] Alladi, K.; Defant, C. Revisiting the Riemann zeta function at positive even integers. Int. J. Number Theory 14 (2018), no. 7, 1849–1856.

[2] Alzer, H; Sondow, J. A parameterized series representation for Apéry's constant $\zeta(3)$. J. Comput. Anal. Appl. 20 (2016), no. 7, 1380–1386.

[3] Alzer, H; Koumandos, S. Series and product representations for some mathematical constants. Period. Math. Hungar. 58 (2009), no. 1, 71–82.

[4] Bailey, D. H.; Borwein, D.; Borwein, J. M. On Eulerian log-gamma integrals and Tornheim-Witten zeta functions. Ramanujan J. 36 (2015), no. 1-2, 43–68.

[5] Borwein, D.; Borwein, J. M.; Girgensohn, R. Explicit evaluation of Euler sums. Proc. Edinburgh Math. Soc. (2) 38 (1995), no. 2, 277–294.

[6] Euler, L. Opera Omnia, Ser 1, Vol. XV (Teubner, Berlin 1917), 217-267.

[7] Hasse, H. Ein Summierungsverfahren für die Riemannsche Zeta-Reihe. Math. Z. 32, 458-464, 1930.

[8] Janous, W. Around Apéry's constant. JIPAM. J. Inequal. Pure Appl. Math. 7 (2006), no. 1, Article 35, 8 pp.

[9] Milgram, M. S. Integral and series representations of Riemann's zeta function and Dirichlet's eta function and a medley of related results. J. Math. 2013, Art. ID 181724, 17 pp.

[10] Ribeiro, P. Another Proof of the Famous Formula for the Zeta Function at Positive Even Integers. Amer. Math. Monthly 125 (2018), no. 9, 839–841.

[11] Sofo, A. Integral identities for sums. Math. Commun. 13 (2008), no. 2, 303–309.

[12] Sofo, A. Summation formula involving harmonic numbers. Anal. Math. 37 (2011), no. 1, 51–64.
[13] Sofo, A. Quadratic alternating harmonic number sums. J. Number Theory 154 (2015), 144–159.
[14] Sofo, A. Integral identities for sums. Math. Commun. 13 (2008), no. 2, 303–309.
[15] Sondow, J. Analytic Continuation of Riemann's Zeta Function and Values at Negative Integers via Euler's Transformation of Series. Proc. Amer. Math. Soc. 120, 421-424, 1994.
[16] Srivastava, H. M.; Choi, J. Zeta and q-Zeta functions and associated series and integrals. Elsevier, Inc., Amsterdam, 2012. xvi+657 pp.
[17] Sondow, J. and Weisstein, E. W. Riemann Zeta Function. From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/RiemannZetaFunction.html.
[18] Yakubovich, S. Certain identities, connection and explicit formulas for the Bernoulli and Euler numbers and the Riemann zeta-values. Analysis (Berlin) 35 (2015), no. 1, 59–71.