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Traveling wave solutions of $(3+1)$-dimensional Boiti–Leon–Manna–Pempinelli equation by using improved $\tanh(\frac{\phi}{2})$-expansion method

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ABSTRACT

Aim of this article is to investigate soliton solutions of recently developed $(3+1)$-dimensional Boiti–Leon–Manna–Pempinelli equation by utilizing newly derived approach namely, improved $\tanh(\frac{\phi}{2})$-expansion method. As a result, we succeed to secure various types of new solutions for this model including kink, periodic rational solutions. Some of the derived solutions has been discussed in the form of 2-,3-dimensional graphs and their contour plots to visualize the wave dynamics graphically. The results generated by this technique proves that it is a straightforward, robust, and effective method to generate variety of solutions and can be applied on different nonlinear models.

1. Introduction

Nonlinear partial differential equations (NLPDEs) play indispensable role in numerous fields of mathematics, physical sciences and engineering. Integrable differential equations gain much attention in the modern era of research for the study of wave propagation especially in plasma physics, ocean and rogue waves, optical fibers, incompressible fluids and many more. Traveling wave solutions in particular solitary wave solutions which are the exact solutions of some NLPEs is the prime objective and most active research area of researchers and scientist to study and understand nonlinear complex physical phenomena.[1–8] It is interesting to point out that with the evolution of soliton theory, many efficient and robust method have been developed and then modified to generate accurate and novel exact solutions of NLPDEs such as Backlund transformation method,[9] Painlevé expansion,[10] Variational iteration method,[11] tanh method,[12] Sine–Cosine method,[13] improved generalized Riccati equation mapping method,[14] Auxiliary equation method,[15] Ansatz method,[16] Functional variable method,[17] G'/G expansion method[18] and many more methods.

In the last decade Boiti–Leon–Manna–Pempinelli (BLMP) equation has gained a lot of attraction by researchers due to the uses of this model in plasma physics, fluid dynamics, ocean engineering, astrophysics, and aerodynamics to explain wave propagation of incompressible fluids.[7,10,19–23] The $(3+1)$-dimensional Boiti–Leon–Manna–Pempinelli (BLMP) equation has imperative impact and significance in the wave propagation in incompressible fluids, moreover when $z=0$, it describes the interaction of Riemann wave propagation.[10]

Boiti–Leon–Manna–Pempinelli (BLMP) model has been introduced in Refs. 24, 25. Later Wazwaz derived new $(3+1)$-dimensional Boiti–Leon–Manna–Pempinelli (BLMP) equation with constant coefficients in Refs. 10, 26.

$$\left(u_x+u_y+u_z\right)_t+\alpha\left(u_x+u_y+u_z\right)_{xxx}+\beta\left(u_x\left(u_x+u_y+u_z\right)\right)_x=0, \tag{1.1}$$

Where, $u=u(x,y,z,t)$, is unknown analytical function with spatial variables x,y,z and temporal variable t, whereas α and β are no-zero constants.

A lot of work has been done on this model. The stair and step solitons of $(2+1)$ and $(3+1)$ dimensional BLMP has been studied in Ref. 24. Bilinear form, lax pairs and Backlund transformation are constructed by Ref. 27. The authors in Refs. 10, 23, secured multiple solitons and complex multi soliton solution by using Painleve test and Hirota's direct method to generate lump solitons, solitary wave solutions and periodic wave solutions and their interactions. New three wave solutions and hyperbolic and trigonometric solutions have been generated for and $(3+1)$ dimensional BLMP in Refs. 28, 29. Moreover, authors in Ref. 26 investigated the interaction solutions among lump wave, N-solitons, periodic and breather wave solutions. Solitary wave, periodic wave and trigonometric wave solutions has been obtained in Ref. 30 with the aid Sine Gordan expansion method and extended tanh function method. Periodic solitons and periodic type solutions of $(3+1)$ dimensional BLMP has been studied in Ref. 31.

The technique, improved tanh $(\frac{\phi}{2})$-expansion method,[32] used here is new and direct and very convenient to handle, and no study has

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not been done so far on this equation by this technique, as both equation and method is new. With the aid of mathematical software, we manage to generate various interesting types of new exact traveling wave solutions.

The prime motive of this article is to thoroughly study newly derived (3 + 1)-dimensional Boiti–Leon–Manna–Pempinelli (BLMP) equation and concurrently reveals the significance of improved tanh ($\frac{\phi}{2}$)-expansion method. It is worth mentioning here that higher dimensional nonlinear models generate large number of exact solutions as compared to lower dimensional equations.[10] We are hopeful that our new abundant exact solutions which are new and have not been reported in literature of this higher dimensional model have great significance for many higher dimensional nonlinear problems in various fields of sciences.

This paper is organized as follows: Section 1, includes introduction, Section 2, provides analysis of Improved tanh ($\frac{\phi}{2}$)-expansion method, Section 3, includes implementation of this method, Section 4, accommodates results and discussion Section 5, conclude the article.

2. Improved tanh ($\frac{\phi}{2}$)-expansion method

Let us consider the nonlinear partial differential equation with independent variables x, t and some dependent function $\dot{u}$:

$$\text{Å}\left(\dot{u}, \dot{u}_x, \dot{u}_t, \dot{u}_{xx}, \dot{u}_{tt}, \ldots\right) = 0, \tag{2.1}$$

Where Å is a polynomial in $\dot{u}$ with its various orders of nonlinear partial derivatives.

Step1. Let

$$\dot{u}(x,t) = \dot{u}(\xi), \tag{2.2}$$

where,

$$\xi = kx + vt, \tag{2.3}$$

is a wave transformation which can convert nonlinear differential Eq. (2.1) into nonlinear ordinary differential equation,

$$\mathcal{H}\left(\dot{u}, k\dot{u}', v\dot{u}', k^2\dot{u}'', v^2\dot{u}'', \ldots\right) = 0, \tag{2.4}$$

where k, v are nonzero.

Step2. We suppose that the following series expansion is the solution of Eq. (2.4).

$$\dot{u}(\xi) = \Lambda(\phi) = \sum_{k=-N}^{N} \text{A}_k[p + \tanh(\phi/2)]^k, \tag{2.5}$$

where $\text{A}_k (0 \leq k \leq N)$ and $\text{A}_{-k} (1 \leq k \leq N)$ are constants, which are to be determined provided $\text{A}_N \neq 0, \text{A}_{-N} \neq 0$. The function $\phi = \phi(\xi)$ satisfies the following ordinary differential equation.

$$\phi'(\xi) = a\sinh(\phi(\xi)) + b\cosh(\phi(\xi)) + c, \quad \text{where a, b, c are real constants.} \tag{2.6}$$

Eq. (2.6) has following special type of solutions:

Family 1: When $a^2 + c^2 - b^2 < 0, b - c \neq 0$ then

$$\phi(\xi) = 2\ \text{arctanh}\left[-\frac{a}{b-c} + \frac{\sqrt{b^2-a^2-c^2}}{b-c}\tan\left(\frac{\sqrt{b^2-a^2-c^2}}{2}(\xi')\right)\right].$$

Family 2: When $a^2 + c^2 - b^2 > 0$ and $b - c \neq 0$, then

$$\phi(\xi) = 2\ \text{arctanh}\left[-\frac{a}{b-c} - \frac{\sqrt{a^2+c^2-b^2}}{b-c}\tanh\left(\frac{\sqrt{a^2+c^2-b^2}}{2}(\xi')\right)\right].$$

Family 3: When $a^2 + c^2 - b^2 < 0$, $b \neq 0$ and $c = 0$, then

$$\phi(\xi) = 2\ \text{arctanh}\left[-\frac{a}{b} + \frac{\sqrt{b^2-a^2}}{b}\tan\left(\frac{\sqrt{b^2-a^2}}{2}(\xi')\right)\right].$$

Family 4: When $a^2 + c^2 - b^2 > 0$, c≠0 and b = 0, then

$$\phi(\xi) = 2\ \text{arctanh}\left[\frac{a}{c} + \frac{\sqrt{a^2+c^2}}{c}\tan\left(\frac{\sqrt{a^2+c^2}}{2}(\xi')\right)\right].$$

Family 5: When $a^2 + c^2 - b^2 < 0$, b-c≠0 and a=0, then

$$\phi(\xi) = 2\ \text{arctanh}\left[\sqrt{\frac{b+c}{b-c}}\tan\left(\frac{\sqrt{b^2-c^2}}{2}(\xi')\right)\right].$$

Family 6: When a = 0 and c = 0, then

$$\phi(\xi) = \ln\left[\tan\left(\frac{b}{2}(\xi')\right)\right].$$

Family 7: When b = 0 and c = 0, then

$$\phi(\xi) = \ln\left[-\tanh\left(\frac{a}{2}(\xi')\right)\right].$$

Family 8: When $a^2 + b^2 = c^2$, then

$$\phi(\xi) = 2\ \text{arctanh}\left[\frac{a}{-b+\sqrt{a^2+b^2}} + \frac{\sqrt{2a}}{-b+\sqrt{a^2+b^2}}\tanh\left(\frac{\sqrt{2a}}{2}(\xi')\right)\right].$$

Family 9: When a = b = c = ka, then

$$\phi(\xi) = 2\ \text{arctanh}\left[e^{ka(\xi')} - 1\right].$$

Family 10: When $a = c = ka$ and $b = -ka$, then

$$\phi(\xi) = 2\ \text{arctanh}\left[\frac{e^{ka(\xi')}}{-1 + e^{ka(\xi')}}\right].$$

Family 11: When $b = a$, then

$$\phi(\xi) = -2\ \text{arctanh}\left[\frac{(a+c)e^{b(\xi')} - 1}{(a-c)e^{b(\xi')} - 1}\right].$$

Family 12: When $b = c$, then

$$\phi(\xi) = 2\ \text{arctanh}\left[\frac{e^{b(\xi')} - c}{a}\right].$$

Family 13: When $a = -c$, and $b = c$ then

$$\phi(\xi) = 2\ \text{arctanh}\left[1 + e^{-c(\xi')}\right].$$

Family 14: When $b = -b$, and $c = -b$ then

$$\phi(\xi) = 2\ \text{arctanh}\left[\frac{b + e^{a(\xi')}}{a}\right].$$

Family 15: When $b = -b$, $a = -b$ and $c = b$ then

$$\phi(\xi) = 2\ \text{arctanh}\left[\frac{1}{e^{b(\xi')} - 1}\right].$$

Family 16: When $b = -c$, then

$$\phi(\xi) = 2\ \text{arctanh}\left[\frac{ae^{a(\xi')}}{ce^{a(\xi')} - 1}\right].$$

Family 17: When $a = 0$ and $b = c$, then

$$\phi(\xi) = 2\ \text{arctanh}[\text{c}(\xi')]$$

Family 18: When $a = 0$, and $b = -c$, then

$$\phi(\xi) = 2\ \text{arctanh}\left[\frac{1}{c(\xi')}\right].$$

Family 19: When $b = 0$, and $a = c$ then

$$\phi(\xi) = 2\ \text{arctanh}\left[1 + \sqrt{2}\tanh\left(\frac{\sqrt{2c}}{2}(\xi')\right)\right].$$

Family 20: When $a = 0$, and $b = 0$ then

$\phi(\xi) = c\xi + C$,

where $\xi' = \xi + C$, $A_k, A_{-k}\,(k = 1, 2, \ldots, N), a, b, c$ are constants to be determined later. Positive integer N in Eq. (2.5) can be found by using homogeneous balance principle between the derivatives of highest order and the highest power of nonlinear terms in Eq. (2.4).

Step4. Substituting Eq. (2.5) along with Eq. (2.6) into Eq. (2.4). We get the polynomial equations. **Step5.** With the help of Maple, we solve the system described in step 4, provides the values of A_0, A_k, A_{-k} where, $i = 1, 2, \ldots .N, a, b, c$. We substitute these values in Eq. (2.5) coupled with solutions of Eq. (2.6) and applying the transformation in Eq. (2.4), we construct several exact solutions of Eq. (2.1), establishing twenty families.[14]

3. Implementation of IThEM

To use improved tanh $(\frac{\phi(\xi)}{2})$-expansion method on Eq. (1.1).

We use following wave transformation, $u(x,t) = u(\xi)$, with $\xi = k_1x + k_2y + k_3z + \omega t$, in Eq. (1.1), substituting $\alpha = \beta = -3$ and after integrating by keeping constant of integration zero, we get the following nonlinear ODE:

$$k_1^3\left(k_3+k_1+k_2\right)\frac{d^3}{d\xi^3}u(\xi)+\omega\left(k_3+k_1+k_2\right)\frac{d}{d\xi}u(\xi) -\frac{3k_1^2\left(k_3+k_1+k_2\right)}{2}\left(\frac{d}{d\xi}u(\xi)\right)^2=0, \tag{3.1}$$

using homogeneous balance principle between $\left(\frac{d^3}{d\xi^3}u(\xi)\right)$ and $\left(\frac{d}{d\xi}u(\xi)\right)^2$ we get $N = 1$. Therefore, the exact series solution has the form,

$$u(\xi) = \Lambda(Y) = \frac{A_{-1}}{p+\tanh\left(\frac{\phi(\xi)}{2}\right)} + A_0 + A_1\left(p+\tanh\left(\frac{\phi(\xi)}{2}\right)\right), \tag{3.2}$$

now, substituting Eq. (3.2) along with Eq. (2.6) into Eq. (3.1) after collecting all terms with the same powers of $\tanh\left(\frac{\phi(\xi)}{2}\right)$ and equating each coefficient to zero, we obtain a system of nonlinear algebraic equations. Solving these equations by using Maple 17, we get following non-trivial solutions. All the abbreviations used in the below mentioned solutions have been expressed in table:

$D = a^2 - b^2 + c^2$	$\Omega = xk_1 + zk_3 + yk_2$
$E = (b-c)\left((b-c)p^2 - b - c\right)$	$F = -a^2 + b^2$
$F' = a^2 + c^2$	$G = b^2 - c^2$

Family 1:

Some trigonometric function solutions are formulated for BLMP equation **for** $a^2 + c^2 - b^2 < 0, b - c \neq 0$:

$a = a, b = c, c = c, \omega = -k_1^3 D, p = p,$

$A_{-1} = 2k_1(-(b-c)p^2 + 2pa - b - c), A_1 = 0,$

$$u_1 = \begin{pmatrix}\sqrt{-D}A_0\tan\left(\left(tDk_1^3-\Omega\right)\sqrt{-D}\right)+2k_1(b-c)^2p^2\\ -4\left(ak_1+A_0/4\right)(b-c)p+2\left(b^2-c^2\right)k_1+aA_0\end{pmatrix} \times\left(\sqrt{-D}\tan\left(\left(tDk_1^3-\Omega\right)\sqrt{-D}\right)+(-b+c)p+a\right)^{-1}, \tag{3.3}$$

$a = a, b = b, c = c, \omega = -Dk_1^3,$

$p = p, A_{-1} = 0, A_1 = 2k_1(b-c),$

$$u_2 = \left(-2\sqrt{-D}\tan\left(\left(tDk_1^3-\Omega\right)\sqrt{-D}/2\right)+2(pb-pc-a)\right)k_1+A_0, \tag{3.4}$$

$a = p(b-c), b = b, c = c, \omega = -4Ek_1^3,$

$p = p, A_{-1} = 2k_1\left((b-c)p^2-b-c\right), A_1 = 2k_1(b-c),$

$$u_3 = \begin{pmatrix}A_0/2\sqrt{-E}\tan\left(2\sqrt{-E}\left(tk_1^3\left(2\left(p^2-1\right)b^2\right.\right.\right.\\ \left.\left.\left.-2bcp^2+\left(p^2+1\right)c^2\right)-\Omega/4\right)\right)\\ -k_1E\left(\tan\left(2\sqrt{-E}\left(tk_1^3\left(2\left(p^2-1\right)b^2\right.\right.\right.\right.\\ \left.\left.\left.\left.-2bcp^2+\left(p^2+1\right)c^2\right)-\Omega/4\right)\right)^2-1\right)\end{pmatrix} \times\left(\sqrt{-E}\tan\left(2\sqrt{-E}\left(tk_1^3\left(2\left(p^2-1\right)b^2-2bcp^2+\left(p^2+1\right)c^2\right)-\Omega/4\right)\right)\right)^{-1}, \tag{3.5}$$

Family 2:

The hyperbolic function solutions can be derive as using the following conditions:

For $a^2 + c^2 - b^2 > 0$ and $b - c \neq 0$:

$a = a, b = b, c = c, \omega = -Dk_1^3, p = p, A_1 = 0,$

$A_{-1} = 2k_1(-(b-c)p^2 + 2pa - b - c),$

$$u_4 = \begin{pmatrix}-\tanh\left(\frac{\left(t(D)k_1^3-\Omega\right)\sqrt{D}}{2}\right)\sqrt{D}A_0+2k_1(b-c)^2p^2\\ -(4ak_1+A_0)(b-c)p+2(b^2-c^2)k_1+aA_0\end{pmatrix} \times\left(-\tanh\left(\left(t(D)k_1^3-\Omega\right)\sqrt{D}/2\right)\sqrt{D}+(-b+c)p+a\right)^{-1}, \tag{3.6}$$

$a = a, b = b, c = c, \omega = -Dk_1^3, p = p, A_{-1} = 0,$

$A_1 = 2k_1(b-c),$

$$u_5 = (2\tanh(1/2(t(D)k_1^3-\Omega)\sqrt{D})\sqrt{D}+2pb-2pc-2a)k_1+A_0 \tag{3.7}$$

Family 3:

When $a^2 + c^2 - b^2 < 0$, b $\neq 0$ and c $= 0$, the trigonometric function solutions generated as:

$a = a, b = b, c = 0, \omega = 4k_1^3F, p = \frac{a}{b},$

$A_{-1} = -2k_1F/b, A_1 = 2bk_1,$

$$u_6 = \left(-2k_1\sqrt{F}\tan\left(-\sqrt{F}\left(4tFk_1^3+\Omega\right)/2\right)+A_0+2k_1\sqrt{F}\right) \times\left(\tan\left(\sqrt{F}\left(4tFk_1^3-\Omega\right)/2\right)\right)^{-1}, \tag{3.8}$$

$a = a, b = b, c = 0, \omega = 4k_1^3F, p = \frac{a}{b}, A_{-1} = -2k_1^3F/b, A_1 = 2bk_1,$

$$u_7 = \begin{pmatrix}-2F\left(\tan\left(\sqrt{F}\left(-4tFk_1^3-\Omega\right)/2\right)^2-1\right)k_1\\ +A_0\tan\left(\sqrt{F}\left(-4tFk_1^3-\Omega\right)/2\right)\sqrt{F}\end{pmatrix} \times\left(A_0\tan\left(\sqrt{F}(-4tFk_1^3-\Omega)/2\right)\sqrt{F}\right)^{-1}, \tag{3.9}$$

$a = a, b = b, c = 0, \omega = k_1^3F, p = p, A_{-1} = 0, A_1 = 2bk_1,$

$$u_8 = \left(-2\tan\left(\sqrt{F}(-tFk_1^3-\Omega)/2\right)\sqrt{F}+2pb-2a\right)k_1+A_0, \tag{3.10}$$

Family 4:

Another choice of hyperbolic function solutions for $a^2 + c^2 - b^2 > 0$, c $\neq 0$ and b $= 0$:

$a = a, b = 0, c = c, \omega = -k_1^3F', p = p, A_1 = 0,$

$A_{-1} = 2k_1pa - \left(-p^2+1\right)c,$

$$u_9 = \begin{pmatrix} \left(\tanh\left(\left(tk_1^3F' - \Omega\right)\sqrt{F'}/2\right)\sqrt{F'} - a\right)A_0 + \\ 2\left(-p^2+1\right)k_1c^2 - p\left(4ak_1 + A_0\right)c \end{pmatrix} \times\left(\tanh\left(\left(tk_1^3F' - \Omega\right)\sqrt{F'}/2\right)\sqrt{F'} - cp - a\right)^{-1}, \tag{3.11}$$

$a = a, b = 0, c = c, \omega = -F'k_1^3, p = p, A_{-1} = 2k_1(2pa - \left(-p^2+1\right)c),$

$A_1 = 0,$

$$u_{10} = \begin{pmatrix} \left(\tanh\left(\left(tk_1^3F' - \Omega\right)\sqrt{F'}/2\right)\sqrt{F'} - a\right)A_0 + \\ 2\left(-p^2+1\right)k_1c^2 - p\left(4ak_1 + A_0\right)c \end{pmatrix} \times\left(\tanh\left(\left(tk_1^3F' - \Omega\right)\sqrt{F'}/2\right)\sqrt{F'} - cp - a\right)^{-1}, \tag{3.12}$$

$a = a, b = 0, c = c, \omega = -k_1^3F', p = p, A_{-1} = 0, A_1 = -2ck_1,$

$$u_{11} = \left(\tanh\left((tF'k_1^3 - \Omega)\sqrt{F'}\right)\sqrt{F'} - 4pc - 4a\right)k_1 + A_0, \tag{3.13}$$

Family 5:

For $a^2 + c^2 - b^2 < 0$, b-c $\neq 0$ and a = 0, trigonometric function solutions has been generated as:

$a = 0, b = b, c = c, \omega = Gk_1^3, p = p,$

$A_{-1} = 2k_1(-bp^2 + cp^2 - b - c), A_1 = 0,$

$$u_{12} = \begin{pmatrix} A_0\sqrt{G}\tan\left(\left(Gtk_1^3 + \Omega\right)\sqrt{G}/2\right)/2 \\ -2\left(k_1(b-c)^2p^2 - A_0p/2 + k_1G\right) \end{pmatrix} \times\left(\sqrt{G}\tan\left(\left(Gtk_1^3 + \Omega\right)\sqrt{G}/2\right) + p(b-c)\right)^{-1}, \tag{3.14}$$

$a = 0, b = b, c = c, \omega = Gk_1^3, p = p, A_{-1} = 0, A_1 = 2bk_1 - 2ck_1,$

$$u_{13} = 2\tan\left(\left(Gtk_1^3 + \Omega\right)\sqrt{G}/2\right)k_1\sqrt{G} + 2p(b-c)k_1 + A_0, \tag{3.15}$$

Family 6:

Mix soliton solution, hyperbolic function solutions has been acquired for a = 0 and c = 0:

$b = b, \omega = b^2k_1^3, p = p, A_1 = 0, A_{-1} = -2bk_1\left(p^2+1\right),$

$$u_{14} = -\frac{2bk_1\left(p^2+1\right)}{p + \tanh\left(\frac{1}{2}ln\left(\tan\left(b^2\left(b^2k_1^3t + \Omega\right)\right)\right)\right)} + A_0, \tag{3.16}$$

$a = 0, b = b, c = 0, \omega = b^2k_1^3, p = p, A_1 = 2bk_1, A_{-1} = 0,$

$$u_{15} = A_0 + 2bk_1\left(p + \tanh\left(\frac{1}{2}ln\left(\tan\left(b^2\left(b^2k_1^3t + \Omega\right)\right)\right)\right)\right), \tag{3.17}$$

$a = 0, b = b, c = 0, \omega = 4b^2k_1^3, p = 0, A_1 = 2bk_1, A_{-1} = -2bk_1,$

$$u_{16} = A_0 + 2bk_1\left(\tanh\left(\frac{1}{2}ln\left(\tan\left(b^2\left(b^2k_1^3t + \Omega\right)/2\right)\right)\right)\right) - \frac{2bk_1}{\tanh\left(\frac{1}{2}ln\left(\tan\left(b\left(4b^2k_1^3t + \Omega\right)b/2\right)\right)\right)}, \tag{3.18}$$

Family 7:

The hyperbolic function solution for b = 0 and c = 0, along with the following conditions:

$a = a, \omega = -a^2k_1^3, p = p, A_{-1} = 4pak_1, A_1 = 0,$

$$u_{17} = \frac{4pak_1}{p + \tanh\left(\frac{1}{2}ln\left(\tanh\left(a\left(a^2k_1^3t - \Omega\right)b/2\right)\right)\right)} + A_0, \tag{3.19}$$

Family 8:

We get mix solutions, trigonometric and hyperbolic function solutions respectively for $a^2 + b^2 = c^2$,

$a = Ib, b = b, c = 0, \omega = 8b^2k_1^3, p = I, A_{-1} = -4bk_1, A_1 = 2bk_1,$

$$u_{18} = \frac{\sqrt{2}\begin{pmatrix} 4bk_1\tan\left(b\sqrt{2}\left(8b^2k_1^3t + \Omega\right)b/2\right)^2 + \\ A_0\tan\left(b\sqrt{2}\left(8b^2k_1^3t + \Omega\right)b/2\right)\sqrt{2} - 4k_1b \end{pmatrix}}{\tan\left(b\sqrt{2}\left(8b^2k_1^3t + \Omega\right)b/2\right)\sqrt{2}}, \tag{3.20}$$

Family 11:

Exponential function solutions for $a = b$, we get as:

$b = b, c = c, \omega = -c^2k_1^3, p = p, A_{-1} = 0, A_1 = 2bk_1 - 2ck_1,$

$$u_{19} = \begin{pmatrix} 2(b-c)\left(((p-1)b - c(p+1))k_1 + A_0/2\right)e^{-c\left(c^2k_1^3t - \Omega\right)} \\ -2(p-1)(b-c)k_1 - A_0 \end{pmatrix} \times\left(-1 + (b-c)e^{-c\left(c^2k_1^3t - \omega\right)}\right)^{-1}, \tag{3.21}$$

$a = 0, b = 0, c = c, \omega = -4c^2k_1^3, p = 0, A_{-1} = -2ck_1, A_1 = -2ck_1,$

$$u_{20} = \begin{pmatrix} -4e^{\left(-8c^3k_1^3t + 2c\Omega\right)}c^3k_1 - A_0 \\ +e^{\left(-8c^3k_1^3t + 2c\Omega\right)}c^2A_0 - 4ck_1 \end{pmatrix}\left(e^{\left(-8c^3k_1^3t + 2c\Omega\right)}c^2 - 1\right)^{-1}, \tag{3.22}$$

Family 12:

For $b = c$, we get exponential function solution as follows:

$a = 1/k_1\sqrt{-\omega/k_1}, c = c, \omega = \omega, p = \dfrac{4ck_1 + A_{-1}}{4\sqrt{-\omega/k_1}}, A_1 = 0, A_{-1} = A_{-1},$

$$u_{21} = \frac{4e^{1/k_1\sqrt{-\omega/k_1}\xi}A_0k_1 + 4\left(\sqrt{-\omega/k_1} + A_0/4\right)A_{-1}}{4e^{1/k_1\sqrt{-\omega/k_1}\xi}k_1 + A_{-1}}, \tag{3.23}$$

Family 13:

For $a = -c$, and $b = c$ we get another type of exponential function solution:

$c = c, \omega = -c^2k_1^3, p = p, A_{-1} = -4pk_1c - 4k_1c, A_1 = 0,$

$$u_{22} = \frac{\left(A_0e^{c\left(c^2k_1^3t - \Omega\right)} - 4(p+1)\left(k_1c - A_0/4\right)\right)}{\left(p + e^{c\left(c^2k_1^3t - \Omega\right)} + 1\right)}, \tag{3.24}$$

Family 14:

For $b = -b$, and $c = -b$ we get another type of exponential function solution:

$a = a, b = 0, c = 0, \omega = -a^2k_1^3, p = p, A_{-1} = 4pak_1, A_1 = 0,$

$$u_{23} = \frac{\left(A_0e^{-a^2\left(a^2k_1^3t - \Omega\right)} + apA_0 + 4pa^2k_1\right)}{\left(ap + e^{-a^2\left(a^2k_1^3t - \Omega\right)}\right)}, \tag{3.25}$$

Family 16:

For $b = -c$, then we different types of exponential function solutions:

$a = a, c = c, \omega = -a^2k_1^3, p = p, A_{-1} = 0, A_1 = -4ck_1,$

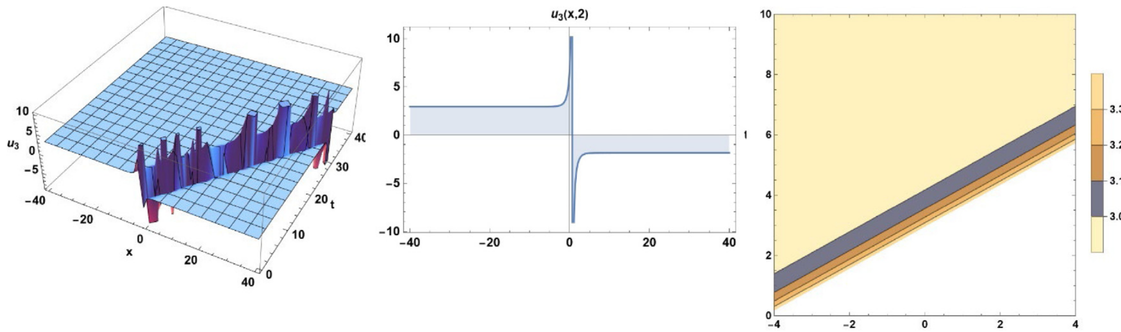

Fig. 1. Graphical evolution of singular kink wave soliton for u_3 using parameters, $b = 0.9, c = 1.5, p = 0.02, k_1 = 0.5, k_2 = 0.5, k_3 = 0.1, A_0 = 0.55, y = 2, z = 1, t = 2$.

$$u_{24} = \frac{\left(4cpk_1 - A_0 - 4c\left(cpk_1 + ak_1 - A_0/4\right)e^{-a^2\left(a^2k_1^3t-\Omega\right)}\right)}{\left(ce^{-a^2\left(a^2k_1^3t-\Omega\right)} - 1\right)}, \quad (3.26)$$

$a = a, c = c, \omega = -a^2k_1^3, p = p, A_{-1} = 4pak_1 + 4p^2ck_1, A_1 = 0,$

$$u_{25} = \begin{pmatrix} -4\left(cpk_1 + ak_1 + A_0/4\right)p \\ +4\left(cpk_1 + A_0/4\right)(cp+a)e^{-a\left(a^2k_1^3t-\Omega\right)} \end{pmatrix} \times \left((cp+a)e^{-a\left(a^2k_1^3t-\Omega\right)} - p\right)^{-1}, \quad (3.27)$$

$a = -2cp, c = c, \omega = -16c^2p^2k_1^3, p = p, A_{-1} = -4p^2ck_1, A_1 = -4ck_1,$

$$u_{26} = \begin{pmatrix} -A_0 + 8c^3pk_1e^{4cp\left(16c^2p^2k_1^3t-\Omega\right)} \\ +c^2A_0e^{4cp\left(16c^2p^2k_1^3t-\Omega\right)} + 8cpk_1 \end{pmatrix}\left(c^2e^{4cp\left(16c^2p^2k_1^3t-\Omega\right)} - 1\right)^{-1}, \quad (3.28)$$

Family 17:

For $a = 0$ and $b = c$, we get various wave solutions given as follows:

$c = c, \omega = \omega, p = p, A_{-1} = 0, A_1 = A_1,$

$$u_{27} = A_0 + A_1\left(p + c\xi\right), \quad (3.29)$$

Family 18:

When $a = 0$, and $b = -c$, we get various rational function solutions as follows:

$c = c, \omega = 0, p = p, A_{-1} = 0, A_1 = -4ck_1,$

$$u_{28} = \frac{-4cpk_1\Omega + (xA_0 - 4)k_1 + A_0(yk_2 + zk_3)}{\Omega}, \quad (3.30)$$

$a = 0, b = -c, c = c, \omega = 0, p = p, A_{-1} = 4cp^2k_1, A_1 = 0,$

$$u_{29} = \frac{4c^2p^2k_1\Omega + pA_0\Omega c + A_0}{pc\Omega + 1}, \quad (3.31)$$

$a = 0, b = -c, c = c, \omega = \omega, p = 0, A_1 = 0, A_{-1} = \frac{2\omega}{3ck_1^2},$

$$u_{30} = \frac{2\omega^2t + 2\Omega\omega + 3A_0k_1^2}{3k_1^2}, \quad (3.32)$$

Family 19:

When $b = 0$, and $a = c$ we get dark solitons:

$c = c, \omega = -2c^2k_1^3, p = p, A_{-1} = 2ck_1(p^2 + 2p - 1), A_1 = 0,$

$$u_{31} = \frac{\begin{pmatrix} \tanh\left(c\sqrt{2}\left(2c^2tk_1^3 - \Omega\right)/2\right)\sqrt{2}A_0 \\ -2cp^2k_1 + (-4ck_1 - A_0)p + 2ck_1 - A_0 \end{pmatrix}}{\tanh\left(c\sqrt{2}\left(2c^2tk_1^3 - \Omega\right)/2\right)\sqrt{2} - p - 1}, \quad (3.33)$$

$c = c, \omega = -2c^2k_1^3, p = p, A_{-1} = 0, A_1 = -2ck_1,$

$$u_{32} = 2\sqrt{2}\tanh\left(c\sqrt{2}\left(2c^2tk_1^3 - \Omega\right)/2\right)ck_1 - 2c(p+1)k_1 + A_0, \quad (3.34)$$

Family 20:

we get hyperbolic function solutions for $a = 0$, and $b = 0$

$c = c, \omega = -c^2k_1^3, p = p, A_{-1} = 2ck_1(p^2 - 1), A_1 = 0,$

$$u_{33} = \frac{2k_1(p^2 - 1)c}{p - \tanh\left(\left(c^2tk_1^3 - \Omega\right)c/2\right)} + A_0, \quad (3.35)$$

$c = c, \omega = -c^2k_1^3, p = p, A_{-1} = 0, A_1 = -2ck_1,$

$$u_{34} = -2ck_1\left(p - \tanh\left(\left(c^2tk_1^3 - \Omega\right)c/2\right)\right) + A_0, \quad (3.36)$$

$c = c, \omega = -4c^2k_1^3, p = 0, A_{-1} = -2ck_1, A_1 = -2ck_1,$

$$u_{35} = 2\frac{ck_1}{\tanh\left(\left(4c^2tk_1^3 - \Omega\right)c/2\right)} + A_0 + 2ck_1\tanh\left(\left(4c^2tk_1^3 - \Omega\right)c/2\right), \quad (3.37)$$

4. Results and discussion

With the help of IThEM, we secured different wave structures of newly derived equation, $(3+1)$-BLMP that includes hyperbolic, trigonometric, exponential, and rational function solutions. All the obtained results are new and generalized solitary waves that comprise kink waves, periodic waves, solitons, singular solitons with suitable choice of free parameters. Uniqueness of our work is evident as we successfully acquired 42 different types of wave solutions however keeping in view the length of the article, we only present some selective ones. These solutions are more generalized and novel and had not been reported in literature previously as we compared with published results,[30] it is worth mentioning our few solutions have similarity with them but most of the solutions are new, and we were able to derive various periodic wave solutions, singular periodic wave solutions, exponential function solutions and rational solutions other than solitons, kink solitons and singular kink solitons, which have not been explained before. Diverse wave structure of various solutions has been well characterized by 3-D, 2-D and their contour plots and we found out that the existence of periodic wave solutions, kink wave solutions and other solitons depends on free parameters. As these answers have not been reported so far, we are sure our work would be a valuable addition in literature to analyze this new model. The diversity and dynamic characteristics of these exact solutions can be well explained by 3-D, and 2-D and their contour plots with the appropriate choice of parameters. Fig. 1- 6 shows 3-D, and 2-D graphs and their contour plots of some obtained results

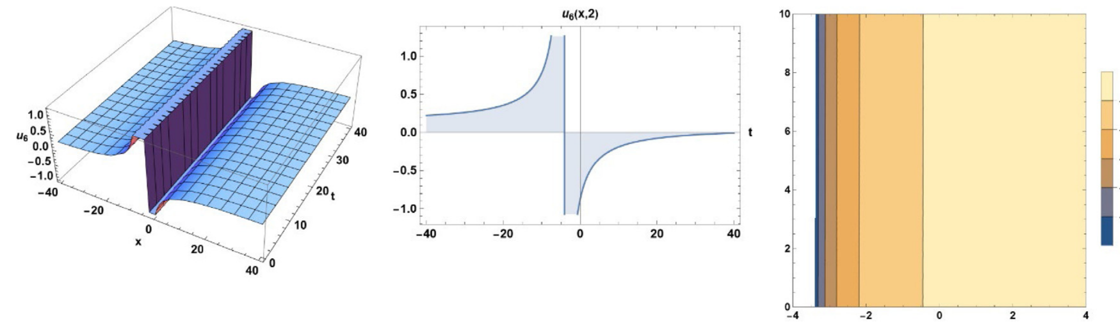

Fig. 2. Graphical evolution of singular kink wave soliton for u_6.using parameters $a = 0.2, b = 0.1, k_1 = 0.1, k_2 = 0.21, k_3 = 0.2, A_0 = 0.1, y = 1, z = 1, t = 2$.

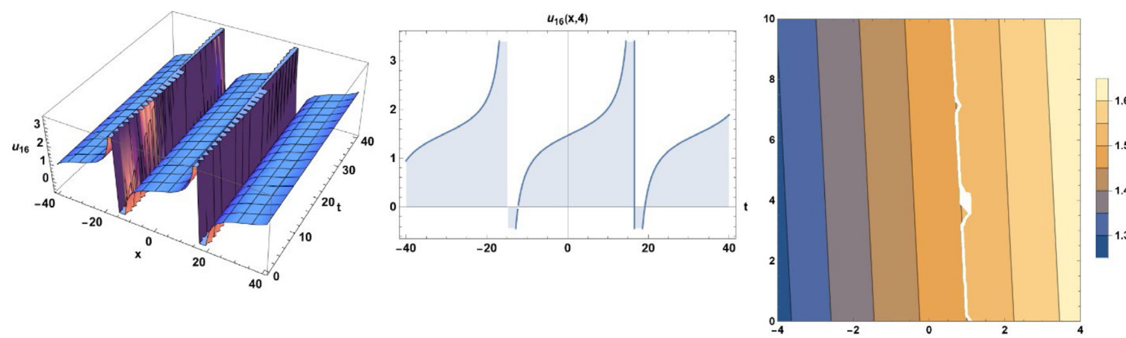

Fig. 3. Graphical evolution of singular periodic wave soliton for u_{16} using parameters $b = 0.5, k_1 = 0.2, k_2 = -0.1, k_3 = 0.3, A_0 = 1.5, y = -1, z = -1, t = 4$.

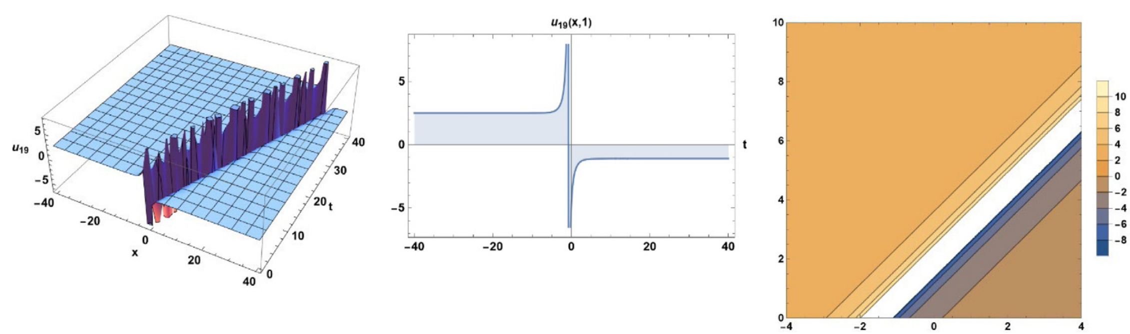

Fig. 4. Graphical evolution of singular kink wave soliton for u_{19}.using parameters $b = 0.1, c = 0.9, p = 0.2, k_1 = 0.5, k_2 = 0.1, k_3 = 0.8, A_0 = 0.7, y = 1, z = 1, t = 1$.

of $(3+1)$- BLMP equation to have a good grasp of physical phenomena of these solutions under appropriate choice of free parameters.

Graphical depiction of Eq. (3.5) expressed as u_3 has been exhibit in Fig. 1, in the form of 3-dimensional, and 2-dimensional and contour plot which demonstrates localized excitation wave pattern as singular kink wave soliton by selecting appropriate parameters. The dynamic behavior of singular kink type solution of Eq. (3.5) is revealed well by suitable parameters.

Graphical depiction of Eq. (3.8) expressed as u_6 has been exhibit in Fig. 2, in the form of 3-dimensional, and 2-dimensional and contour plot which demonstrates localized excitation wave pattern as singular kink soliton by selecting suitable parameters.

Graphical depiction of Eq. (3.18) expressed as u_{16} has been exhibit in Fig. 3, in the form of 3-dimensional, and 2-dimensional and their contour plot which demonstrates localized excitation wave pattern as singular periodic wave soliton by selecting appropriate parameters.

Graphical depiction of Eq. (3.21) expressed as u_{19} has been exhibit in Fig. 4, in the form of 3-dimensional, 2-dimensional and their contour plot which demonstrates localized excitation wave pattern as singular kink soliton by selecting suitable parameters.

Graphical depiction of Eq. (3.23) expressed as u_{21} has been exhibit in Fig. 5, in the form of 3 dimensional, and 2 dimensional and their contour plot which demonstrates localized excitation wave pattern as periodic wave solution by selecting suitable parameters

Graphical depiction of Eq. (3.34) expressed as u_{32} has been exhibit in Fig. 6, in the form of 3 dimensional, and 2 dimensional and their contour plot which demonstrates localized excitation wave pattern as kink shape soliton by selecting appropriate parameters.

5. Conclusions

In this article, improved tanh $(\frac{\phi}{2})$-expansion method is applied to perceive general solutions of newly derived $(3+1)$-dimensional Boiti–Leon–Manna–Pempinelli equation. As a result, some totally new solutions have been obtained which are several solitary wave solutions including hyperbolic wave solutions, periodic wave solutions, exponential solutions. These new solutions maybe worthwhile in the field of ocean engineering, astrophysics, and aerodynamics, plasma physics and fluid mechanics to explain wave propagation of incompressible fluids. Each type of solitary wave has its importance in nonlinear media such

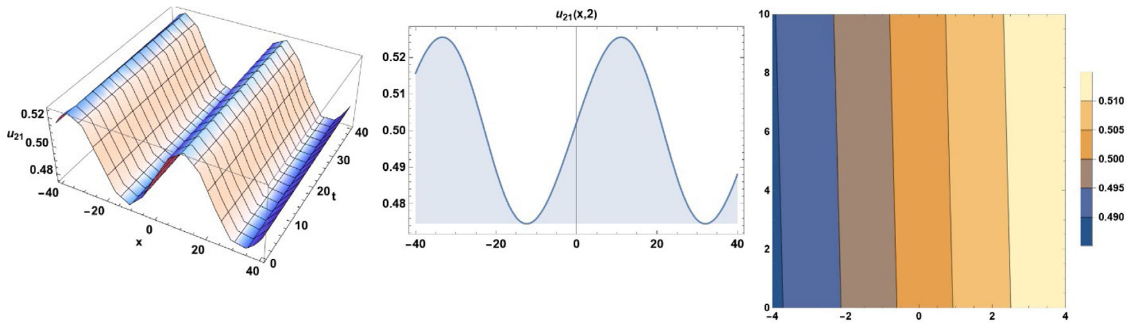

Fig. 5. 3D and 2D-graphs of periodic wave solution for u_{21}.using parameters . $c = 2, k_1 = 5, k_2 = 1, k_3 = 2, A_0 = 0.5, A_{-1} = 0.9, p = 2, y = 1, z = 1, t = 2$.

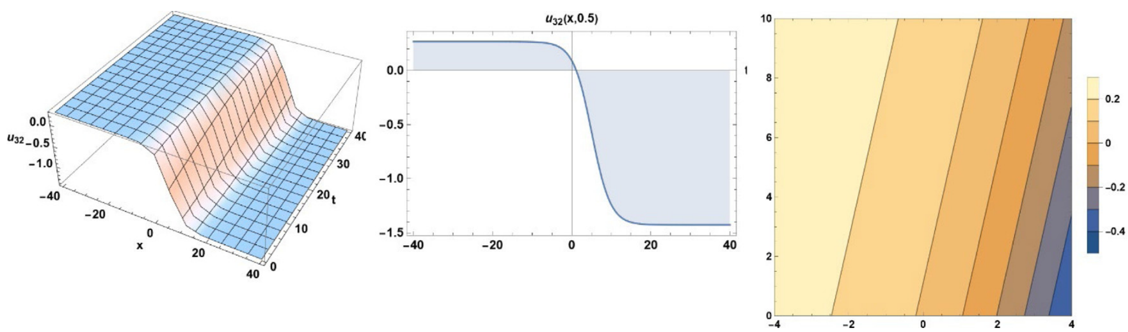

Fig. 6. Graphical evolution of kink wave soliton for u_{32} using parameters $c = 3, k_1 = 0.1, k_2 = 0.5, k_3 = 1, A_0 = 0.5, p = 0.8, y = 1, z = -1, t = 0.5$.

as kink solitons which propagates in nonlinear physical phenomena having high order nonlinearity, high order nonlinear effects and self-steepening. These solitons have been studied extensively due to its perfect propagation through nonlinear media.[33] Singular solitons are also very important type of solitons that appears with singularity. These solitons likely provide information about formation of rouge waves, also another type of solitary waves are periodic wave solutions that plays notable role in the study of chemistry, physics, biology and many more.[34] This newly derived method, IThEM is more effective than many other techniques such as tanh method and extended tanh method,[35],[36] sine–cosine method,[37] ansatz method,[38] Improved tan $(\frac{\phi}{2})$-expansion method[39] to generate more general and abundant solutions. This technique has developed recently and has not been used much previously, results show that this scheme is robust and effective to find plenty new solutions of different types. It can be applied to many nonlinear PDEs arising in different fields of sciences to generate new type of solutions. The nature of these results has been analyzed physically by 2D and 3D graph simulation and their corresponding contour plots with the aid of computational software.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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