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Article

Analysis and Applications of Some New Fractional Integral Inequalities

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Abstract: This paper presents a novel parameterized fractional integral identity. By using this auxiliary result and the $\mathfrak{s}$-convexity property of the mapping, a series of fractional variants of certain classical inequalities, including Simpson's, midpoint, and trapezoidal-type inequalities, have been derived. Additionally, some applications of our main outcomes to special means of real numbers have been explored. Moreover, we have derived a new generic numerical scheme for solving non-linear equations, demonstrating an application of our main results in numerical analysis.

Keywords: $\mathfrak{s}$-convex mappings; Simpson's $\frac{1}{3}$ formula; midpoint formula; trapezoidal formula; integral inequalities; fractional calculus; basins of attraction

MSC: 26A33; 26A51; 26D07; 26D10; 26D15; 26D20; 58C30; 65H05

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1. Introduction and Preliminaries

The notion of convex sets and convex functions has numerous applications in the fields of both pure and applied sciences. In addition, the theory of convexity has undergone rapid advancements in recent years owing to its numerous applications and its close connection with the theory of inequalities. Solutions to mathematical problems can be approximated using the application of inequalities in cases where there is difficulty in finding the exact values. There is a strong relation between convexity and the theory of inequalities, as convex functions can be directly applied to derive many inequalities.

According to Dragomir and Pearce [1], the Hermite–Hadamard inequality is one of the most renowned results in the class of classical convex functions. This inequality possesses a clear intrinsic geometrical interpretation and finds numerous applications. Although the result was initially identified by Hadamard (1865–1963), it was primarily attributed to Hermite (1822–1901) [2,3]. The statement of this inequality is as follows:

Suppose that $\aleph : \mathcal{I} \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a convex mapping, and let $\tau_1, \tau_2 \in \mathcal{I}$ such that $\tau_1 < \tau_2$. Then,

$$\aleph\left(\frac{\tau_1 + \tau_2}{2}\right) \leq \frac{1}{\tau_2 - \tau_1}\int_{\tau_1}^{\tau_2} \aleph(\lambda)d\lambda \leq \frac{\aleph(\tau_1) + \aleph(\tau_2)}{2}. \quad (1)$$

The two sides of the Hermite–Hadamard inequality, namely the midpoint and trapezoidal-type inequalities, are utilized for the estimation of error bounds for certain quadrature rules. These inequalities were first derived in [4,5] and are defined as follows:

Suppose that $\aleph : [\tau_1, \tau_2] \to \mathbb{R}$ is a differentiable mapping on (τ_1, τ_2), with $\tau_1 < \tau_2$. If $|\aleph'|$ is convex on $[\tau_1, \tau_2]$, then:

$$\left| \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \aleph(\lambda) d\lambda - \aleph\left(\frac{\tau_1 + \tau_2}{2}\right) \right| \leq \frac{\tau_2 - \tau_1}{8} \left[|\aleph'(\tau_1)| + |\aleph'(\tau_2)| \right].$$

Let $\aleph : [\tau_1, \tau_2] \to \mathbb{R}$ be a differentiable mapping on (τ_1, τ_2), with $\tau_1 < \tau_2$. If $|\aleph'|$ is convex on $[\tau_1, \tau_2]$, then:

$$\left| \frac{\aleph(\tau_1) + \aleph(\tau_2)}{2} - \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \aleph(\lambda) d\lambda \right| \leq \frac{\tau_2 - \tau_1}{8} \left[|\aleph'(\tau_1)| + |\aleph'(\tau_2)| \right].$$

Another significant inequality in the literature is known as Simpson's integral inequality [6], which yields an error bound for the well-known Simpson's rule and is defined as:

Let $\aleph : [\tau_1, \tau_2] \to \mathbb{R}$ be four times continuously differentiable mapping on (τ_1, τ_2) and $\|\aleph^{(4)}\|_\infty < \infty$, then:

$$\left| \frac{1}{3} \left[\frac{\aleph(\tau_1) + \aleph(\tau_2)}{2} + 2\aleph\left(\frac{\tau_1 + \tau_2}{2}\right) \right] - \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} \aleph(\lambda) d\lambda \right| \leq \frac{1}{2880} \|\aleph^{(4)}\|_\infty (\tau_2 - \tau_1)^4.$$

Awan et al. [7] obtained some new generalized variants of Simpson-type inequalities based on differentiable, strongly $(\mathfrak{s}, \mathfrak{m})$-convex mappings. Further generalizations, extensions, and refinements of Simpson's integral inequality can be found in [8–11].

Over time, researchers have extended the definition of convex functions to derive different variants of the Hermite–Hadamard inequality. On the other hand, the concept of $\mathfrak{s}$-convexity [12,13] is split into two notions, which are described below, with the basic condition that $0 < \mathfrak{s} \leq 1$.

A function $\aleph : [0, \infty) \to \mathbb{R}$ is said to be an $\mathfrak{s}$-convex function in the first sense, denoted by $K_\mathfrak{s}^1$, if

$$\aleph(\varkappa_1 \tau_1 + \varkappa_2 \tau_2) \leq \varkappa_1{}^\mathfrak{s} \aleph(\tau_1) + \varkappa_2{}^\mathfrak{s} \aleph(\tau_2), \tag{2}$$

holds for all $\tau_1, \tau_2 \in [0, \infty)$ and all $\varkappa_1, \varkappa_2 \geq 0$ and $\varkappa_1{}^\mathfrak{s} + \varkappa_2{}^\mathfrak{s} = 1$.

A function $\aleph : [0, \infty) \to \mathbb{R}$ is said to be an $\mathfrak{s}$-convex function in the second sense, or $\mathfrak{s}$-Breckner convex, if the inequality (2) holds for all $\tau_1, \tau_2 \in [0, \infty)$ and all $\varkappa_1, \varkappa_2 \geq 0$ with $\varkappa_1 + \varkappa_2 = 1$. We denote this as $K_\mathfrak{s}^2$. Of course, both $\mathfrak{s}$-convexities reduce to standard convexity when $\mathfrak{s} = 1$.

The geometrical meaning of $\mathfrak{s}$-convexity $(0 < \mathfrak{s} < 1)$ is that the graph of the function lies below a curved chord L that is located between any two points.

Example 1. *Let $0 < \mathfrak{s} < 1$ and $a, b, c \in \mathbb{R}$. By defining, for $u \in [0, \infty)$,*

$$\aleph(u) = \begin{cases} a & if \;\; u = 0, \\ bu^\mathfrak{s} + c & if \;\; u > 0, \end{cases}$$

we have the following:

1. *If $b \geq 0$ and $c \leq a$, then $\aleph \in K_\mathfrak{s}^1$.*
2. *If $b \geq 0$ and $c < a$, then $\aleph$ is non-decreasing on $(0, \infty)$ but not on $[0, \infty)$.*
3. *If $b \geq 0$ and $0 \leq c \leq a$, then $\aleph \in K_\mathfrak{s}^2$.*
4. *If $b > 0$ and $c < 0$, then $\aleph \notin K_\mathfrak{s}^2$.*

For $\mathfrak{s}$-convexity in the first and second senses, Dragomir and Fitzpatrick [14] described the respective Hermite–Hadamard-type inequalities as follows:

Suppose that $\aleph : [0,\infty) \to [0,\infty)$ is an $\mathfrak{s}$-convex mapping in the first sense, where $\mathfrak{s} \in (0,1]$, and let $\tau_1, \tau_2 \in [0,\infty)$ and $\tau_1 < \tau_2$. Then, the following inequalities hold:

$$\aleph\left(\frac{\tau_1+\tau_2}{2}\right) \leq \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2} \aleph(\lambda)d\lambda \leq \frac{\aleph(\tau_1)+\mathfrak{s}\aleph(\tau_2)}{\mathfrak{s}+1}. \tag{3}$$

Suppose that $\aleph : [0,\infty) \to [0,\infty)$ is an $\mathfrak{s}$-convex mapping in the second sense, where $\mathfrak{s} \in (0,1]$, and let $\tau_1, \tau_2 \in [0,\infty)$ and $\tau_1 < \tau_2$. Then, the following inequalities hold:

$$2^{\mathfrak{s}-1}\aleph\left(\frac{\tau_1+\tau_2}{2}\right) \leq \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2} \aleph(\lambda)d\lambda \leq \frac{\aleph(\tau_1)+\aleph(\tau_2)}{\mathfrak{s}+1}. \tag{4}$$

Further generalizations and extensions of classical convex functions can be found in [15–19].

The study of integrals and derivatives of arbitrary real order is known as fractional calculus. The goal of fractional integrals is to address various problems involving special functions of mathematical science, as well as their extensions and generalizations to one or more variables. Additionally, fractional-order derivatives are much better at describing the memory and hereditary properties of various processes compared to classical derivatives. In fact, the latest advancements in fractional calculus have been driven by current applications in physics, differential and integral equations, signal processing, fluid mechanics, mathematical biology, and electrochemistry. There is no doubt that various diverse problems in mathematics, engineering, and science can be addressed through the application of fractional calculus [20–22]. A detailed history of fractional calculus can be found in [23].

Sarikaya and Ertuğral [24] introduced the idea of generalized fractional integrals and derived Hadamard-type inequalities. The generalized fractional integrals from both the left and right sides of the interval $[\tau_1, \tau_2]$ are defined as:

$${}_{\tau_1^+}I_{\varphi}\aleph(y) = \int_{\tau_1}^{y} \frac{\varphi(y-\lambda)}{y-\lambda}\aleph(\lambda)d\lambda, \quad y > \tau_1, \tag{5}$$

and

$${}_{\tau_2^-}I_{\varphi}\aleph(y) = \int_{y}^{\tau_2} \frac{\varphi(\lambda-y)}{\lambda-y}\aleph(\lambda)d\lambda, \quad y < \tau_2, \tag{6}$$

where $\varphi : [0,\infty) \to [0,\infty)$ is the mapping satisfying the following condition:

$$\int_0^1 \frac{\varphi(\lambda)}{\lambda}d\lambda < \infty.$$

For some suitable choices of the mapping φ in (5) and (6), we can obtain Riemann–Liouville fractional integrals, k-Riemann–Liouville fractional integrals, Katugampola fractional integrals, conformable fractional integrals, and Hadamard fractional integrals as special cases.

From (5) and (6), the following fractional integrals are obtained:

1. For $\varphi(\lambda) = \lambda$, the resulting integrals are Riemann integrals:

$$I_{\tau_1^+}\aleph(y) = \int_{\tau_1}^{y} \aleph(\lambda)d\lambda, \quad y > \tau_1,$$
$$I_{\tau_2^-}\aleph(y) = \int_{y}^{\tau_2} \aleph(\lambda)d\lambda, \quad y < \tau_2.$$

2. By setting $\varphi(\lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)}$ and $\alpha > 0$, the resulting integrals are Riemann–Liouville integrals:

$$I^{\alpha}_{\tau_1^+}\aleph(y) = \frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{y}(y-\lambda)^{\alpha-1}\aleph(\lambda)d\lambda, \quad y > \tau_1,$$
$$I^{\alpha}_{\tau_2^-}\aleph(y) = \frac{1}{\Gamma(\alpha)}\int_{y}^{\tau_2}(\lambda-y)^{\alpha-1}\aleph(\lambda)d\lambda, \quad y < \tau_2,$$

where Γ is the gamma mapping.

3. By taking $\varphi(\lambda) = \frac{1}{k\Gamma_k(\alpha)}\lambda^{\frac{\alpha}{k}}$ and $\alpha, k > 0$, the resulting integrals are k-Riemann–Liouville fractional integrals provided in [25] and defined as:

$$I^{\alpha}_{\tau_1^+,k}\aleph(y) = \frac{1}{k\Gamma_k(\alpha)}\int_{\tau_1}^{y}(y-\lambda)^{\frac{\alpha}{k}-1}\aleph(\lambda)d\lambda, \quad y > \tau_1,$$
$$I^{\alpha}_{\tau_2^-,k}\aleph(y) = \frac{1}{k\Gamma_k(\alpha)}\int_{y}^{\tau_2}(\lambda-y)^{\frac{\alpha}{k}-1}\aleph(\lambda)d\lambda, \quad y < \tau_2,$$

where

$$\Gamma_k(\alpha) = \int_0^\infty \lambda^{\alpha-1}e^{\frac{-\lambda^k}{k}}d\lambda, \quad Re(\alpha) > 0,$$

and

$$\Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1}\Gamma\left(\frac{\alpha}{k}\right), \quad \Gamma_k(\alpha+k) = \alpha\Gamma_k(\alpha), \quad Re(\alpha) > 0; \quad k > 0.$$

Now, let us recall some special functions, which we will use in our calculations:
The Euler gamma mapping, or Euler integral of the second kind, is defined as:

$$\Gamma(\alpha) = \int_0^\infty \lambda^{\alpha-1}e^{-\lambda}d\lambda, \quad Re(\alpha) > 0.$$

The beta mapping, or Euler integral of the first kind with two variables, is defined as:

$$\mathrm{B}(v_1, v_2) = \int_0^1 \lambda^{v_1-1}(1-\lambda)^{v_2-1}d\lambda, \quad Re(v_1) > 0, \quad Re(v_2) > 0. \tag{7}$$

In terms of gamma mapping, it is defined as:

$$\mathrm{B}(v_1, v_2) = \frac{\Gamma(v_1)\Gamma(v_2)}{\Gamma(v_1+v_2)}.$$

The incomplete beta mapping, which is a generalization of the beta mapping, is defined in [26] as:

$$\mathrm{B}_x(v_1, v_2) = \mathrm{B}(x : v_1, v_2) = \int_0^x \lambda^{v_1-1}(1-\lambda)^{v_2-1}d\lambda, \quad Re(v_1) > 0, \quad Re(v_2) > 0. \tag{8}$$

When $x = 1$ in (8), it coincides with the beta mapping.
The hypergeometric mapping is:

$${}_2F_1(v_1, v_2; \mathfrak{c}; z) = \frac{1}{B(v_2, \mathfrak{c}-v_2)}\int_0^1 \lambda^{v_2-1}(1-\lambda)^{\mathfrak{c}-v_2-1}(1-z\lambda)^{-v_1}d\lambda, \quad Re(\mathfrak{c}) > Re(v_2) > 0, \quad |z| < 1. \tag{9}$$

This research article is organized as follows. In Section 2, we derive a new general parameterized integral identity for differentiable mappings. We also present various additional results that can be deduced from this new identity. In Section 3, we derive some new parameterized inequalities involving generalized fractional integrals for differentiable $\mathfrak{s}$-convex mappings of the second kind, utilizing the identity derived in Section 2. Some detailed graphical visualizations of our main findings are presented in Section 4, which shows the significance and validity of our results. In Section 5, some applications to special means of real numbers and quadrature formulas are presented. As an application, we also derive a new generalized numerical scheme. To the best of our knowledge, this is the first study in the literature pertaining to applications of integral inequalities in numerical analysis. We hope that the ideas and techniques presented in this paper will inspire interested readers working in this field.

2. A Parameterized Integral Identity Involving Generalized Fractional Integrals

In this section, a parameterized identity involving generalized fractional integrals is derived. Further, for some suitable choices of the given parameters, Simpson's, midpoint, and trapezoidal-type identities are also derived.

Lemma 1. *For a differentiable mapping $\aleph : [\tau_1, \tau_2] \to \mathbb{R}$ on (τ_1, τ_2) with continuous and integrable derivative $\aleph'$ on $[\tau_1, \tau_2]$, the following equality holds for $\rho, \sigma \geq 0$ and $n \in \mathbb{N}$:*

$$\begin{aligned}
&(1-\sigma)\aleph(\tau_1) + (1-\rho)\aleph(\tau_2) + \sigma\aleph\left(\frac{n\tau_1+\tau_2}{n+1}\right) + \rho\aleph\left(\frac{\tau_1+n\tau_2}{n+1}\right) \\
&- \frac{1}{\Delta(1)}\left[{}_{\tau_1^+}I_{\varphi}\aleph\left(\frac{n\tau_1+\tau_2}{n+1}\right) + {}_{\tau_2^-}I_{\varphi}\aleph\left(\frac{\tau_1+n\tau_2}{n+1}\right)\right] \\
&= \frac{\tau_2-\tau_1}{n+1}\frac{1}{\Delta(1)}\left[\int_0^1 (\Delta(\lambda)-\Delta(1)\rho)\aleph'\left(\frac{1-\lambda}{n+1}\tau_1 + \frac{n+\lambda}{n+1}\tau_2\right)d\lambda \right. \\
&\left. + \int_0^1 (\Delta(1)\sigma - \Delta(\lambda))\aleph'\left(\frac{n+\lambda}{n+1}\tau_1 + \frac{1-\lambda}{n+1}\tau_2\right)d\lambda\right], \qquad (10)
\end{aligned}$$

where $\Delta : [0,1] \to \mathbb{R}$ is defined as

$$\Delta(\lambda) = \int_0^\lambda \frac{\varphi\left(\left(\frac{\tau_2-\tau_1}{n+1}\right)\mu\right)}{\mu} d\mu.$$

Proof. Let

$$Y_1 = \int_0^1 (\Delta(\lambda) - \Delta(1)\rho)\aleph'\left(\frac{1-\lambda}{n+1}\tau_1 + \frac{n+\lambda}{n+1}\tau_2\right)d\lambda.$$

By applying integration by parts, we obtain

$$
\begin{aligned}
&= \frac{n+1}{\tau_2-\tau_1}(\Delta(\lambda)-\Delta(1)\rho)\aleph\left(\frac{1-\lambda}{n+1}\tau_1+\frac{n+\lambda}{n+1}\tau_2\right)\Bigg|_0^1 \\
&- \frac{n+1}{\tau_2-\tau_1}\int_0^1 \frac{\varphi\left(\left(\frac{\tau_2-\tau_1}{n+1}\right)\lambda\right)}{\lambda}\aleph\left(\frac{1-\lambda}{n+1}\tau_1+\frac{n+\lambda}{n+1}\tau_2\right)d\lambda \\
&= \frac{n+1}{\tau_2-\tau_1}\Bigg[\Delta(1)\left((1-\rho)\aleph(\tau_2)+\rho\aleph\left(\frac{\tau_1+n\tau_2}{n+1}\right)\right) \\
&- \int_{\frac{\tau_1+n\tau_2}{n+1}}^{\tau_2} \aleph(\mu)\varphi\left(\mu-\frac{\tau_1+n\tau_2}{n+1}\right)\frac{1}{\mu-\frac{\tau_1+n\tau_2}{n+1}}d\mu\Bigg] \\
&= \frac{n+1}{\tau_2-\tau_1}\left[\Delta(1)\left((1-\rho)\aleph(\tau_2)+\rho\aleph\left(\frac{\tau_1+n\tau_2}{n+1}\right)\right)-\left({}_{\tau_2^-}I_\varphi\aleph\left(\frac{\tau_1+n\tau_2}{n+1}\right)\right)\right]. \qquad (11)
\end{aligned}
$$

and

$$
\begin{aligned}
Y_2 &= \int_0^1 (\Delta(1)\sigma-\Delta(\lambda))\aleph'\left(\frac{n+\lambda}{n+1}\tau_1+\frac{1-\lambda}{n+1}\tau_2\right)d\lambda \\
&= -\frac{n+1}{\tau_2-\tau_1}(\Delta(1)\sigma-\Delta(\lambda))\aleph\left(\frac{n+\lambda}{n+1}\tau_1+\frac{1-\lambda}{n+1}\tau_2\right)\Bigg|_0^1 \\
&+ \frac{n+1}{\tau_2-\tau_1}\int_0^1 \frac{\varphi\left(\left(\frac{\tau_2-\tau_1}{n+1}\right)\lambda\right)}{\lambda}\aleph\left(\frac{n+\lambda}{n+1}\tau_1+\frac{1-\lambda}{n+1}\tau_2\right)d\lambda \\
&= \frac{n+1}{\tau_2-\tau_1}\Bigg[\Delta(1)\left((1-\sigma)\aleph(\tau_1)+\sigma\aleph\left(\frac{n\tau_1+\tau_2}{n+1}\right)\right) \\
&- \int_{\tau_1}^{\frac{n\tau_1+\tau_2}{n+1}} \aleph(\mu)\varphi\left(\frac{n\tau_1+\tau_2}{n+1}-\mu\right)\frac{1}{\frac{n\tau_1+\tau_2}{n+1}-\mu}d\mu\Bigg] \\
&= \frac{n+1}{\tau_2-\tau_1}\left[\Delta(1)\left((1-\sigma)\aleph(\tau_1)+\sigma\aleph\left(\frac{n\tau_1+\tau_2}{n+1}\right)\right)-\left({}_{\tau_1^+}I_\varphi\aleph\left(\frac{n\tau_1+\tau_2}{n+1}\right)\right)\right]. \qquad (12)
\end{aligned}
$$

Now, by adding (11) and (12) and multiplying by $\frac{\tau_2-\tau_1}{n+1}\frac{1}{\Delta(1)}$, we obtain

$$
\begin{aligned}
=&(1-\sigma)\aleph(\tau_1)+(1-\rho)\aleph(\tau_2)+\sigma\aleph\left(\frac{n\tau_1+\tau_2}{n+1}\right)+\rho\aleph\left(\frac{\tau_1+n\tau_2}{n+1}\right) \\
&-\frac{1}{\Delta(1)}\left[{}_{\tau_1^+}I_\varphi\aleph\left(\frac{n\tau_1+\tau_2}{n+1}\right)+{}_{\tau_2^-}I_\varphi\aleph\left(\frac{\tau_1+n\tau_2}{n+1}\right)\right].
\end{aligned}
$$

The proof is completed. □

Remark 1. *From Lemma 1:*

1. *By setting $\rho=\sigma=\frac{n+1}{n+2}$, the resulting identity is identical to Lemma 3 in [27] for $n=1$*
2. *By setting $\rho=\sigma=0$, the resulting identity is identical to Corollary 5.2 in [28] for $n=1$.*

Corollary 1. *By setting $\varphi(\lambda) = \lambda$ in Lemma 1, the following equality for Riemann integrals is obtained:*

$$\frac{1}{n+1}\left[(1-\sigma)\aleph(\tau_1)+(1-\rho)\aleph(\tau_2)+\sigma\aleph\left(\frac{n\tau_1+\tau_2}{n+1}\right)+\rho\aleph\left(\frac{\tau_1+n\tau_2}{n+1}\right)\right]-\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\aleph(\lambda)d\lambda$$
$$=\frac{\tau_2-\tau_1}{(n+1)^2}\left[\int_0^1(\lambda-\rho)\aleph'\left(\frac{1-\lambda}{n+1}\tau_1+\frac{n+\lambda}{n+1}\tau_2\right)d\lambda\right.$$
$$\left.+\int_0^1(\sigma-\lambda)\aleph'\left(\frac{n+\lambda}{n+1}\tau_1+\frac{1-\lambda}{n+1}\tau_2\right)d\lambda\right].$$

Remark 2. *From Corollary 1:*

1. *By setting $\rho = \sigma = \frac{n+1}{n+2}$, the resulting identity is identical to Lemma 1 in [10] for $n = 1$.*
2. *By setting $\rho = \sigma = 0$, the resulting identity is identical to Corollary 5.2 in [28] for $n = 1$.*

Corollary 2. *By setting $\varphi(\lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)}, \alpha > 0$ in Lemma 1, the following equality is obtained for Riemann–Liouville fractional integrals:*

$$(1-\sigma)\aleph(\tau_1)+(1-\rho)\aleph(\tau_2)+\sigma\aleph\left(\frac{n\tau_1+\tau_2}{n+1}\right)+\rho\aleph\left(\frac{\tau_1+n\tau_2}{n+1}\right)$$
$$-\frac{(n+1)^\alpha\Gamma(\alpha+1)}{(\tau_2-\tau_1)^\alpha}\left[I^\alpha_{\tau_1^+}\aleph\left(\frac{n\tau_1+\tau_2}{n+1}\right)+I^\alpha_{\tau_2^-}\aleph\left(\frac{\tau_1+n\tau_2}{n+1}\right)\right]$$
$$=\frac{\tau_2-\tau_1}{n+1}\left[\int_0^1(\lambda^\alpha-\rho)\aleph'\left(\frac{1-\lambda}{n+1}\tau_1+\frac{n+\lambda}{n+1}\tau_2\right)d\lambda\right.$$
$$\left.+\int_0^1(\sigma-\lambda^\alpha)\aleph'\left(\frac{n+\lambda}{n+1}\tau_1+\frac{1-\lambda}{n+1}\tau_2\right)d\lambda\right].$$

Remark 3. *From Corollary 2:*

1. *By setting $\rho = \sigma = \frac{n+1}{n+2}$, the resulting identity is identical to Lemma 2.1 in [29] for $n = 1$.*
2. *By setting $\rho = \sigma = 0$, the resulting identity is identical to Corollary 5.3 in [28] for $n = 1$.*

Corollary 3. *By setting $\varphi(\lambda) = \frac{\lambda^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ for $\alpha, k > 0$ in Lemma 1, the following equality is obtained for k-Riemann–Liouville fractional integrals:*

$$(1-\sigma)\aleph(\tau_1)+(1-\rho)\aleph(\tau_2)+\sigma\aleph\left(\frac{n\tau_1+\tau_2}{n+1}\right)+\rho\aleph\left(\frac{\tau_1+n\tau_2}{n+1}\right)$$
$$-\frac{(n+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{(\tau_2-\tau_1)^{\frac{\alpha}{k}}}\left[I^\alpha_{\tau_1^+,k}\aleph\left(\frac{n\tau_1+\tau_2}{n+1}\right)+I^\alpha_{\tau_2^-,k}\aleph\left(\frac{\tau_1+n\tau_2}{n+1}\right)\right]$$
$$=\frac{\tau_2-\tau_1}{n+1}\left[\int_0^1\left(\lambda^{\frac{\alpha}{k}}-\rho\right)\aleph'\left(\frac{1-\lambda}{n+1}\tau_1+\frac{n+\lambda}{n+1}\tau_2\right)d\lambda\right.$$
$$\left.+\int_0^1\left(\sigma-\lambda^{\frac{\alpha}{k}}\right)\aleph'\left(\frac{n+\lambda}{n+1}\tau_1+\frac{1-\lambda}{n+1}\tau_2\right)d\lambda\right].$$

Remark 4. *From Corollary 3:*

1. *By setting $\rho = \sigma = \frac{n+1}{n+2}$, the resulting identity is identical to Corollary 1 in [27] for $n = 1$.*
2. *By setting $\rho = \sigma = 0$, the resulting identity is identical to Corollary 5.4 in [28] for $n = 1$.*

3. Some Parameterized Inequalities Involving Generalized Fractional Integrals

In this section, we establish some parameterized inequalities involving generalized fractional integrals for differentiable $\mathfrak{s}$-convex mappings of the second kind.

Theorem 1. *Let all the conditions of Lemma 1 be satisfied. If $|\aleph'|$ is an $\mathfrak{s}$-convex mapping on $[\tau_1,\tau_2]$ for $\mathfrak{s}\in(0,1]$ and $n\in\mathbb{N}$, then:*

$$\begin{aligned}&\bigg|(1-\sigma)\aleph(\tau_1)+(1-\rho)\aleph(\tau_2)+\sigma\aleph\left(\frac{n\tau_1+\tau_2}{n+1}\right)+\rho\aleph\left(\frac{\tau_1+n\tau_2}{n+1}\right)\\&-\frac{1}{\Delta(1)}\left[{}_{\tau_1^+}I_\varphi\aleph\left(\frac{n\tau_1+\tau_2}{n+1}\right)+{}_{\tau_2^-}I_\varphi\aleph\left(\frac{\tau_1+n\tau_2}{n+1}\right)\right]\bigg|\\&\le\frac{\tau_2-\tau_1}{(n+1)^{\mathfrak{s}+1}}\frac{1}{\Delta(1)}\Big[|\aleph'(\tau_1)|\Big(\Pi_1^\varphi(\rho,\mathfrak{s})+\Pi_2^\varphi(\sigma,\mathfrak{s},n)\Big)+|\aleph'(\tau_2)|\Big(\Pi_2^\varphi(\rho,\mathfrak{s},n)+\Pi_1^\varphi(\sigma,\mathfrak{s})\Big)\Big],\end{aligned}\tag{13}$$

where

$$\Pi_1^\varphi(\delta,\mathfrak{s})=\int_0^1(1-\lambda)^{\mathfrak{s}}|\Delta(\lambda)-\Delta(1)\delta|d\lambda$$

and

$$\Pi_2^\varphi(\delta,\mathfrak{s},n)=\int_0^1(n+\lambda)^{\mathfrak{s}}|\Delta(\lambda)-\Delta(1)\delta|d\lambda.$$

Proof. By taking the modulus in Lemma 1 and applying the $\mathfrak{s}$-convexity of $|\aleph'|$, we obtain

$$\begin{aligned}&\bigg|(1-\sigma)\aleph(\tau_1)+(1-\rho)\aleph(\tau_2)+\sigma\aleph\left(\frac{n\tau_1+\tau_2}{n+1}\right)+\rho\aleph\left(\frac{\tau_1+n\tau_2}{n+1}\right)\\&-\frac{1}{\Delta(1)}\left[{}_{\tau_1^+}I_\varphi\aleph\left(\frac{n\tau_1+\tau_2}{n+1}\right)+{}_{\tau_2^-}I_\varphi\aleph\left(\frac{\tau_1+n\tau_2}{n+1}\right)\right]\bigg|\\&=\bigg|\frac{\tau_2-\tau_1}{n+1}\frac{1}{\Delta(1)}\left[\int_0^1(\Delta(\lambda)-\Delta(1)\rho)\aleph'\left(\frac{1-\lambda}{n+1}\tau_1+\frac{n+\lambda}{n+1}\tau_2\right)d\lambda\right.\\&\left.+\int_0^1(\Delta(1)\sigma-\Delta(\lambda))\aleph'\left(\frac{n+\lambda}{n+1}\tau_1+\frac{1-\lambda}{n+1}\tau_2\right)d\lambda\right]\bigg|\\&\le\frac{\tau_2-\tau_1}{n+1}\frac{1}{\Delta(1)}\left[\int_0^1|\Delta(\lambda)-\Delta(1)\rho|\left|\aleph'\left(\frac{1-\lambda}{n+1}\tau_1+\frac{n+\lambda}{n+1}\tau_2\right)\right|d\lambda\right.\\&\left.+\int_0^1|\Delta(1)\sigma-\Delta(\lambda)|\left|\aleph'\left(\frac{n+\lambda}{n+1}\tau_1+\frac{1-\lambda}{n+1}\tau_2\right)\right|d\lambda\right]\\&\le\frac{\tau_2-\tau_1}{n+1}\frac{1}{\Delta(1)}\left[\int_0^1|\Delta(\lambda)-\Delta(1)\rho|\left[\left(\frac{1-\lambda}{n+1}\right)^{\mathfrak{s}}|\aleph'(\tau_1)|+\left(\frac{n+\lambda}{n+1}\right)^{\mathfrak{s}}|\aleph'(\tau_2)|\right]d\lambda\right.\\&\left.+\int_0^1|\Delta(1)\sigma-\Delta(\lambda)|\left[\left(\frac{n+\lambda}{n+1}\right)^{\mathfrak{s}}|\aleph'(\tau_1)|+\left(\frac{1-\lambda}{n+1}\right)^{\mathfrak{s}}|\aleph'(\tau_2)|\right]d\lambda\right]\\&\le\frac{\tau_2-\tau_1}{(n+1)^{\mathfrak{s}+1}}\frac{1}{\Delta(1)}\left[|\aleph'(\tau_1)|\left(\int_0^1|\Delta(\lambda)-\Delta(1)\rho|(1-\lambda)^{\mathfrak{s}}d\lambda+\int_0^1|\Delta(1)\sigma-\Delta(\lambda)|(n+\lambda)^{\mathfrak{s}}d\lambda\right)\right.\\&\left.+|\aleph'(\tau_2)|\left(\int_0^1|\Delta(\lambda)-\Delta(1)\rho|(n+\lambda)^{\mathfrak{s}}d\lambda+\int_0^1|\Delta(1)\sigma-\Delta(\lambda)|(1-\lambda)^{\mathfrak{s}}d\lambda\right)\right]\\&\le\frac{\tau_2-\tau_1}{(n+1)^{\mathfrak{s}+1}}\frac{1}{\Delta(1)}\Big[|\aleph'(\tau_1)|\Big(\Pi_1^\varphi(\rho,\mathfrak{s})+\Pi_2^\varphi(\sigma,\mathfrak{s},n)\Big)+|\aleph'(\tau_2)|\Big(\Pi_2^\varphi(\rho,\mathfrak{s},n)+\Pi_1^\varphi(\sigma,\mathfrak{s})\Big)\Big].\end{aligned}$$

The proof is completed. □

Theorem 2. *Let all the conditions of Lemma 1 be satisfied. If* $|\aleph'|^q$ *is an* $\mathfrak{s}$*-convex mapping on* $[\tau_1, \tau_2]$ *for* $\mathfrak{s} \in (0,1]$, $n \in \mathbb{N}$ *and* $q > 1$, *then:*

$$\begin{aligned}
&\bigg|(1-\sigma)\aleph(\tau_1) + (1-\rho)\aleph(\tau_2) + \sigma\aleph\left(\frac{n\tau_1+\tau_2}{n+1}\right) + \rho\aleph\left(\frac{\tau_1+n\tau_2}{n+1}\right) \\
&-\frac{1}{\Delta(1)}\left[{}_{\tau_1^+}I_\varphi\aleph\left(\frac{n\tau_1+\tau_2}{n+1}\right) + {}_{\tau_2^-}I_\varphi\aleph\left(\frac{\tau_1+n\tau_2}{n+1}\right)\right]\bigg| \\
&\leq \frac{\tau_2-\tau_1}{n+1}\frac{1}{\Delta(1)}\left[\left(\int_0^1 |\Delta(\lambda)-\Delta(1)\rho|^{\frac{q}{q-1}}d\lambda\right)^{\frac{q-1}{q}}\right. \\
&\left(\frac{1}{(\mathfrak{s}+1)(n+1)^{\mathfrak{s}}}|\aleph'(\tau_1)|^q + \frac{(n+1)^{\mathfrak{s}+1}-n^{\mathfrak{s}+1}}{(\mathfrak{s}+1)(n+1)^{\mathfrak{s}}}|\aleph'(\tau_2)|^q\right)^{\frac{1}{q}} \\
&+\left(\int_0^1 |\Delta(1)\sigma-\Delta(\lambda)|^{\frac{q}{q-1}}d\lambda\right)^{\frac{q-1}{q}} \\
&\left.\left(\frac{(n+1)^{\mathfrak{s}+1}-n^{\mathfrak{s}+1}}{(\mathfrak{s}+1)(n+1)^{\mathfrak{s}}}|\aleph'(\tau_1)|^q + \frac{1}{(\mathfrak{s}+1)(n+1)^{\mathfrak{s}}}|\aleph'(\tau_2)|^q\right)^{\frac{1}{q}}\right]. \qquad (14)
\end{aligned}$$

Proof. By using Lemma 1 and the Hölder integral inequality, we have

$$\begin{aligned}
&\bigg|(1-\sigma)\aleph(\tau_1) + (1-\rho)\aleph(\tau_2) + \sigma\aleph\left(\frac{n\tau_1+\tau_2}{n+1}\right) + \rho\aleph\left(\frac{\tau_1+n\tau_2}{n+1}\right) \\
&-\frac{1}{\Delta(1)}\left[{}_{\tau_1^+}I_\varphi\aleph\left(\frac{n\tau_1+\tau_2}{n+1}\right) + {}_{\tau_2^-}I_\varphi\aleph\left(\frac{\tau_1+n\tau_2}{n+1}\right)\right]\bigg| \\
&= \bigg|\frac{\tau_2-\tau_1}{n+1}\frac{1}{\Delta(1)}\left[\int_0^1 (\Delta(\lambda)-\Delta(1)\rho)\aleph'\left(\frac{1-\lambda}{n+1}\tau_1 + \frac{n+\lambda}{n+1}\tau_2\right)d\lambda\right. \\
&\left.+\int_0^1 (\Delta(1)\sigma-\Delta(\lambda))\aleph'\left(\frac{n+\lambda}{n+1}\tau_1 + \frac{1-\lambda}{n+1}\tau_2\right)d\lambda\right]\bigg| \\
&\leq \frac{\tau_2-\tau_1}{n+1}\frac{1}{\Delta(1)}\left[\left(\int_0^1 |\Delta(\lambda)-\Delta(1)\rho|^{\frac{q}{q-1}}d\lambda\right)^{\frac{q-1}{q}}\left(\int_0^1 \left|\aleph'\left(\frac{1-\lambda}{n+1}\tau_1 + \frac{n+\lambda}{n+1}\tau_2\right)\right|^q d\lambda\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1 |\Delta(1)\sigma-\Delta(\lambda)|^{\frac{q}{q-1}}d\lambda\right)^{\frac{q-1}{q}}\left(\int_0^1 \left|\aleph'\left(\frac{n+\lambda}{n+1}\tau_1 + \frac{1-\lambda}{n+1}\tau_2\right)\right|^q d\lambda\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Since $|\aleph'|^q$ is $\mathfrak{s}$-convex, we obtain

$$\leq \frac{\tau_2-\tau_1}{n+1}\frac{1}{\Delta(1)}\left[\left(\int_0^1 |\Delta(\lambda)-\Delta(1)\rho|^{\frac{q}{q-1}}d\lambda\right)^{\frac{q-1}{q}}\right.$$
$$\left(\int_0^1 \left[\left(\frac{1-\lambda}{n+1}\right)^{\mathfrak{s}}|\aleph'(\tau_1)|^q+\left(\frac{n+\lambda}{n+1}\right)^{\mathfrak{s}}|\aleph'(\tau_2)|^q\right]d\lambda\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1 |\Delta(1)\sigma-\Delta(\lambda)|^{\frac{q}{q-1}}d\lambda\right)^{\frac{q-1}{q}}$$
$$\left.\left(\int_0^1 \left[\left(\frac{n+\lambda}{n+1}\right)^{\mathfrak{s}}|\aleph'(\tau_1)|^q+\left(\frac{1-\lambda}{n+1}\right)^{\mathfrak{s}}|\aleph'(\tau_2)|^q\right]d\lambda\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{\tau_2-\tau_1}{n+1}\frac{1}{\Delta(1)}\left[\left(\int_0^1 |\Delta(\lambda)-\Delta(1)\rho|^{\frac{q}{q-1}}d\lambda\right)^{\frac{q-1}{q}}\right.$$
$$\left(\frac{1}{(\mathfrak{s}+1)(n+1)^{\mathfrak{s}}}|\aleph'(\tau_1)|^q+\frac{(n+1)^{\mathfrak{s}+1}-n^{\mathfrak{s}+1}}{(\mathfrak{s}+1)(n+1)^{\mathfrak{s}}}|\aleph'(\tau_2)|^q\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1 |\Delta(1)\sigma-\Delta(\lambda)|^{\frac{q}{q-1}}d\lambda\right)^{\frac{q-1}{q}}$$
$$\left.\left(\frac{(n+1)^{\mathfrak{s}+1}-n^{\mathfrak{s}+1}}{(\mathfrak{s}+1)(n+1)^{\mathfrak{s}}}|\aleph'(\tau_1)|^q+\frac{1}{(\mathfrak{s}+1)(n+1)^{\mathfrak{s}}}|\aleph'(\tau_2)|^q\right)^{\frac{1}{q}}\right].$$

The proof is completed. □

Theorem 3. *Let all the conditions of Lemma 1 be satisfied. If $|\aleph'|^q$ is an $\mathfrak{s}$-convex mapping on $[\tau_1,\tau_2]$ for $\mathfrak{s}\in(0,1]$, $n\in\mathbb{N}$ and $q>1$, then:*

$$\left|(1-\sigma)\aleph(\tau_1)+(1-\rho)\aleph(\tau_2)+\sigma\aleph\left(\frac{n\tau_1+\tau_2}{n+1}\right)+\rho\aleph\left(\frac{\tau_1+n\tau_2}{n+1}\right)\right.$$
$$\left.-\frac{1}{\Delta(1)}\left[{}_{\tau_1^+}I_{\varphi}\aleph\left(\frac{n\tau_1+\tau_2}{n+1}\right)+{}_{\tau_2^-}I_{\varphi}\aleph\left(\frac{\tau_1+n\tau_2}{n+1}\right)\right]\right|$$
$$\leq \frac{\tau_2-\tau_1}{n+1}\frac{1}{\Delta(1)}\left[\left(\int_0^1 |\Delta(\lambda)-\Delta(1)\rho|d\lambda\right)^{1-\frac{1}{q}}\left(\frac{|\aleph'(\tau_1)|^q\Pi_1^{\varphi}(\rho,\mathfrak{s})+|\aleph'(\tau_2)|^q\Pi_2^{\varphi}(\rho,\mathfrak{s},n)}{(n+1)^{\mathfrak{s}}}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1 |\Delta(1)\sigma-\Delta(\lambda)|d\lambda\right)^{1-\frac{1}{q}}\left(\frac{|\aleph'(\tau_1)|^q\Pi_2^{\varphi}(\sigma,\mathfrak{s},n)+|\aleph'(\tau_2)|^q\Pi_1^{\varphi}(\sigma,\mathfrak{s})}{(n+1)^{\mathfrak{s}}}\right)^{\frac{1}{q}}\right], \quad (15)$$

where $\Pi_1^{\varphi}(\delta,\mathfrak{s})$ and $\Pi_2^{\varphi}(\delta,\mathfrak{s},n)$ are defined in Theorem 1.

Proof. By using Lemma 1 and the power mean integral inequality, we deduce that

$$\Bigg|(1-\sigma)\aleph(\tau_1)+(1-\rho)\aleph(\tau_2)+\sigma\aleph\left(\frac{n\tau_1+\tau_2}{n+1}\right)+\rho\aleph\left(\frac{\tau_1+n\tau_2}{n+1}\right)$$

$$-\frac{1}{\Delta(1)}\left[{}_{\tau_1^+}I_\varphi\aleph\left(\frac{n\tau_1+\tau_2}{n+1}\right)+{}_{\tau_2^-}I_\varphi\aleph\left(\frac{\tau_1+n\tau_2}{n+1}\right)\right]\Bigg|$$

$$=\Bigg|\frac{\tau_2-\tau_1}{n+1}\frac{1}{\Delta(1)}\Bigg[\int_0^1(\Delta(\lambda)-\Delta(1)\rho)\aleph'\left(\frac{1-\lambda}{n+1}\tau_1+\frac{n+\lambda}{n+1}\tau_2\right)d\lambda$$

$$+\int_0^1(\Delta(1)\sigma-\Delta(\lambda))\aleph'\left(\frac{n+\lambda}{n+1}\tau_1+\frac{1-\lambda}{n+1}\tau_2\right)d\lambda\Bigg]\Bigg|$$

$$\leq\frac{\tau_2-\tau_1}{n+1}\frac{1}{\Delta(1)}\Bigg[\left(\int_0^1|\Delta(\lambda)-\Delta(1)\rho|d\lambda\right)^{1-\frac{1}{q}}\left(\int_0^1|\Delta(\lambda)-\Delta(1)\rho|\left|\aleph'\left(\frac{1-\lambda}{n+1}\tau_1+\frac{n+\lambda}{n+1}\tau_2\right)\right|^q d\lambda\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1|\Delta(1)\sigma-\Delta(\lambda)|d\lambda\right)^{1-\frac{1}{q}}\left(\int_0^1|\Delta(1)\sigma-\Delta(\lambda)|\left|\aleph'\left(\frac{n+\lambda}{n+1}\tau_1+\frac{1-\lambda}{n+1}\tau_2\right)\right|^q d\lambda\right)^{\frac{1}{q}}\Bigg].$$

Since $|\aleph'|^q$ is $\mathfrak{s}$-convex, we obtain

$$\leq\frac{\tau_2-\tau_1}{n+1}\frac{1}{\Delta(1)}\Bigg[\left(\int_0^1|\Delta(\lambda)-\Delta(1)\rho|d\lambda\right)^{1-\frac{1}{q}}$$

$$\left(|\aleph'(\tau_1)|^q\int_0^1|\Delta(\lambda)-\Delta(1)\rho|\left(\frac{1-\lambda}{n+1}\right)^{\mathfrak{s}}d\lambda+|\aleph'(\tau_2)|^q\int_0^1|\Delta(\lambda)-\Delta(1)\rho|\left(\frac{n+\lambda}{n+1}\right)^{\mathfrak{s}}d\lambda\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1|\Delta(1)\sigma-\Delta(\lambda)|d\lambda\right)^{1-\frac{1}{q}}$$

$$\left(|\aleph'(\tau_1)|^q\int_0^1|\Delta(1)\sigma-\Delta(\lambda)|\left(\frac{n+\lambda}{n+1}\right)^{\mathfrak{s}}d\lambda+|\aleph'(\tau_2)|^q\int_0^1|\Delta(1)\sigma-\Delta(\lambda)|\left(\frac{1-\lambda}{n+1}\right)^{\mathfrak{s}}d\lambda\right)^{\frac{1}{q}}\Bigg]$$

$$\leq\frac{\tau_2-\tau_1}{n+1}\frac{1}{\Delta(1)}\Bigg[\left(\int_0^1|\Delta(\lambda)-\Delta(1)\rho|d\lambda\right)^{1-\frac{1}{q}}\left(\frac{|\aleph'(\tau_1)|^q\Pi_1^\varphi(\rho,\mathfrak{s})+|\aleph'(\tau_2)|^q\Pi_2^\varphi(\rho,\mathfrak{s},n)}{(n+1)^{\mathfrak{s}}}\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1|\Delta(1)\sigma-\Delta(\lambda)|d\lambda\right)^{1-\frac{1}{q}}\left(\frac{|\aleph'(\tau_1)|^q\Pi_2^\varphi(\sigma,\mathfrak{s},n)+|\aleph'(\tau_2)|^q\Pi_1^\varphi(\sigma,\mathfrak{s})}{(n+1)^{\mathfrak{s}}}\right)^{\frac{1}{q}}\Bigg].$$

The proof is completed. □

Remark 5. *From Theorems 1, 2, and 3:*

1. *By setting $\varphi(\lambda)=\lambda$, the inequalities for Riemann integrals are obtained.*
2. *By setting $\varphi(\lambda)=\frac{\lambda^\alpha}{\Gamma(\alpha)},\alpha>0$, the inequalities for Riemann–Liouville fractional integrals are obtained.*
3. *By setting $\varphi(\lambda)=\frac{\lambda^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)},\alpha,k>0$, the inequalities for k-Riemann–Liouville fractional integrals are obtained.*

4. Examples and Graphical Analysis

In this section, we validate the main results of Section 3 through various simulations and numerical examples. It is important to note that by specifying the values for $\varphi(\lambda)$ in Theorems 1–3, we recover several new and novel fractional versions of inequalities, including those involving Riemann integrals, Riemann–Liouville fractional integrals, and k-Riemann–Liouville fractional integrals. Further, by choosing several values for the parameters ρ and σ, we provide graphical visualizations of Simpson's, midpoint, and trapezoidal-type inequalities in Figures 1–3. In addition, from these simulations, one can visualize the comparison between error bounds involving different fractional operators and generalized convexity. The following assumptions are utilized in all the graphs:

$$\aleph(\lambda) = \lambda^{\mathfrak{s}},$$

where $\mathfrak{s} \in (0,1]$, $[\tau_1, \tau_2] = [1,3]$, $n = 1$, $\alpha \in (0,1]$, $k = 2$, and $q = 2$

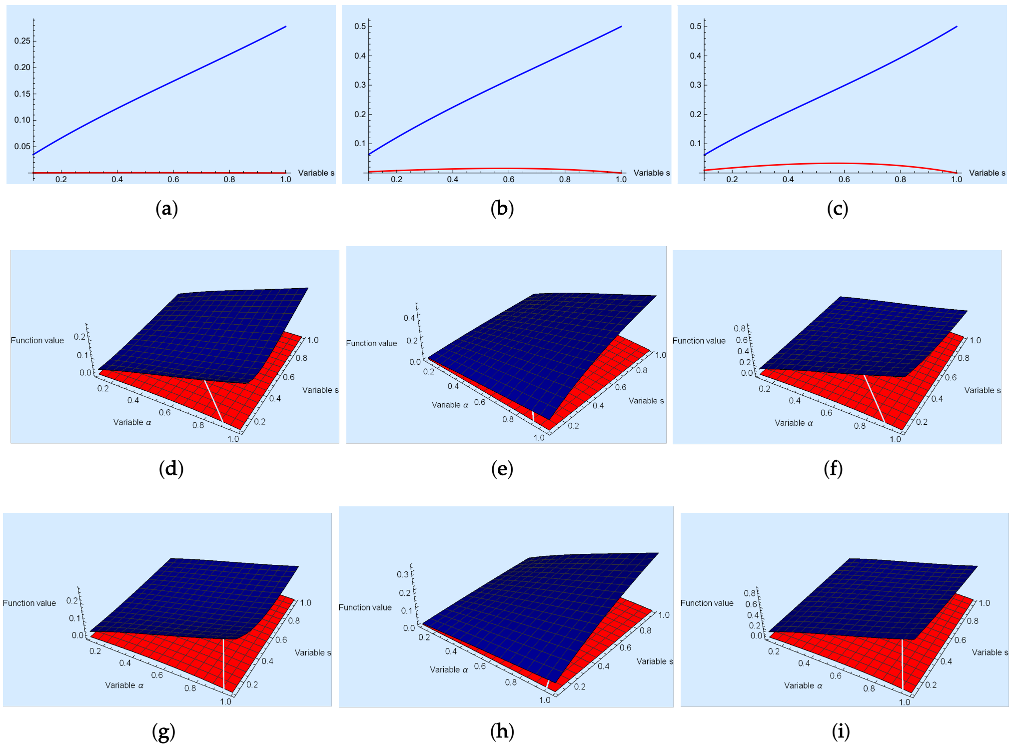


Figure 1. Visual analysis of left (red) and right (blue) sides of (**a**,**d**,**g**) Simpson's inequalities, (**b**,**e**,**h**) midpoint inequalities, (**c**,**f**,**i**) trapezoidal-type inequalities. In figures (**a**,**d**,**g**), the Simpson's inequalities are derived by setting $\varphi(\lambda) = \lambda$, $\varphi(\lambda) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)}, \alpha > 0$ and $\varphi(\lambda) = \frac{\lambda^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}, \alpha, k > 0$, respectively, for Remark 5(1) for the choices of the parameters $\sigma = \rho = \frac{2}{3}$. Similar cases hold for the midpoint inequalities in figures (**b**,**e**,**h**) and trapezoidal-type inequalities in figures (**c**,**f**,**i**) for the parametric values $\sigma = \rho = 1$ and $\sigma = \rho = 0$, respectively.

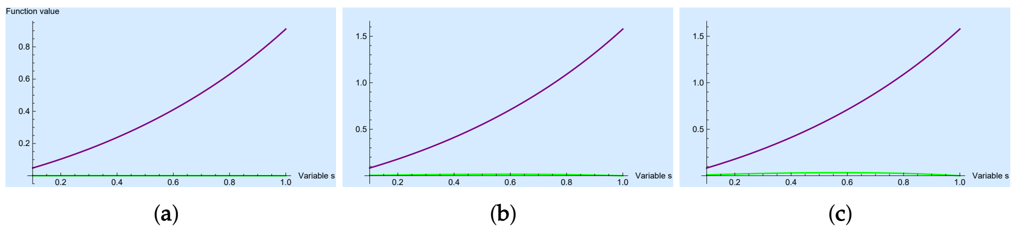


Figure 2. *Cont.*

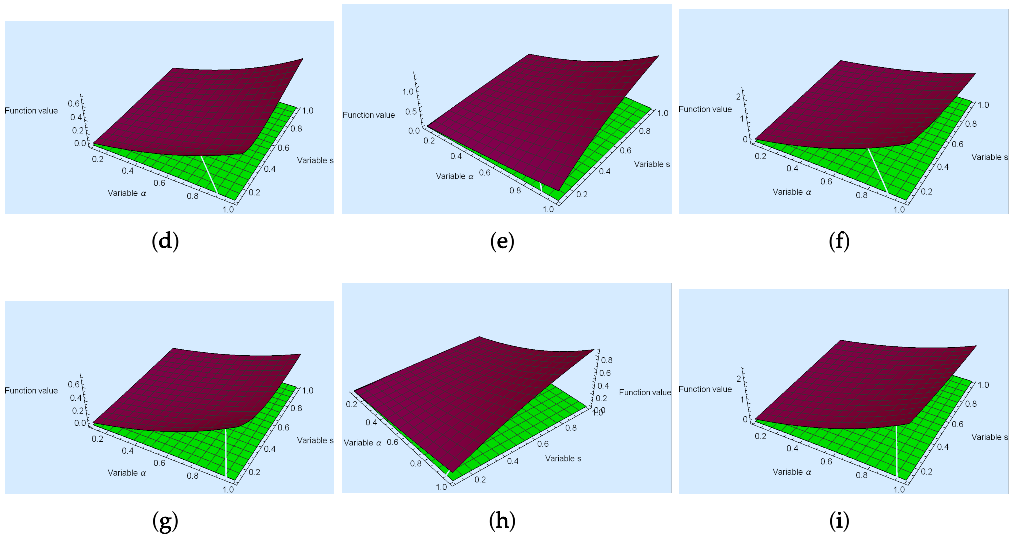


Figure 2. Visual analysis of left (green) and right (purple) sides of (**a**,**d**,**g**) Simpson's inequalities, (**b**,**e**,**h**) midpoint inequalities, (**c**,**f**,**i**) trapezoidal-type inequalities. In figures (**a**,**d**,**g**), the Simpson's inequalities are derived by setting $\varphi(\lambda) = \lambda$, $\varphi(\lambda) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)}, \alpha > 0$ and $\varphi(\lambda) = \frac{\lambda^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}, \alpha, k > 0$, respectively, for Remark 5(2) for the choices of the parameters $\sigma = \rho = \frac{2}{3}$. Similar cases hold for the midpoint inequalities in figures (**b**,**e**,**h**) and trapezoidal-type inequalities in figures (**c**,**f**,**i**) for the parametric values $\sigma = \rho = 1$ and $\sigma = \rho = 0$, respectively.

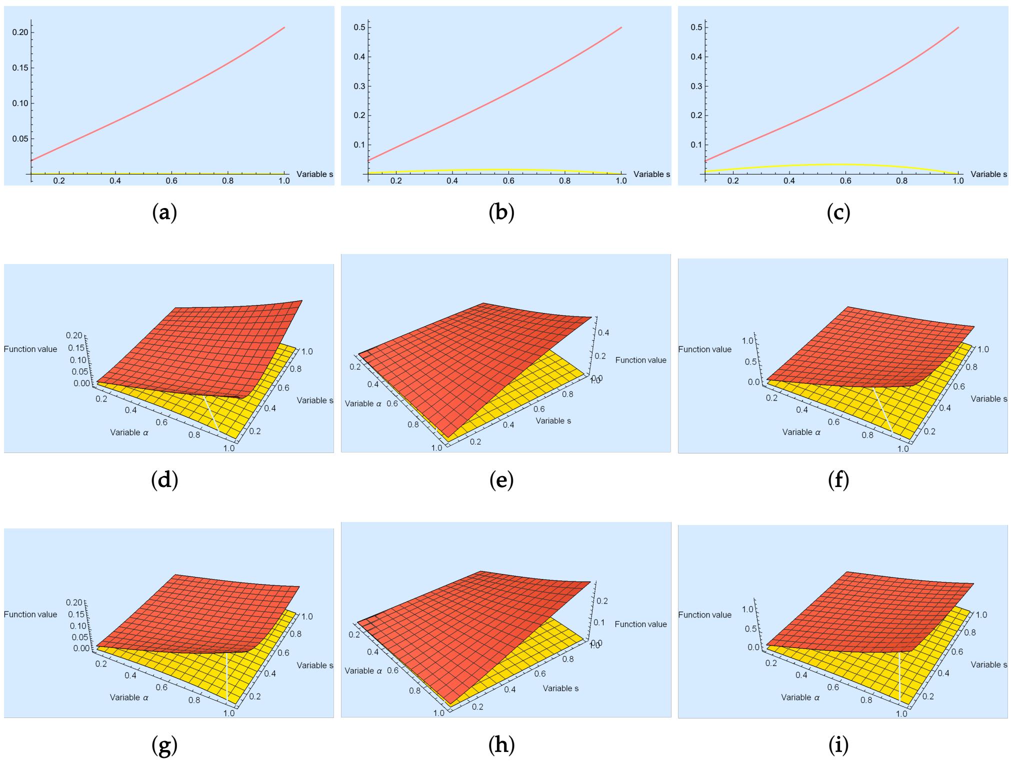


Figure 3. Visual analysis of left (yellow) and right (orange) sides of (**a**,**d**,**g**) Simpson's inequalities, (**b**,**e**,**h**) midpoint inequalities, (**c**,**f**,**i**) trapezoidal-type inequalities. In figures (**a**,**d**,**g**), the Simpson's inequalities are derived by setting $\varphi(\lambda) = \lambda$, $\varphi(\lambda) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)}, \alpha > 0$ and $\varphi(\lambda) = \frac{\lambda^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}, \alpha, k > 0$, respectively, for Remark 5(3) for the choices of the parameters $\sigma = \rho = \frac{2}{3}$. Similar cases hold for the midpoint inequalities in figures (**b**,**e**,**h**) and trapezoidal-type inequalities in figures (**c**,**f**,**i**) for the parametric values $\sigma = \rho = 1$ and $\sigma = \rho = 0$, respectively.

5. Applications

In this section, some interesting applications to special means, quadrature formulas, and in numerical analysis for solving non-linear equations are discussed, which highlight the significance and validation of our main findings.

5.1. Special Means

Before we present applications to special means, let us recall some classical concepts. For further details, see [30].

Let $\zeta : \mathcal{I} \to \mathcal{I}_1 \subseteq [0, \infty)$ be a non-negative convex mapping on $\mathcal{I}$. Then, $\zeta^{\mathfrak{s}}$ is $\mathfrak{s}$-convex on $\mathcal{I}$, $0 < \mathfrak{s} < 1$.

For positive real numbers $\tau_1, \tau_2, \tau_1 \neq \tau_2$, the following means are well known in the literature:

1. The arithmetic mean

$$A(\tau_1, \tau_2) = \frac{\tau_1 + \tau_2}{2}, \quad \tau_1, \tau_2 \in \mathbb{R}.$$

2. The harmonic mean

$$H(\tau_1, \tau_2) = \frac{2\tau_1\tau_2}{\tau_1 + \tau_2}, \quad \tau_1, \tau_2 \in \mathbb{R} \setminus \{0\}.$$

3. The generalized log mean

$$L_{\mathfrak{m}}(\tau_1, \tau_2) = \left(\frac{\tau_2{}^{\mathfrak{m}+1} - \tau_1{}^{\mathfrak{m}+1}}{(\mathfrak{m}+1)(\tau_2 - \tau_1)} \right)^{\frac{1}{\mathfrak{m}}}, \quad \mathfrak{m} \in \mathbb{R} \setminus \{-1, 0\}, \quad \tau_1, \tau_2 > 0.$$

Now, we will derive some inequalities for special means by utilizing the results from Section 3 for the following assumptions:

Consider $\aleph : [\tau_1, \tau_2] \to \mathbb{R}, \tau_1 < \tau_2$ such that $\aleph(\lambda) = \lambda^{\mathfrak{s}}$ for $\mathfrak{s} \in (0, 1]$, $\varphi(\lambda) = \lambda$ and $n = 1$. Then, from Theorem 1:

1. For $\rho = \sigma = \frac{2}{3}$, we obtain

$$\begin{aligned} &\left| \frac{1}{3} A(\tau_1^{\mathfrak{s}}, \tau_2^{\mathfrak{s}}) + \frac{2}{3} A^{\mathfrak{s}}(\tau_1, \tau_2) - L_{\mathfrak{s}}^{\mathfrak{s}}(\tau_1, \tau_2) \right| \\ &\leq \frac{(\tau_2 - \tau_1)|\mathfrak{s}|}{2^{\mathfrak{s}}} \frac{\left(1 - 2^{2+\mathfrak{s}} 3^{1+\mathfrak{s}} - 3^{2+\mathfrak{s}} + 5^{2+\mathfrak{s}} + 2^{\mathfrak{s}} 3^{1+\mathfrak{s}} \mathfrak{s}\right)}{3^{2+\mathfrak{s}}(1+\mathfrak{s})(2+\mathfrak{s})} A\left(|\tau_1^{\mathfrak{s}-1}|, |\tau_2^{\mathfrak{s}-1}|\right). \end{aligned} \tag{16}$$

2. For $\rho = \sigma = 1$, we obtain

$$|L_{\mathfrak{s}}^{\mathfrak{s}}(\tau_1, \tau_2) - A^{\mathfrak{s}}(\tau_1, \tau_2)| \leq \frac{(\tau_2 - \tau_1)|\mathfrak{s}|}{2^{\mathfrak{s}+1}} \frac{(-1 + 2^{1+\mathfrak{s}})}{(1+\mathfrak{s})(2+\mathfrak{s})} A\left(|\tau_1^{\mathfrak{s}-1}|, |\tau_2^{\mathfrak{s}-1}|\right). \tag{17}$$

3. For $\rho = \sigma = 1$, we obtain

$$|A(\tau_1^{\mathfrak{s}}, \tau_2^{\mathfrak{s}}) - L_{\mathfrak{s}}^{\mathfrak{s}}(\tau_1, \tau_2)| \leq \frac{(\tau_2 - \tau_1)|\mathfrak{s}|}{2^{\mathfrak{s}+1}} \frac{(1 + 2^{\mathfrak{s}}\mathfrak{s})}{2 + 3\mathfrak{s} + \mathfrak{s}^2} A\left(|\tau_1^{\mathfrak{s}-1}|, |\tau_2^{\mathfrak{s}-1}|\right). \tag{18}$$

Similarly, from Theorem 2:

1. For $\rho = \sigma = \frac{2}{3}$, we obtain

$$\left|\frac{1}{3}A(\tau_1^{\mathfrak{s}}, \tau_2^{\mathfrak{s}}) + \frac{2}{3}A^{\mathfrak{s}}(\tau_1, \tau_2) - L_{\mathfrak{s}}^{\mathfrak{s}}(\tau_1, \tau_2)\right| \leq \frac{(\tau_2 - \tau_1)|\mathfrak{s}|}{2}\left(\frac{4(q-1)(3)^{\frac{1-2q}{q-1}}(2)^{\frac{1}{q-1}}}{2q-1}\right)^{\frac{q-1}{q}}\left(\frac{1}{(\mathfrak{s}+1)2^{\mathfrak{s}}}\right)^{\frac{1}{q}} \left[1 + \left(2^{\mathfrak{s}+1} - 1\right)^{\frac{1}{q}}\right] A\left(|\tau_1^{\mathfrak{s}-1}|, |\tau_2^{\mathfrak{s}-1}|\right). \tag{19}$$

2. For $\rho = \sigma = 1$, we obtain

$$|L_{\mathfrak{s}}^{\mathfrak{s}}(\tau_1, \tau_2) - A^{\mathfrak{s}}(\tau_1, \tau_2)| \leq \frac{(\tau_2 - \tau_1)|\mathfrak{s}|}{2}\left(\frac{1-q}{1-2q}\right)^{\frac{q-1}{q}}\left(\frac{1}{(\mathfrak{s}+1)2^{\mathfrak{s}}}\right)^{\frac{1}{q}}\left[1 + \left(2^{\mathfrak{s}+1} - 1\right)^{\frac{1}{q}}\right] A\left(|\tau_1^{\mathfrak{s}-1}|, |\tau_2^{\mathfrak{s}-1}|\right). \tag{20}$$

3. For $\rho = \sigma = 0$, we obtain

$$|A(\tau_1^{\mathfrak{s}}, \tau_2^{\mathfrak{s}}) - L_{\mathfrak{s}}^{\mathfrak{s}}(\tau_1, \tau_2)| \leq \frac{(\tau_2 - \tau_1)|\mathfrak{s}|}{2}\left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}}\left(\frac{1}{(\mathfrak{s}+1)2^{\mathfrak{s}}}\right)^{\frac{1}{q}}\left[1 + \left(2^{\mathfrak{s}+1} - 1\right)^{\frac{1}{q}}\right] A\left(|\tau_1^{\mathfrak{s}-1}|, |\tau_2^{\mathfrak{s}-1}|\right). \tag{21}$$

Similarly, from Theorem 3:

1. For $\rho = \sigma = \frac{2}{3}$, we obtain

$$\left|\frac{1}{3}A(\tau_1^{\mathfrak{s}}, \tau_2^{\mathfrak{s}}) + \frac{2}{3}A^{\mathfrak{s}}(\tau_1, \tau_2) - L_{\mathfrak{s}}^{\mathfrak{s}}(\tau_1, \tau_2)\right| \leq \left(\frac{5}{18}\right)^{1-\frac{1}{q}}\frac{(\tau_2 - \tau_1)|\mathfrak{s}|}{2} \left[\left(\frac{2 \times 5^{\mathfrak{s}} \times 9^{-1-\mathfrak{s}}\left(50 \times 3^{\mathfrak{s}} - 7 \times 3^{1+2\mathfrak{s}} \times 5^{-\mathfrak{s}} - 2^{3+\mathfrak{s}}3^{1+2\mathfrak{s}}5^{-\mathfrak{s}} - 2 \times 3^{1+2\mathfrak{s}}5^{-\mathfrak{s}}\mathfrak{s} + 2^{1+\mathfrak{s}}3^{1+2\mathfrak{s}}5^{-\mathfrak{s}}\mathfrak{s}\right)}{(1+\mathfrak{s})(2+\mathfrak{s})}\right)^{\frac{1}{q}} + \left(\frac{3^{-2-\mathfrak{s}}\left(2 + 3^{1+\mathfrak{s}} + 2 \times 3^{1+\mathfrak{s}}\mathfrak{s}\right)}{2 + 3\mathfrak{s} + \mathfrak{s}^2}\right)^{\frac{1}{q}}\right] A\left(|\tau_1^{\mathfrak{s}-1}|, |\tau_2^{\mathfrak{s}-1}|\right). \tag{22}$$

2. For $\rho = \sigma = 1$, we obtain

$$|L_{\mathfrak{s}}^{\mathfrak{s}}(\tau_1, \tau_2) - A^{\mathfrak{s}}(\tau_1, \tau_2)| \leq \frac{(\tau_2 - \tau_1)|\mathfrak{s}|}{2}\left(\frac{1}{1+p}\right)^{1-\frac{1}{q}}\left[\left(\frac{-3 + 2^{2+\mathfrak{s}} - \mathfrak{s}}{2 + 3\mathfrak{s} + \mathfrak{s}^2}\right)^{\frac{1}{q}} + \left(\frac{1}{2+\mathfrak{s}}\right)^{\frac{1}{q}}\right] A\left(|\tau_1^{\mathfrak{s}-1}|, |\tau_2^{\mathfrak{s}-1}|\right). \tag{23}$$

3. For $\rho = \sigma = 0$, we obtain

$$|A(\tau_1^{\mathfrak{s}}, \tau_2^{\mathfrak{s}}) - L_{\mathfrak{s}}^{\mathfrak{s}}(\tau_1, \tau_2)| \leq \frac{(\tau_2 - \tau_1)|\mathfrak{s}|}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{2 + 3\mathfrak{s} + \mathfrak{s}^2}\right)^{\frac{1}{q}}\left[\left(1 + 2^{1+\mathfrak{s}}\mathfrak{s}\right)^{\frac{1}{q}} + 1\right] A\left(|\tau_1^{\mathfrak{s}-1}|, |\tau_2^{\mathfrak{s}-1}|\right). \tag{24}$$

For $\tau_1, \tau_2 \in \mathbb{R} \setminus \{0\}$, $\tau_1 < \tau_2$, ${\tau_1}^{-1} > {\tau_2}^{-1}$ and by substituting $\tau_1 \to (\tau_2)^{-1}$ and $\tau_2 \to (\tau_1)^{-1}$ in (16), we obtain

$$\begin{aligned}&\left|\frac{1}{3}H^{-1}(\tau_2^{\mathfrak{s}}, \tau_1^{\mathfrak{s}}) + \frac{2}{3}H^{-\mathfrak{s}}(\tau_1, \tau_2) - L_{\mathfrak{s}}^{\mathfrak{s}}\left({\tau_2}^{-1}, {\tau_1}^{-1}\right)\right| \\ &\leq \frac{({\tau_1}^{-1} - {\tau_2}^{-1})|\mathfrak{s}|}{2^{\mathfrak{s}}} \frac{(1 - 2^{2+\mathfrak{s}}3^{1+\mathfrak{s}} - 3^{2+\mathfrak{s}} + 5^{2+\mathfrak{s}} + 2^{\mathfrak{s}}3^{1+\mathfrak{s}}\mathfrak{s})}{3^{2+\mathfrak{s}}(1+\mathfrak{s})(2+\mathfrak{s})} H^{-1}\left(|\tau_1^{\mathfrak{s}-1}|, |\tau_2^{\mathfrak{s}-1}|\right). \quad (25)\end{aligned}$$

Remark 6. *By setting the same assumptions as those followed in inequality* (25)*, we can also obtain the inequalities involving harmonic means for* (17)–(24).

5.2. Quadrature Formulas

In this section, for different choices of the parameters ρ and σ, we provide a range of Simpson's, midpoint, and trapezoidal-type inequalities. These inequalities provide error bounds for several quadrature formulas.

Remark 7. *From Theorem 1, the following inequalities are obtained:*

1. *By setting $\rho = \sigma = \frac{n+1}{n+2}$, we have the Simpson's inequality for generalized fractional integrals:*

$$\begin{aligned}&\left|\frac{1}{(n+1)(n+2)}[\aleph(\tau_1) + \aleph(\tau_2)] + \frac{1}{n+2}\left[\aleph\left(\frac{n\tau_1 + \tau_2}{n+1}\right) + \aleph\left(\frac{\tau_1 + n\tau_2}{n+1}\right)\right]\right. \\ &\left.- \frac{1}{\Delta(1)(n+1)}\left[{}_{\tau_1^+}I_{\varphi}\aleph\left(\frac{n\tau_1 + \tau_2}{n+1}\right) + {}_{\tau_2^-}I_{\varphi}\aleph\left(\frac{\tau_1 + n\tau_2}{n+1}\right)\right]\right| \\ &\leq \frac{\tau_2 - \tau_1}{(n+1)^{\mathfrak{s}+2}} \frac{1}{\Delta(1)}\left[\left(\Pi_1^{\varphi}\left(\frac{n+1}{n+2}, \mathfrak{s}\right) + \Pi_2^{\varphi}\left(\frac{n+1}{n+2}, \mathfrak{s}\right)\right)(|\aleph'(\tau_1)| + |\aleph'(\tau_2)|)\right], \quad (26)\end{aligned}$$

where

$$\Pi_1^{\varphi}\left(\frac{n+1}{n+2}, \mathfrak{s}\right) = \int_0^1 (1-\lambda)^{\mathfrak{s}}\left|\Delta(\lambda) - \Delta(1)\frac{n+1}{n+2}\right| d\lambda,$$

and

$$\Pi_2^{\varphi}\left(\frac{n+1}{n+2}, \mathfrak{s}\right) = \int_0^1 (n+\lambda)^{\mathfrak{s}}\left|\Delta(\lambda) - \Delta(1)\frac{n+1}{n+2}\right| d\lambda,$$

and the inequality (26) *is identical to Theorem 4 in [27] by taking $n = 1$ and $\mathfrak{s} = 1$.*

2. *By setting $\rho = \sigma = n$, we have the midpoint inequality for generalized fractional integrals:*

$$\begin{aligned}&\left|\frac{n}{n+1}\left[\aleph\left(\frac{n\tau_1 + \tau_2}{n+1}\right) + \aleph\left(\frac{\tau_1 + n\tau_2}{n+1}\right)\right] - \frac{n-1}{n+1}[\aleph(\tau_1) + \aleph(\tau_2)]\right. \\ &\left.- \frac{1}{\Delta(1)(n+1)}\left[{}_{\tau_1^+}I_{\varphi}\aleph\left(\frac{n\tau_1 + \tau_2}{n+1}\right) + {}_{\tau_2^-}I_{\varphi}\aleph\left(\frac{\tau_1 + n\tau_2}{n+1}\right)\right]\right| \\ &\leq \frac{\tau_2 - \tau_1}{(n+1)^{\mathfrak{s}+2}} \frac{1}{\Delta(1)}\left[\left(\Pi_1^{\varphi}(n, \mathfrak{s}) + \Pi_2^{\varphi}(n, \mathfrak{s})\right)(|\aleph'(\tau_1)| + |\aleph'(\tau_2)|)\right], \quad (27)\end{aligned}$$

where

$$\Pi_1^{\varphi}(n, \mathfrak{s}) = \int_0^1 (1-\lambda)^{\mathfrak{s}}|\Delta(\lambda) - \Delta(1)n| d\lambda,$$

and

$$\Pi_2^{\varphi}(n,\mathfrak{s}) = \int_0^1 (n+\lambda)^{\mathfrak{s}} |\Delta(\lambda) - \Delta(1)n| d\lambda.$$

3. *By setting $\rho = \sigma = 0$, we have the trapezoidal-type inequality for generalized fractional integrals:*

$$\left| \frac{\aleph(\tau_1) + \aleph(\tau_2)}{n+1} - \frac{1}{\Delta(1)(n+1)} \left[{}_{\tau_1^+}I_{\varphi}\aleph\left(\frac{n\tau_1 + \tau_2}{n+1}\right) + {}_{\tau_2^-}I_{\varphi}\aleph\left(\frac{\tau_1 + n\tau_2}{n+1}\right) \right] \right|$$
$$\leq \frac{\tau_2 - \tau_1}{(n+1)^{\mathfrak{s}+2}} \frac{1}{\Delta(1)} \left[\left(\Pi_1^{\varphi}(\mathfrak{s}) + \Pi_2^{\varphi}(n,\mathfrak{s}) \right) \left(|\aleph'(\tau_1)| + |\aleph'(\tau_2)| \right) \right], \tag{28}$$

where

$$\Pi_1^{\varphi}(\mathfrak{s}) = \int_0^1 (1-\lambda)^{\mathfrak{s}} |\Delta(\lambda)| d\lambda,$$

and

$$\Pi_2^{\varphi}(n,\mathfrak{s}) = \int_0^1 (n+\lambda)^{\mathfrak{s}} |\Delta(\lambda)| d\lambda,$$

and inequality (28) is identical to Theorem 5.5 in [28] by taking $n = 1$ and $\mathfrak{s} = 1$.

Remark 8. *The inequalities that we derived in Remark 7 can also be established for Theorems 2 and 3 and for all the inequalities that can be derived for different choices of the function $\phi(\lambda)$, which are discussed in Remark 5.*

5.3. A Family of Numerical Schemes to Solve Non-Linear Equations

The aim of this section is to present a new iterative scheme as an application of our main results.

Consider a non-linear equation

$$\aleph(\omega) = 0. \tag{29}$$

One of the most significant problems in applied mathematics is finding the solutions to equations of the form (29). There are several methods known in the literature that can be used to find the solutions to equations of the form (29). For further details, see [31–34]. In this section, as an application of our main outcomes, we present a new generalized form of an iterative scheme that can be used to find the solution to (29).

In [35], Weerakoon and Fernando proposed the idea of obtaining quadrature rules through an iterative method. Indeed, they used Newton's method in the integral form given in [36]. The integral representation of Newton's method is

$$\aleph(\omega) = \aleph(\omega_{\mathfrak{m}}) + \int_{\omega_{\mathfrak{m}}}^{\omega} \aleph'(\lambda) d\lambda. \tag{30}$$

We now present a new generalized iterative scheme by applying the technique of Weerakoon and Fernando in [35] as follows Algorithm 1:

Algorithm 1: Generalized Iterative Scheme

Let $\sigma, \rho \geq 0$ and a non-linear function $\aleph(\omega) = 0$. Then, we have

$$\omega_{m+1} = \omega_m - \frac{2\aleph(\omega_m)}{(1-\sigma)\aleph'(\omega_m) + (1-\rho)\aleph'(v_m) + (\rho+\sigma)\aleph'\left(\frac{\omega_m+v_m}{2}\right)}. \tag{31}$$

Proof. Substituting $\varphi(\lambda) = \lambda$, $n = 1$ and $\mathfrak{s} = 1$ in Theorem 1, using (29) and (30), we deduce

$$\omega = \omega_m - \frac{2\aleph(\omega_m)}{(1-\sigma)\aleph'(\omega_m) + (1-\rho)\aleph'(\omega) + (\rho+\sigma)\aleph'\left(\frac{\omega_m+\omega}{2}\right)}. \tag{32}$$

Now, (32) allows us to suggest the following generic iterative scheme for finding the solution to equations of the form (29).

$$\omega_{m+1} = \omega_m - \frac{2\aleph(\omega_m)}{(1-\sigma)\aleph'(\omega_m) + (1-\rho)\aleph'(v_m) + (\rho+\sigma)\aleph'\left(\frac{\omega_m+v_m}{2}\right)}, \tag{33}$$

where v_m is some explicit method. This completes the proof. □

Remark 9. *For different values of ρ and σ in (31), we derive different classical numerical schemes:*

1. *By setting $\rho = 0$ and $\sigma = 0$, the trapezoidal Newton method is derived and is given in [35].*
2. *By setting $\rho = 1$ and $\sigma = 1$, the midpoint Newton method is derived and is given in [37].*
3. *By setting $\rho = \frac{1}{2}$ and $\sigma = \frac{1}{2}$, the average trapezoidal midpoint Newton method is derived and is given in [38].*
4. *By setting $\rho = \frac{2}{3}$ and $\sigma = \frac{2}{3}$, the Simpson-Newton method is derived and is given in [39].*

We now discuss the convergence analysis of Algorithm 1.

Theorem 4. *Let r be a simple zero of a sufficiently differentiable function $\aleph : \mathcal{I} \subseteq \mathbb{R} \to \mathbb{R}$, where $r \in \mathcal{I}$, provided that ω_0 is in close proximity to r. Then, the generalized iterative scheme given by Algorithm 1 exhibits a quadratic order of convergence, satisfying the following error equation:*

$$e_{m+1} = -\frac{1}{2}c_2(\sigma-\rho)e_m^2 + \left(\frac{1}{2}c_3 - \frac{9}{8}\sigma c_3 - \rho c_2^2 + c_2^2 + \frac{3}{8}\rho c_3 + \sigma c_2^2 - \frac{1}{4}\sigma^2 c_2^2 + \frac{1}{2}\sigma\rho c_2^2 - \frac{1}{4}\rho^2 c_2^2\right)e_m^3 + O(e_m^4),$$

where $c_k = \frac{1}{k!}\frac{\aleph^k(r)}{\aleph'(r)}$, $k = 1, 2, 3, \ldots$ and $e_m = \omega_m - r$.

Proof. From Algorithm 1, we have

$$\omega_{m+1} = \omega_m - \frac{2\aleph(\omega_m)}{(1-\sigma)\aleph'(\omega_m) + (1-\rho)\aleph'(v_m) + (\rho+\sigma)\aleph'\left(\frac{\omega_m+v_m}{2}\right)}, \tag{34}$$

where v_m is some explicit method, so we take the Newton–Raphson method.

$$v_m = \omega_m - \frac{\aleph(\omega_m)}{\aleph'(\omega_m)}. \tag{35}$$

Since r is a simple zero of $\aleph$, which is sufficiently differentiable, using a Taylor-series expansion of $\aleph(\omega_m)$ and $D_q\aleph(\omega_m)$ about r, we obtain

$$\aleph(\omega_m) = \aleph'(r)[e_m + c_2e_m^2 + c_3e_m^3 + c_4e_m^4 + O(e_m^5)], \tag{36}$$

$$\aleph'(\omega_m) = \aleph'(r)[1 + 2c_2e_m + 3c_3e_m^2 + 4c_4e_m^3 + O(e_m^4)]. \quad (37)$$

By utilizing (36) and (37), we obtain

$$v_m = c_2e_m^2 + (-2c_2^2 + 2c_3)e_m^3 + O(e_m^4). \quad (38)$$

Now, using (38), we deduce that

$$\aleph(v_m) = c_1 + 2c_1c_2^2e_m^2 - 4c_1(c_2^2 - c_3)c_2e_m^3 + O(e_m^4). \quad (39)$$

Also, we have

$$\aleph\left(\frac{\omega_m + v_m}{2}\right) = c_1 + c_1c_2e_m + \frac{1}{4}c_1(4c_2^2 + 3c_3)e_m^2 - \frac{1}{2}c_1(4c_2^3 - 7c_2c_3 - c_4)e_m^3 + O(e_m^4). \quad (40)$$

Using (36)–(40), we obtain

$$e_{m+1} = -\frac{1}{2}c_2(\sigma - \rho)e_m^2 + \left(\frac{1}{2}c_3 - \frac{9}{8}\sigma c_3 - \rho c_2^2 + c_2^2 + \frac{3}{8}\rho c_3 + \sigma c_2^2 - \frac{1}{4}\sigma^2c_2^2 + \frac{1}{2}\sigma\rho c_2^2 - \frac{1}{4}\rho^2c_2^2\right)e_m^3 + O(e_m^4).$$

This completes the proof. □

Now, we consider the following numerical scheme (Algorithm 2).

Algorithm 2: A new iterative scheme of order 3

For a given x_0, compute the approximate solution x_{n+1} using the following two-step iterative scheme:

$$v_m = x_n - \frac{\aleph(\omega_m)}{\aleph'(\omega_m)},$$

$$\omega_{m+1} = \omega_m - \frac{3\aleph(\omega_m)}{\aleph'(\omega_m) + \aleph'(v_m) + \aleph'\left(\frac{\omega_m + v_m}{2}\right)}. \quad (41)$$

Note that Algorithm 2 can be deduced using our generic iterative Algorithm 1 by taking $\rho = \frac{1}{3} = \sigma$. We would like to mention here that to the best of our knowledge, this iterative scheme is new in the literature.

The convergence analysis of Algorithm 2 can easily be checked from the convergence analysis of Algorithm 1 by taking $\rho = \frac{1}{3} = \sigma$. It can be seen that it satisfies the following error equation:

$$e_{m+1} = \left(c_2^2 + \frac{1}{4}c_3\right)e_m^3 + O(e_m^4).$$

This shows that the iterative scheme provided by (41) exhibits a cubic order of convergence.

5.3.1. Comparison Analysis

In this section, we present some examples that demonstrate the effectiveness of our suggested approach. We compare our proposed method (Algorithm 1) with well-known techniques, including the Newton method (NM) [32], Abbasbandy method (AM) [40], Halley method (HM) [32], and Chun method (CM) [33]. To determine the approximate root, we employed a tolerance of $\epsilon = 10^{-15}$. The following termination conditions were utilized for the computer algorithms:

1. $|\omega_{m+1} - \omega_m| < \epsilon$;
2. $|\aleph(\omega_{m+1})| < \epsilon$.

The numerical tests were conducted on an Intel(R) Core(TM) i5 processor with 1.60 GHz and 16 GB RAM. Maple 2020 was used for coding, while the graphical analysis was carried out using Matlab 2021. For the comparison analysis, we consider the following four types of examples.

1. $\aleph(\omega) = \omega^3 + 4\omega^2 - 15$;
2. $\aleph(\omega) = \omega \exp(\omega^2) - \sin^2 \omega + 3\cos\omega + 5$;
3. $\aleph(\omega) = 10\omega \exp(-\omega^2) - 1$;
4. $\aleph(\omega) = \exp(-\omega) + \cos\omega$.

After carrying out the numerical tests with the software, we prepared tables and visual illustrations of Algorithm 1 for the above-mentioned examples, which are presented below (Tables 1–4).

Table 1. Comparison results for $\aleph(\omega) = \omega^3 + 4\omega^2 - 15$.

Methods	ω_0	IT	ω_m	$\aleph(\omega_m)$	δ
NM	2	6	1.6319808055660635175	0	0
AM	2	4	1.6319808055660635175	0	0
HM	2	4	1.6319808055660635175	0	0
CM	2	4	1.6319808055660635175	0	0
ALG	2	4	1.6319808055660635175	0	0

Table 2. Comparison results for $\aleph(\omega) = \omega \exp(\omega^2) - \sin^2 \omega + 3\cos\omega + 5$.

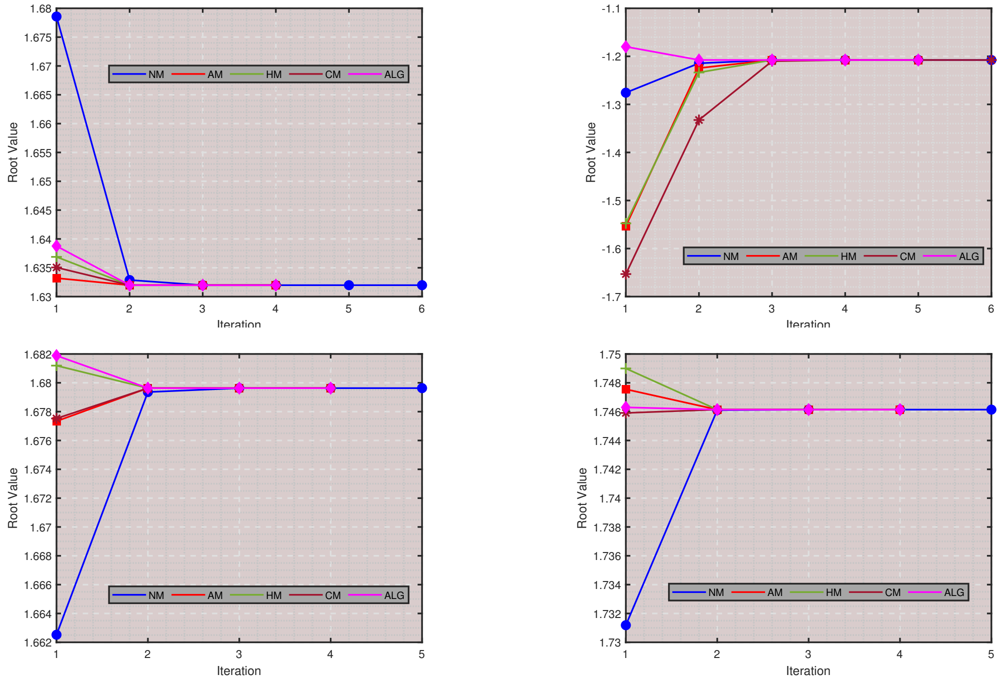
Methods	ω_0	IT	ω_m	$\aleph(\omega_m)$	δ
NM	−1	6	−1.2076478271309189270	4.0×10^{-19}	7.58×10^{-17}
AM	−1	5	−1.2076478271309189270	4.0×10^{-19}	0
HM	−1	5	−1.2076478271309189270	0	0
CM	−1	6	−1.2076478271309189270	4.0×10^{-19}	0
ALG	−1	5	−1.2076478271309189270	4.0×10^{-19}	0

Table 3. Comparison results for $\aleph(\omega) = 10\omega \exp(-\omega^2) - 1$.

Methods	ω_0	IT	ω_m	$\aleph(\omega_m)$	δ
NM	1.8	5	1.6796306104284499407	-9×10^{-20}	4.7395×10^{-15}
AM	1.8	4	1.6796306104284499407	-9×10^{-20}	1.0×10^{-19}
HM	1.8	4	1.6796306104284499407	-9×10^{-20}	0
CM	1.8	4	1.6796306104284499407	2.0×10^{-19}	0
ALG	1.8	4	1.6796306104284499407	-9×10^{-20}	0

Table 4. Comparison results for $\aleph(\omega) = \exp(-\omega) + \cos\omega$.

Methods	ω_0	IT	ω_m	$\aleph(\omega_m)$	δ
NM	2	5	1.7461395304080124177	6.0×10^{-20}	1.0×10^{-19}
AM	2	4	1.7461395304080124177	-6×10^{-20}	1.0×10^{-19}
HM	2	4	1.7461395304080124177	6.0×10^{-20}	1.0×10^{-19}
CM	2	3	1.7461395304080124177	-6×10^{-20}	4.63×10^{-17}
ALG	2	4	1.7461395304080124177	-6×10^{-20}	1.0×10^{-19}



5.3.2. Basins of Attraction

In this section, we discuss the basins of attraction for Algorithm 1. We apply our proposed method to $\mathbb{R} \times \mathbb{R} = [-2, 2] \times [-2, 2]$ with a grid of 500×500 points, employing a tolerance of $|\aleph(\omega_{\mathfrak{m}})| < 1 \times 10^{-10}$ and a maximum of 20 iterations. Additionally, we present graphical representations of the CPU time consumed to generate the basins of attraction per iteration. For this analysis, we consider a famous problem involving finding the roots of $\aleph(\omega) = \omega^{\mathfrak{m}} - 1$. We consider $\mathfrak{m} = 2, 3, 4$. Figures 4–6 gives the visual analysis of basins of attraction for $\aleph(\omega) = \omega^2 - 1$, $\aleph(\omega) = \omega^3 - 1$ and $\aleph(\omega) = \omega^4 - 1$ and also CPU time consumed to generate the basins of attraction per iteration respectively.

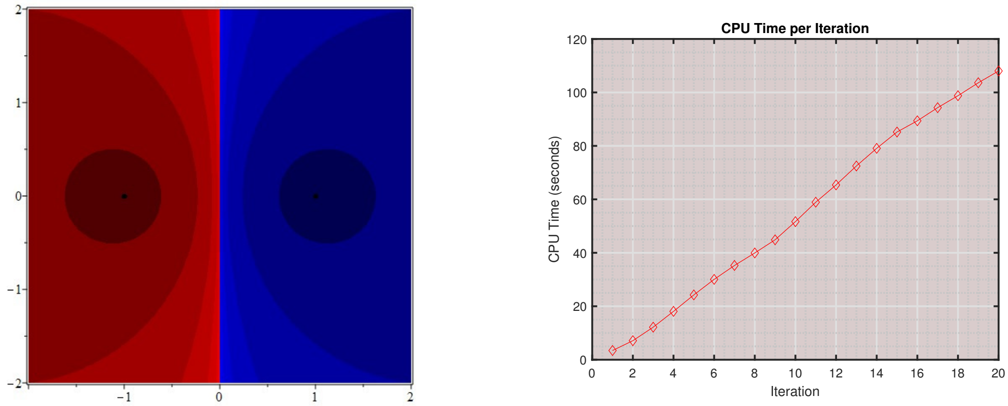


Figure 4. (**Left**) Basins of attraction for $\aleph(\omega) = \omega^2 - 1$. (**Right**) CPU time consumed to generate the basins of attraction per iteration.

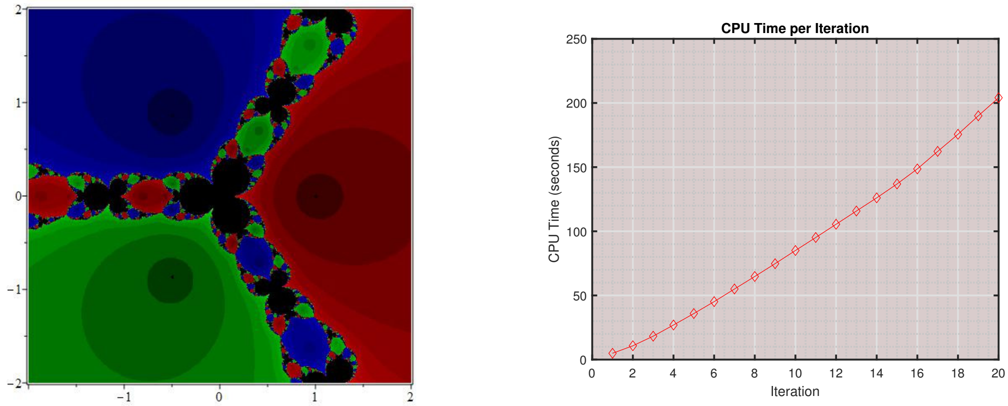


Figure 5. (**Left**) Basins of attraction $\aleph(\omega) = \omega^3 - 1$. (**Right**) CPU time consumed to generate the basins of attraction per iteration.

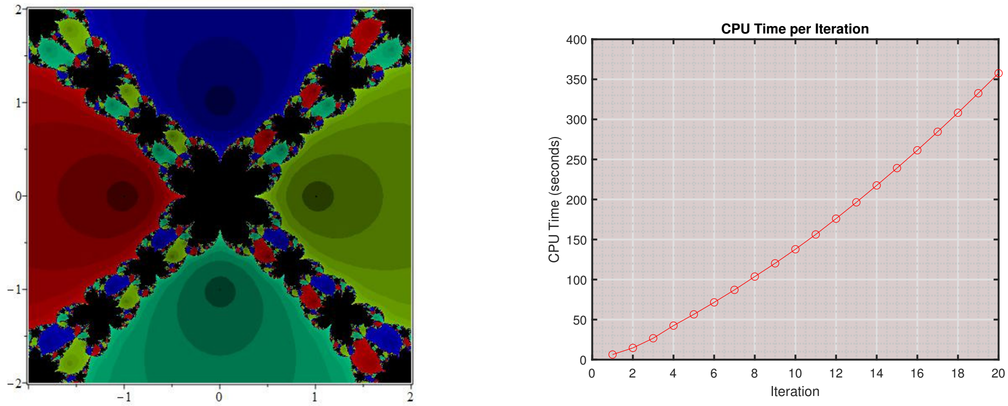


Figure 6. (**Left**) Basins of attraction $\aleph(\omega) = \omega^4 - 1$. (**Right**) CPU time consumed to generate the basins of attraction per iteration.

6. Conclusions

In conclusion, this research paper presents a novel extension of a parameterized identity. By utilizing this identity, some parametric inequalities for differentiable $\mathfrak{s}$-convex mappings through the use of generalized fractional integral operators are obtained. The derived Simpson's, midpoint, and trapezoidal-type inequalities, incorporating different integral operators and parameters, serve as additional contributions to the existing literature on this topic. We also discuss the applicability of the main results to means of real numbers and in numerical analysis for solving non-linear equations. The graphical analysis provided in this paper further supports the importance and practical relevance of our findings. The results of this research paper could potentially have applications in various areas of mathematics, physics, and engineering. The extension of the proposed iterative method to systems of equations could be an interesting future research problem.

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