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OPERATOR CONVEXITY OF AN INTEGRAL TRANSFORM WITH APPLICATIONS

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ABSTRACT

For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0,\infty)$ we consider the following *integral transform*

$$\mathcal{D}(w,\mu)(t) := \int_0^\infty w(\lambda)(\lambda+t)^{-1}d\mu(\lambda),$$

where the integral is assumed to exist for $t > 0$.

We show among others that $\mathcal{D}(w,\mu)$ is operator convex on $(0,\infty)$. From this we derive that, if $f : [0,\infty) \to \mathbb{R}$ is an operator monotone function on $[0,\infty)$, then the function $\left[f(0) - f(t)\right]t^{-1}$ is operator convex on $(0,\infty)$. Also, if $f : [0,\infty) \to \mathbb{R}$ is an operator convex function on $[0,\infty)$, then the function $\left[f(0) + f'_+(0)t - f(t)\right]t^{-2}$ is operator convex on $(0,\infty)$. Some lower and upper bounds for the Jensen's difference

$$\frac{\mathcal{D}(w,\mu)(A) + \mathcal{D}(w,\mu)(B)}{2} - \mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right)$$

under some natural assumptions for the positive operators A and B are given. Examples for power, exponential and logarithmic functions are also provided.

KEYWORDS

Operator monotone functions, operator convex functions, operator inequalities, Löwner–Heinz inequality, logarithmic operator inequalities

MATHEMATICS SUBJECT CLASSIFICATION (2020)

Primary 47A63; Secondary 47A60

1. INTRODUCTION

Consider a complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x\rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function f on $(0,\infty)$ is said to be operator

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monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$. We have the following representation of operator monotone functions [8], see for instance [1, p. 144-145]:

THEOREM 1.1. A function $f : [0, \infty) \to \mathbb{R}$ is operator monotone in $[0, \infty)$ if and only if it has the representation

$$f(t) = f(0) + bt + \int_0^\infty \frac{t\lambda}{t+\lambda} d\mu(\lambda), \tag{1.1}$$

where $b \geq 0$ and a positive measure μ on $[0, \infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda} d\mu(\lambda) < \infty. \tag{1.2}$$

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f\big((1-\lambda)A + \lambda B\big) \leq (\geq)(1-\lambda)f(A) + \lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda \in [0, 1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [1, p. 147]:

THEOREM 1.2. A function $f : [0, \infty) \to \mathbb{R}$ is operator convex in $[0, \infty)$ with $f'_+(0) \in \mathbb{R}$ if and only if it has the representation

$$f(t) = f(0) + f'_+(0)t + ct^2 + \int_0^\infty \frac{t^2\lambda}{t+\lambda} d\mu(\lambda), \tag{1.3}$$

where $c \geq 0$ and a positive measure μ on $[0, \infty)$ such that (1.2) holds.

We have the following integral representation for the power function when $t > 0$, $r \in (0, 1]$, see for instance [1, p. 145]

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \int_0^\infty \frac{\lambda^{r-1}}{\lambda + t} d\lambda.$$

Observe that for $t > 0$, $t \neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda+t)(\lambda+1)} = \frac{\ln t}{t-1} + \frac{1}{1-t} \ln\left(\frac{u+t}{u+1}\right)$$

for all $u > 0$.

By taking the limit over $u \to \infty$ in this equality, we derive

$$\frac{\ln t}{t-1} = \int_0^\infty \frac{d\lambda}{(\lambda+t)(\lambda+1)},$$

which gives the representation for the logarithm

$$\ln t = (t-1) \int_0^\infty \frac{d\lambda}{(\lambda+1)(\lambda+t)}$$

for all $t > 0$.

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda > 0$, the following *integral transform*

$$\mathcal{D}(w, \mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\mu(\lambda), \quad t > 0, \tag{1.4}$$

where μ is a positive measure on $(0, \infty)$ and the integral (1.4) exists for all $t > 0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\lambda, \quad t > 0. \tag{1.5}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda) = \lambda^{r-1}$, $r \in (0, 1]$, then

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \mathcal{D}(w_r)(t), \quad t > 0. \tag{1.6}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda) = (\lambda+1)^{-1}$, $t > 0$, we have the representation

$$\ln t = (t-1)\mathcal{D}(w_{\ln})(t), \quad t > 0. \tag{1.7}$$

Assume that $T > 0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w,\mu)(T) := \int_0^\infty w(\lambda)(\lambda+T)^{-1} d\mu(\lambda),$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T) := \int_0^\infty w(\lambda)(\lambda+T)^{-1} d\lambda, \tag{1.8}$$

for $T > 0$.

From (1.6) we have the representation

$$T^{r-1} = \frac{\sin(r\pi)}{\pi}\mathcal{D}(w_r)(T) \tag{1.9}$$

where $T > 0$ and from (1.7)

$$(T-1)^{-1}\ln T = \mathcal{D}(w_{\ln})(T) \tag{1.10}$$

provided $T > 0$ and $T-1$ is invertible.

In this paper, we show among others that $\mathcal{D}(w,\mu)$ is operator convex on $(0,\infty)$. From this we derive that, if $f : [0,\infty) \to \mathbb{R}$ is an operator monotone function on $[0,\infty)$, then the function $[f(0)-f(t)]t^{-1}$ is operator convex on $(0,\infty)$. Also, if $f : [0,\infty) \to \mathbb{R}$ is an operator convex function on $[0,\infty)$, then the function $\left[f(0)+f'_+(0)t-f(t)\right]t^{-2}$ is operator convex on $(0,\infty)$. Some lower and upper bounds for the Jensen's difference

$$\frac{\mathcal{D}(w,\mu)(A)+\mathcal{D}(w,\mu)(B)}{2} - \mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right)$$

under some natural assumptions for the positive operators A and B are given. Examples for power, exponential and logarithmic functions are also provided.

2. PRELIMINARY RESULTS

We start with the following elementary identity that give a simple proof for the fact that the function $f(t) = t^{-1}$ is operator convex on $(0,\infty)$, see for instance [6, p. 8]:

LEMMA 2.1. For any $A, B > 0$ we have

$$\frac{A^{-1}+B^{-1}}{2} - \left(\frac{A+B}{2}\right)^{-1} = \frac{\left(A^{-1}-B^{-1}\right)\left(A^{-1}+B^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)}{2} \geq 0. \tag{2.1}$$

If more assumptions are made for the operators A and B, then one can obtain the following lower and upper bounds:

COROLLARY 2.2. Assume that $0 < \alpha \leq A \leq \beta$ and $0 < \gamma \leq B \leq \delta$ for some constants $\alpha, \beta, \gamma, \delta$. Then

$$\begin{aligned}\frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)^2 &\leq \frac{A^{-1}+B^{-1}}{2} - \left(\frac{A+B}{2}\right)^{-1} \\ &\leq \frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)^2.\end{aligned} \tag{2.2}$$

Proof. We have $\beta^{-1} \leq A^{-1} \leq \alpha^{-1}$ and $\delta^{-1} \leq B^{-1} \leq \gamma^{-1}$, which gives

$$\beta^{-1}+\delta^{-1} \leq A^{-1}+B^{-1} \leq \alpha^{-1}+\gamma^{-1},$$

namely

$$\left(\alpha^{-1}+\gamma^{-1}\right)^{-1} \leq \left(A^{-1}+B^{-1}\right)^{-1} \leq \left(\beta^{-1}+\delta^{-1}\right)^{-1}.$$

By multiplying both sides by $\left(A^{-1}-B^{-1}\right)$ and dividing by 2, we get

$$\frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)^2 \leq \frac{\left(A^{-1}-B^{-1}\right)\left(A^{-1}+B^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)}{2} \leq \frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)^2. \quad \square$$

We know that for $T>0$, we have the operator inequalities

$$0<\left\|T^{-1}\right\|^{-1} \leq T \leq \|T\|. \tag{2.3}$$

Indeed, it is well known that, if $P \geq 0$, then

$$|\langle Px, y\rangle|^2 \leq \langle Px, x\rangle\langle Py, y\rangle$$

for all $x, y \in H$.

Therefore, if $T>0$, then

$$\begin{aligned} 0 \leq \langle x, x\rangle^2 &= \left\langle T^{-1}Tx, x\right\rangle^2 = \left\langle Tx, T^{-1}x\right\rangle^2 \\ &\leq \langle Tx, x\rangle\left\langle TT^{-1}x, T^{-1}x\right\rangle = \langle Tx, x\rangle\left\langle x, T^{-1}x\right\rangle \end{aligned}$$

for all $x \in H$.

If $x \in H$, $\|x\|=1$, then

$$1 \leq \langle Tx, x\rangle\left\langle x, T^{-1}x\right\rangle \leq \langle Tx, x\rangle \sup_{\|x\|=1}\left\langle x, T^{-1}x\right\rangle = \langle Tx, x\rangle\left\|T^{-1}\right\|,$$

which implies the following operator inequality

$$\left\|T^{-1}\right\|^{-1} \leq T.$$

The second inequality in (2.3) is obvious.

REMARK 2.3. If $A, B>0$ and $B-A>0$, then by taking $\alpha=\left\|A^{-1}\right\|^{-1}$, $\beta=\|A\|$, $\gamma=\left\|B^{-1}\right\|^{-1}$ and $\delta=\|B\|$ in (2.2), we get

$$\begin{aligned} \frac{1}{2}\left(\left\|A^{-1}\right\|+\left\|B^{-1}\right\|\right)^{-1}\left(A^{-1}-B^{-1}\right)^2 &\leq \frac{A^{-1}+B^{-1}}{2}-\left(\frac{A+B}{2}\right)^{-1} \\ &\leq \frac{1}{2}\left(\|A\|^{-1}+\|B\|^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)^2. \end{aligned} \tag{2.4}$$

A continuous function $g: \mathcal{SA}_I(H) \rightarrow \mathcal{B}(H)$ is said to be *Gâteaux differentiable* in $A \in \mathcal{SA}_I(H)$, the class of selfadjoint operators on I, along the direction $B \in \mathcal{B}(H)$ if the following limit exists in the strong topology of $\mathcal{B}(H)$

$$\nabla g_A(B) := \lim_{s \rightarrow 0} \frac{g(A+sB)-g(A)}{s} \in \mathcal{B}(H). \tag{2.5}$$

If the limit (2.5) exists for all $B \in \mathcal{B}(H)$, then we say that g is *Gâteaux differentiable* in A and we can write $g \in \mathcal{G}(A)$. If this is true for any A in an open set $\mathcal{S}$ from $\mathcal{SA}_I(H)$ we write that $g \in \mathcal{G}(\mathcal{S})$.

If g is a continuous function on I, by utilising the continuous functional calculus the corresponding function of operators will be denoted in the same way.

For two distinct operators $A, B \in \mathcal{SA}_I(H)$ we consider the segment of selfadjoint operators

$$[A, B] := \left\{(1-t)A+tB \mid t \in [0,1]\right\}.$$

We observe that $A, B \in [A, B]$ and $[A, B] \subset \mathcal{SA}_I(H)$.

We have the following gradient inequalities, see for instance:

LEMMA 2.4. Let f be an operator convex function on I and $A, B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}\left([A, B]\right)$, then

$$\nabla_B f(B-A) \geq f(B)-f(A) \geq \nabla_A f(B-A). \tag{2.6}$$

Let $T, S > 0$. The function $f(t) = t^{-1}$ is operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S) := \lim_{t\to 0}\left[\frac{f(T+tS)-f(T)}{t}\right] = -T^{-1}ST^{-1} \tag{2.7}$$

for $T, S > 0$.

Using (2.7) for the operator convex function $f(t) = t^{-1}$, we get

$$-D^{-1}(D-C)D^{-1} \geq D^{-1} - C^{-1} \geq -C^{-1}(D-C)C^{-1}$$

that is equivalent to

$$D^{-1}(D-C)D^{-1} \leq C^{-1} - D^{-1} \leq C^{-1}(D-C)C^{-1} \tag{2.8}$$

for all $C, D > 0$.

If

$$m \leq D - C \leq M$$

for some constants m, M, then

$$mD^{-2} \leq D^{-1}(D-C)D^{-1}$$

and

$$C^{-1}(D-C)C^{-1} \leq MC^{-2}$$

and by (2.8) we derive

$$mD^{-2} \leq C^{-1} - D^{-1} \leq MC^{-2}. \tag{2.9}$$

Moreover, if $C \geq \alpha > 0$ and $D \leq \delta$, then we get

$$C^{-2} \leq \alpha^{-2} \text{ and } D^{-2} \geq \delta^{-2},$$

which implies that

$$\frac{m}{\delta^2} \leq C^{-1} - D^{-1} \leq \frac{M}{\alpha^2} \tag{2.10}$$

COROLLARY 2.5. Assume that $0 < \alpha \leq A \leq \beta$, $0 < \gamma \leq B \leq \delta$ and $0 < m \leq B - A \leq M$ for some constants $\alpha, \beta, \gamma, \delta, m, M$. Then

$$\begin{aligned}
0 &< \frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\frac{m^2}{\delta^4} \\
&\leq \frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)^2 \leq \frac{A^{-1}+B^{-1}}{2} - \left(\frac{A+B}{2}\right)^{-1} \\
&\leq \frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)^2 \leq \frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\frac{M^2}{\alpha^4}.
\end{aligned} \tag{2.11}$$

Proof. From (2.10) we have

$$0 < \frac{m}{\delta^2} \leq A^{-1} - B^{-1} \leq \frac{M}{\alpha^2},$$

which implies that

$$0 < \frac{m^2}{\delta^4} \leq \left(A^{-1}-B^{-1}\right)^2 \leq \frac{M^2}{\alpha^4}$$

and by (2.2) we get (2.11). □

REMARK 2.6. If the positive operators A, B are separated, namely $0 < \alpha \leq A \leq \beta < \gamma \leq B \leq \delta$ for some constants $\alpha, \beta, \gamma, \delta$, then obviously $0 < \gamma - \beta \leq B - A \leq \delta - \alpha$ and by (2.11) for $m = \gamma - \beta$ and $M = \delta - \alpha$, we get

$$\begin{aligned}
0 &< \frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\frac{(\gamma-\beta)^2}{\delta^4} \leq \frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)^2 \\
&\leq \frac{A^{-1}+B^{-1}}{2} - \left(\frac{A+B}{2}\right)^{-1} \\
&\leq \frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)^2 \leq \frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\frac{(\delta-\alpha)^2}{\alpha^4}.
\end{aligned} \tag{2.12}$$

If $0 < \|A\| \|B^{-1}\| < 1$, then

$$0 < \|A^{-1}\|^{-1} \le A \le \|A\| < \|B^{-1}\|^{-1} \le B \le \|B\|$$

and by (2.12) we get

$$\begin{aligned} 0 &< \frac{1}{2}\left(\|A^{-1}\| + \|B^{-1}\|\right)^{-1} \frac{\left(\|B^{-1}\|^{-1} - \|A\|\right)^2}{\|B\|^4} \\ &\le \frac{1}{2}\left(\|A^{-1}\| + \|B^{-1}\|\right)^{-1}\left(A^{-1} - B^{-1}\right)^2 \\ &\le \frac{A^{-1} + B^{-1}}{2} - \left(\frac{A+B}{2}\right)^{-1} \\ &\le \frac{1}{2}\left(\|A\|^{-1} + \|B\|^{-1}\right)^{-1}\left(A^{-1} - B^{-1}\right)^2 \\ &\le \frac{1}{2}\left(\beta^{-1} + \delta^{-1}\right)^{-1}\left(\|B\| - \|A^{-1}\|^{-1}\right)^2 \|A^{-1}\|^4 . \end{aligned} \tag{2.13}$$

We can present now our main results.

3. MAIN RESULTS

We have

THEOREM 3.1. For all $A, B > 0$ we have

$$\begin{aligned} &\frac{\mathcal{D}(w,\mu)(A) + \mathcal{D}(w,\mu)(B)}{2} - \mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right) \\ &= \frac{1}{2}\int_0^\infty \left((\lambda+A)^{-1} - (\lambda+B)^{-1}\right)\left((\lambda+A)^{-1} + (\lambda+B)^{-1}\right)^{-1} \times \left((\lambda+A)^{-1} - (\lambda+B)^{-1}\right) w(\lambda) d\mu(\lambda) \\ &\ge 0. \end{aligned} \tag{3.1}$$

The function $\mathcal{D}(w,\mu)$ is an operator convex function on $(0,\infty)$

Proof. We have for all $A, B > 0$

$$\begin{aligned} &\frac{\mathcal{D}(w,\mu)(A) + \mathcal{D}(w,\mu)(B)}{2} - \mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right) \\ &= \int_0^\infty w(\lambda)\left[\frac{(\lambda+A)^{-1} + (\lambda+B)^{-1}}{2} - \left(\lambda + \frac{A+B}{2}\right)^{-1}\right] d\mu(\lambda). \end{aligned} \tag{3.2}$$

Since, by (2.1)

$$\begin{aligned} &\frac{(\lambda+A)^{-1} + (\lambda+B)^{-1}}{2} - \left(\lambda + \frac{A+B}{2}\right)^{-1} \\ &= \frac{1}{2}\left((\lambda+A)^{-1} - (\lambda+B)^{-1}\right)\left((\lambda+A)^{-1} + (\lambda+B)^{-1}\right)^{-1} \times \left((\lambda+A)^{-1} - (\lambda+B)^{-1}\right) \\ &\ge 0 \end{aligned}$$

for all $\lambda \ge 0$, then by (3.2) we obtain the representation (3.1).

Since $\mathcal{D}(w,\mu)$ is continuous in $\mathcal{B}(H)$ and satisfies Jensen's inequality (3.1), it follows that $\mathcal{D}(w,\mu)$ is an operator convex function on $(0,\infty)$. □

The case of operator monotone functions is as follows:

COROLLARY 3.2. Assume that $f : [0,\infty) \to \mathbb{R}$ is an operator monotone function on $[0,\infty)$. Then the function $[f(t) - f(0)]\, t^{-1}$ is operator convex on $(0,\infty)$. For all $A, B > 0$ we have

$$\frac{f(A)A^{-1} + f(B)B^{-1}}{2} - f\left(\frac{A+B}{2}\right)\left(\frac{A+B}{2}\right)^{-1} \ge f(0)\left[\frac{\left(A^{-1} - B^{-1}\right)\left(A^{-1} + B^{-1}\right)^{-1}\left(A^{-1} - B^{-1}\right)}{2}\right]. \tag{3.3}$$

If $f(0)=0$, then $f(t)t^{-1}$ is operator convex on $(0,\infty)$ and

$$\frac{f(A)A^{-1}+f(B)B^{-1}}{2} \geq f\left(\frac{A+B}{2}\right)\left(\frac{A+B}{2}\right)^{-1}$$

for all $A, B>0$.

Proof. From (1.1) we have

$$\frac{f(t)-f(0)}{t}-b=\mathcal{D}(\ell,\mu)(t), \tag{3.4}$$

for some μ, a positive measure on $(0,\infty)$, where $\ell(\lambda)=\lambda$, $\lambda\geq 0$. By utilising Theorem 3.1 and Lemma 2.1 we deduce the desired results. □

COROLLARY 3.3. Assume that $f:[0,\infty)\to\mathbb{R}$ is an operator convex function on $[0,\infty)$. Then the function $\left[f(t)-f(0)-f'_+(0)t\right]t^{-2}$ is operator convex on $(0,\infty)$. For all $A, B>0$ we have

$$\begin{aligned}&\frac{f(A)A^{-2}+f(B)B^{-2}}{2}-f\left(\frac{A+B}{2}\right)\left(\frac{A+B}{2}\right)^{-2}\\ &\geq f(0)\left[\frac{A^{-2}+B^{-2}}{2}-\left(\frac{A+B}{2}\right)^{-2}\right]+f'_+(0)\left[\frac{\left(A^{-1}-B^{-1}\right)\left(A^{-1}+B^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)}{2}\right].\end{aligned} \tag{3.5}$$

If $f(0)=0$, then $\left[f(t)-f'_+(0)t\right]t^{-2}$ is operator convex on $(0,\infty)$ and

$$\frac{f(A)A^{-2}+f(B)B^{-2}}{2}-f\left(\frac{A+B}{2}\right)\left(\frac{A+B}{2}\right)^{-2}\geq f'_+(0)\left[\frac{\left(A^{-1}-B^{-1}\right)\left(A^{-1}+B^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)}{2}\right] \tag{3.6}$$

for all $A, B>0$.

Proof. From (1.3) we have

$$\left[f(t)-f(0)-f'_+(0)t\right]t^{-2}-c=\mathcal{D}(\ell,\mu)(t),$$

for some μ, a positive measure on $(0,\infty)$, where $\ell(\lambda)=\lambda$, $\lambda\geq 0$. By utilising Theorem 3.1 and Lemma 2.1 we deduce the desired results. □

When more assumptions are imposed on the operators A and B, then the following improvement and refinement of Jensen's inequality hold:

THEOREM 3.4. Assume that $0<\alpha\leq A\leq\beta$, $0<\gamma\leq B\leq\delta$ and $0<m\leq B-A\leq M$ for some constants $\alpha,\beta,\gamma,\delta,m,M$. Then

$$\begin{aligned}0&<-\frac{m^2\gamma\alpha}{12(\alpha+\gamma)}\mathcal{D}'''(w,\mu)(\delta)\\ &\leq\frac{\mathcal{D}(w,\mu)(A)+\mathcal{D}(w,\mu)(B)}{2}-\mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right)\\ &\leq\frac{M^2}{2(\beta+\delta)}\left[-\mathcal{D}'(w,\mu)(\alpha)+\left(\frac{\delta+\beta}{2}-\alpha\right)\mathcal{D}''(w,\mu)(\alpha)-\frac{1}{6}(\beta-\alpha)(\delta-\alpha)\mathcal{D}'''(w,\mu)(\alpha)\right].\end{aligned} \tag{3.7}$$

Proof. We have $0<\alpha+\lambda\leq A+\lambda\leq\beta+\lambda$, $0<\gamma+\lambda\leq B+\lambda\leq\delta+\lambda$ and $0<m\leq B+\lambda-A-\lambda=B-A\leq M$ for all $\lambda\geq 0$. By (2.11) we get

$$\begin{aligned}0&<\frac{1}{2}\left(\frac{1}{\alpha+\lambda}+\frac{1}{\gamma+\lambda}\right)^{-1}\frac{m^2}{(\delta+\lambda)^4}\\ &\leq\frac{(A+\lambda)^{-1}+(B+\lambda)^{-1}}{2}-\left(\lambda+\frac{A+B}{2}\right)^{-1}\\ &\leq\frac{1}{2}\left(\frac{1}{\beta+\lambda}+\frac{1}{\delta+\lambda}\right)^{-1}\frac{M^2}{(\alpha+\lambda)^4}.\end{aligned} \tag{3.8}$$

We have that

$$\left(\frac{1}{\beta+\lambda}+\frac{1}{\delta+\lambda}\right)^{-1}=\frac{(\beta+\lambda)(\delta+\lambda)}{\beta+\delta+2\lambda}\le\frac{(\beta+\lambda)(\delta+\lambda)}{\beta+\delta} \tag{3.9}$$

and

$$\left(\frac{1}{\alpha+\lambda}+\frac{1}{\gamma+\lambda}\right)^{-1}=\frac{(\gamma+\lambda)(\alpha+\lambda)}{\alpha+\gamma+2\lambda}=g(\lambda).$$

We have

$$g'(\lambda)=\frac{(\alpha+\gamma+2\lambda)^2-2(\gamma+\lambda)(\alpha+\lambda)}{(\alpha+\gamma+2\lambda)^2}=\frac{(\alpha+\lambda)^2+(\gamma+\lambda)^2}{(\alpha+\gamma+2\lambda)^2}>0,$$

which shows that g is increasing on $[0,\infty)$.

Therefore

$$g(\lambda)\ge g(0)=\frac{\gamma\alpha}{\alpha+\gamma}\text{ for all }\lambda\ge 0. \tag{3.10}$$

By (3.8)–(3.10) we derive that

$$\begin{aligned}0&<\frac{1}{2}\frac{\gamma\alpha}{\alpha+\gamma}\frac{m^2}{(\delta+\lambda)^4}\\&\le\frac{(A+\lambda)^{-1}+(B+\lambda)^{-1}}{2}-\left(\lambda+\frac{A+B}{2}\right)^{-1}\\&\le\frac{1}{2}\frac{(\beta+\lambda)(\delta+\lambda)}{\beta+\delta}\frac{M^2}{(\alpha+\lambda)^4},\end{aligned}$$

which implies that

$$\begin{aligned}0&<\frac{1}{2}m^2\frac{\gamma\alpha}{\alpha+\gamma}\int_0^\infty\frac{w(\lambda)}{(\delta+\lambda)^4}d\mu(\lambda)\\&\le\int_0^\infty\left[\frac{(A+\lambda)^{-1}+(B+\lambda)^{-1}}{2}-\left(\lambda+\frac{A+B}{2}\right)^{-1}\right]w(\lambda)d\mu(\lambda)\\&\le\frac{1}{2}\frac{M^2}{\beta+\delta}\int_0^\infty\frac{(\beta+\lambda)(\delta+\lambda)}{(\alpha+\lambda)^4}w(\lambda)d\mu(\lambda).\end{aligned} \tag{3.11}$$

We observe that, by the definition of $\mathcal{D}(w,\mu)(t)$, and the properties of the derivatives of integrals with a parameter, we have

$$\begin{aligned}\mathcal{D}'(w,\mu)(t)&:=-\int_0^\infty\frac{w(\lambda)}{(\lambda+t)^2}d\mu(\lambda),\\\mathcal{D}''(w,\mu)(t)&:=2\int_0^\infty\frac{w(\lambda)}{(\lambda+t)^3}d\mu(\lambda),\end{aligned}$$

and

$$\mathcal{D}'''(w,\mu)(t):=-6\int_0^\infty\frac{w(\lambda)}{(\lambda+t)^4}d\mu(\lambda),$$

which gives that

$$\int_0^\infty\frac{w(\lambda)}{(\lambda+\delta)^4}d\mu(\lambda)=-\frac{1}{6}\mathcal{D}'''(w,\mu)(\delta). \tag{3.12}$$

Also, we observe that

$$\begin{aligned}\frac{(\beta+\lambda)(\delta+\lambda)}{(\alpha+\lambda)^4}&=\frac{(\beta-\alpha+\lambda+\alpha)(\delta-\alpha+\lambda+\alpha)}{(\alpha+\lambda)^4}\\&=(\beta-\alpha)(\delta-\alpha)\frac{1}{(\alpha+\lambda)^4}+(\delta+\beta-2\alpha)\frac{1}{(\alpha+\lambda)^3}+\frac{1}{(\alpha+\lambda)^2}.\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_0^\infty \frac{(\beta+\lambda)(\delta+\lambda)}{(\alpha+\lambda)^4} w(\lambda)d\mu(\lambda) \\
&= \int_0^\infty \frac{w(\lambda)d\mu(\lambda)}{(\alpha+\lambda)^2} + (\delta+\beta-2\alpha)\int_0^\infty \frac{w(\lambda)d\mu(\lambda)}{(\alpha+\lambda)^3} + (\beta-\alpha)(\delta-\alpha)\int_0^\infty \frac{w(\lambda)d\mu(\lambda)}{(\alpha+\lambda)^4} \\
&= -\mathcal{D}'(w,\mu)(\alpha) + \left(\frac{\delta+\beta}{2}-\alpha\right)\mathcal{D}''(w,\mu)(\alpha) - \frac{1}{6}(\beta-\alpha)(\delta-\alpha)\mathcal{D}'''(w,\mu)(\alpha).
\end{aligned} \tag{3.13}$$

By making use of (3.11)-(3.13), we deduce (3.7). □

COROLLARY 3.5. If the positive operators A, B are separated, namely $0 < \alpha \le A \le \beta < \gamma \le B \le \delta$ for some constants α, β, γ, δ, then

$$\begin{aligned}
0 &< -\frac{(\gamma-\beta)^2\gamma\alpha}{12(\alpha+\gamma)}\mathcal{D}'''(w,\mu)(\delta) \\
&\le \frac{\mathcal{D}(w,\mu)(A)+\mathcal{D}(w,\mu)(B)}{2} - \mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right) \\
&\le \frac{(\delta-\alpha)^2}{2(\beta+\delta)}\left[-\mathcal{D}'(w,\mu)(\alpha) + \left(\frac{\delta+\beta}{2}-\alpha\right)\mathcal{D}''(w,\mu)(\alpha) - \frac{1}{6}(\beta-\alpha)(\delta-\alpha)\mathcal{D}'''(w,\mu)(\alpha)\right].
\end{aligned} \tag{3.14}$$

We have:

COROLLARY 3.6. Assume that $f : [0,\infty) \to \mathbb{R}$ is an operator monotone function on $[0,\infty)$ with $f(0) = 0$, $0 < \alpha \le A$, $0 < \gamma \le B \le \delta$ and $0 < m \le B - A$ for some constants α, γ, δ, m. Then we have the refinement of Jensen's inequality

$$\begin{aligned}
0 &< -\frac{m^2\gamma\alpha}{12(\alpha+\gamma)}\left[\frac{f'''(\delta)\delta^3 - 3f''(\delta)\delta^2 + 6f'(\delta)\delta - 6f(\delta)}{\delta^4}\right] \\
&\le \frac{\mathcal{D}(w,\mu)(A)+\mathcal{D}(w,\mu)(B)}{2} - \mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right).
\end{aligned} \tag{3.15}$$

Proof. From (3.4) for $f(0) = 0$ we have

$$\mathcal{D}'(\ell,\mu)(t) = \frac{f'(t)t - f(t)}{t^2},$$
$$\mathcal{D}''(\ell,\mu)(t) = \frac{f''(t)t^2 - 2f'(t)t + 2f(t)}{t^3}$$

and

$$\mathcal{D}'''(\ell,\mu)(t) = \frac{f'''(t)t^3 - 3f''(t)t^2 + 6f'(t)t - 6f(t)}{t^4}.$$

Employing the first part of (3.14) we derive (3.15). □

4. SOME EXAMPLES

By employing the first inequality in Theorem 3.4, we derive (3.15). If $g(t) = t^{r-1}$ for $t > 0$, $r \in (0,1)$, then

$$g'(t) = (r-1)t^{r-2}, \quad g''(t) = (r-1)(r-2)t^{r-3},$$

and

$$g'''(t) = (r-1)(r-2)(r-3)t^{r-4}.$$

From (1.6) we get

$$\mathcal{D}(w_r)(t) = \frac{\pi}{\sin(r\pi)}t^{r-1}, \quad t > 0.$$

Then by (3.7) we get

$$\begin{aligned} 0 &< \frac{(1-r)(2-r)(3-r)m^2\gamma\alpha}{12(\alpha+\gamma)\delta^{4-r}} \\ &\le \frac{A^{r-1}+B^{r-1}}{2} - \left(\frac{A+B}{2}\right)^{r-1} \\ &\le \frac{M^2}{2(\beta+\delta)\alpha^{4-r}}\left[(1-r)\alpha^2+\left(\frac{\delta+\beta}{2}-\alpha\right)\alpha(1-r)(2-r)+\frac{1}{6}(\beta-\alpha)(\delta-\alpha)(1-r)(2-r)(3-r)\right] \end{aligned} \tag{4.1}$$

provided that $0 < \alpha \le A \le \beta$, $0 < \gamma \le B \le \delta$ and $0 < m \le B - A \le M$ for some constants α, β, γ, δ, m, M.

If we take $r \to 0+$ in (4.1), then we get

$$0 < \frac{m^2\gamma\alpha}{2(\alpha+\gamma)\delta^4} \le \frac{A^{-1}+B^{-1}}{2} - \left(\frac{A+B}{2}\right)^{-1} \le \frac{M^2\delta\beta}{2(\beta+\delta)\alpha^4} \tag{4.2}$$

which is the same as (2.11).

If $0 < \alpha \le A \le \beta < \gamma \le B \le \delta$ for some constants α, β, γ, δ, then

$$\begin{aligned} 0 &< \frac{(1-r)(2-r)(3-r)(\gamma-\beta)^2\gamma\alpha}{12(\alpha+\gamma)\delta^{4-r}} \\ &\le \frac{A^{r-1}+B^{r-1}}{2} - \left(\frac{A+B}{2}\right)^{r-1} \\ &\le \frac{(\delta-\alpha)^2}{2(\beta+\delta)\alpha^{4-r}}\left[(1-r)\alpha^2+\left(\frac{\delta+\beta}{2}-\alpha\right)\alpha(1-r)(2-r)+\frac{1}{6}(\beta-\alpha)(\delta-\alpha)(1-r)(2-r)(3-r)\right], \end{aligned} \tag{4.3}$$

where $r \in (0, 1)$.

If we take $r \to 0+$ in (4.3), then we get, see also (2.12),

$$0 < \frac{(\gamma-\beta)^2\gamma\alpha}{2(\alpha+\gamma)\delta^4} \le \frac{A^{-1}+B^{-1}}{2} - \left(\frac{A+B}{2}\right)^{-1} \le \frac{(\delta-\alpha)^2\delta\beta}{2(\beta+\delta)\alpha^4}. \tag{4.4}$$

We define the *upper incomplete Gamma function* as [12]

$$\Gamma(a, z) := \int_z^\infty t^{a-1}e^{-t}dt,$$

which for $z = 0$ gives *Gamma function*

$$\Gamma(a) := \int_0^\infty t^{a-1}e^{-t}dt \text{ for } \operatorname{Re} a > 0.$$

We have the integral representation [13]

$$\Gamma(a, z) = \frac{z^a e^{-z}}{\Gamma(1-a)}\int_0^\infty \frac{t^{-a}e^{-t}}{z+t}dt \tag{4.5}$$

for $\operatorname{Re} a < 1$ and $|\operatorname{ph} z| < \pi$.

Now, we consider the weight $w_{\cdot^{-a}e^{-\cdot}}(\lambda) := \lambda^{-a}e^{-\lambda}$ for $\lambda > 0$. Then by (4.5) we have

$$\mathcal{D}\left(w_{\cdot^{-a}e^{-\cdot}}\right)(t) = \int_0^\infty \frac{\lambda^{-a}e^{-\lambda}}{t+\lambda}d\lambda = \Gamma(1-a)t^{-a}e^t\Gamma(a, t) \tag{4.6}$$

for $a < 1$ and $t > 0$.

For $a = 0$ in (4.6) we get

$$\mathcal{D}\left(w_{e^{-\cdot}}\right)(t) = \int_0^\infty \frac{e^{-\lambda}}{t+\lambda}d\lambda = \Gamma(1)e^t\Gamma(0, t) = e^tE_1(t) \tag{4.7}$$

for $t > 0$, where

$$E_1(t) := \int_t^\infty \frac{e^{-u}}{u}du. \tag{4.8}$$

Let $a = 1 - n$, with n a natural number with $n \geq 0$, then by (4.6) we have

$$\begin{aligned} \mathcal{D}\left(w_{\cdot^{n-1}e^{-\cdot}}\right)(t) &= \int_0^\infty \frac{\lambda^{n-1}e^{-\lambda}}{t+\lambda}d\lambda = \Gamma(n)t^{n-1}e^t\Gamma(1-n,t) \\ &= (n-1)!t^{n-1}e^t\Gamma(1-n,t). \end{aligned} \tag{4.9}$$

If we define the generalized exponential integral [14] by

$$E_p(z) := z^{p-1}\Gamma(1-p,z) = z^{p-1}\int_z^\infty \frac{e^{-t}}{t^p}dt$$

then

$$t^{n-1}\Gamma(1-n,t) = E_n(t)$$

for $n \geq 1$ and $t > 0$.

Using the identity [14, Eq 8.19.7], for $n \geq 2$

$$E_n(z) = \frac{(-z)^{n-1}}{(n-1)!}E_1(z) + \frac{e^{-z}}{(n-1)!}\sum_{k=0}^{n-2}(n-k-2)!(-z)^k,$$

we get

$$\begin{aligned} \mathcal{D}\left(w_{\cdot^{n-1}e^{-\cdot}}\right)(t) &= (n-1)!e^tE_n(t) \\ &= (n-1)!e^t\left[\frac{(-t)^{n-1}}{(n-1)!}E_1(t) + \frac{e^{-t}}{(n-1)!}\sum_{k=0}^{n-2}(n-k-2)!(-t)^k\right] \\ &= \sum_{k=0}^{n-2}(-1)^k(n-k-2)!t^k + (-1)^{n-1}t^{n-1}e^tE_1(t) \end{aligned} \tag{4.10}$$

for $n \geq 2$ and $t > 0$.

For $n = 2$, we also get

$$\mathcal{D}\left(w_{\cdot e^{-\cdot}}\right)(t) = \int_0^\infty \lambda e^{-\lambda}(t+\lambda)^{-1}d\lambda = 1 - t\exp(t)E_1(t) \tag{4.11}$$

for $t > 0$.

PROPOSITION 4.1. For all $a < 1$, the function $t^{-a}e^t\Gamma(a,t)$ is operator convex on $(0,\infty)$. In particular, $e^tE_n(t)$ is operator convex on $(0,\infty)$. As a consequence $e^tE_1(t)$ is operator convex and $te^tE_1(t)$ is operator concave on $(0,\infty)$.

We can also consider the weight $w_{(\cdot^2+a^2)^{-1}}(\lambda) := \frac{1}{\lambda^2+a^2}$ for $\lambda > 0$ and $a > 0$. Then, by simple calculations, we get

$$\begin{aligned} \mathcal{D}\left(w_{(\cdot^2+a^2)^{-1}}\right)(t) &:= \int_0^\infty \frac{1}{(\lambda+t)(\lambda^2+a^2)}d\lambda \\ &= \frac{1}{(t^2+a^2)}\left(\frac{\pi t}{2a} - \ln t + \ln a\right) \end{aligned}$$

for $t > 0$ and $a > 0$.

For $a = 1$ we also have

$$\begin{aligned} \mathcal{D}\left(w_{(\cdot^2+1)^{-1}}\right)(t) &:= \int_0^\infty \frac{1}{(\lambda+t)(\lambda^2+1)}d\lambda \\ &= \frac{1}{t^2+1}\left(\frac{\pi}{2}t - \ln t\right) \end{aligned}$$

for $t > 0$.

PROPOSITION 4.2. For all $a > 0$, the functions

$$\frac{1}{(t^2+a^2)}\left(\frac{\pi t}{2a} - \ln t + \ln a\right)$$

are operator convex on $(0, \infty)$. In particular,

$$\frac{1}{t^2+1}\left(\frac{\pi}{2}t - \ln t\right)$$

is operator convex on $(0, \infty)$.

The interested reader may state other similar results by employing the examples of monotone operator functions provided in [3], [4], [5], [10] and [11].

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