Exact Solitary Wave Solutions of Some Non-Linear Partial Differential Equations arising in Wave Propagation and Optical Fibers

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Abstract:

One of the most intriguing areas of applied mathematics is the study of non-linear partial differential equations (NLPDEs). They play a pivotal role in describing, modelling, and predicting many real-life phenomena. Due to the abstract nature, the fundamental problem is to find their exact solutions. Several methods have been proposed for this purpose. The study aims to find out unexplored exact solitary wave solutions to some NLPDEs arising in the fields of wave propagation and optical fiber. We shall be dealing with nonlinear dispersive PDEs. They are the ones where we could expect to have special type of exact solutions known as solitary wave solutions or solitons. Since solitons have been proven to be the exact solutions of many families of NLPDEs, their complete understanding would lead us to a broad understanding of the real-life phenomena themselves. In this thesis, modified extended tanh method, improved tanh $(\frac{\phi}{2})$-expansion method, generalized auxiliary equation mapping method, and improved generalized Riccati equation method have been used to solve few distinguished NLPDEs and NLFPDEs. The results obtained by these methods are new and have not been reported in literature previously proves the efficacy and productiveness of these methods. The main objective of this research is to find new exact solutions and graphical visualization of these results of PDE of integer and fractional order. This project has two aspects of its significance. One is purely mathematical, and the other is its applications in other fields of science and technology. The new solutions would help scientists in developing cost-effective simulators to understand complex qualitative features of many phenomena in the fields of wave propagation and signal processing.

Declaration by author

I, Mehwish Rani, declare that the Ph.D. thesis entitled "Exact Solitary Wave Solutions of Some Non-Linear Partial Differential Equations arising in Wave Propagation and Optical fibers" is no more than 80,000 words in length including quotes and exclusive of tables, figures, appendices, bibliography, references, and footnotes.

This thesis contains no material that has been submitted previously, in whole or in part, for the award of any other academic degree or diploma. Except where otherwise indicated, this thesis is my own work.

I have conducted my research in alignment with the Australian Code for the Responsible Conduct of Research and Victoria University's Higher Degree by Research Policy and Procedures.

Signature: [redacted] Date: 24/11/2023

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Certificates

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MEHWISH RANI

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Participation - Technology Stream Presentation

ASSOCIATE PROFESSOR RANDALL W ROBINSON
Executive Director (Interim)
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List of Publications:

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1. M. Rani, N. Ahmed, S. S. Dragomir, S. T. Mohyud-Din, I. Khan, and K. S. Nisar, "Some newly explored exact solitary wave solutions to nonlinear inhomogeneous Murnaghan's rod equation of fractional order," Journal of Taibah University for Science, vol. 15, no. 1, pp. 97–110, Jan. 2021.

2. M. Rani, N. Ahmed, S. S. Dragomir, and S. T. Mohyud-Din, "New travelling wave solutions to (2+1)-Heisenberg ferromagnetic spin chain equation using Atangana's conformable derivative," Physica Scripta, vol. 96, no. 9, Sep. 2021.

3. M. Rani, N. Ahmed, S. S. Dragomir, and S. T. Mohyud-Din, "Traveling wave solutions of 3+1-dimensional Boiti–Leon–Manna–Pempinelli equation by using improved tanh(ϕ2)-expansion method," Partial Differential Equations in Applied Mathematics, vol. 6, p. 100394, Dec. 2022.

4. M. Rani, N. Ahmed, and S. S. Dragomir, "New exact solutions for nonlinear fourth-order Ablowitz–Kaup–Newell–Segur water wave equation by the improved tanh(φ(ξ) 2)-expansion method," https://doi.org/10.1142/S0217979223500443, vol. 37, no. 5, Sep. 2022.

5. N. Ahmed, M. Rani, S. S. Dragomir, and A. Akgul, "New exact solutions to space-time fractional telegraph equation with conformable derivative," International Journal of Modern Physics B, 2023.

6. N. Ahmed, M. Rani, S. S. Dragomir, and B. Bin Mohsin, "Optical soliton solutions of fokas system and (2 + 1) Davey-Stewartson system by mapping method," Physica Scripta, vol. 99, no. 3, p. 035209, Mar. 2024.

Results ready for submission:

Computational analysis of travelling wave solutions to some nonlinear dispersive equations using modified extended tanh expansion method.

List of Abbreviations

Notation	Meanings
METEM	Modified extended tanh expansion method
IThEM	Improved tanh($\frac{\phi(\xi)}{2}$)-expansion method
GAEMM	Generalized Auxiliary equation mapping method
IGREM	Improved generalized Riccati equation method
DDE	Doubly dispersive equation
NLPDEs	Nonlinear Partial Differential Equations
NLFPDEs	Nonlinear Fractional Partial Differential Equations
ShGEEM	Extended Sinh-Gordon equation expansion method
NLSE	Nonlinear Schrödinger Equation
HFM	Heisenberg ferromagnet model
WBBME	Wazwaz-Benjamin-Bona-Mahony equation
DBME	Dodd-Bullough-Mikhailov equation
SGE	Sinh-Gordan equation
LE	Liouville equation
BLMP	$(3+1)$-dimensional Boiti-Leon-Manna-Pempinelli equation
AKNS	fourth order Ablowitz-Kaup-Newell-Segur water wave
KdV	Korteweg–de Vries
DS	Davey-Stewartson
Δ	$p^2 - 4qr$

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Synopsis of Thesis:

Thesis Title: Exact Solitary Wave Solutions of Some Non-Linear Partial Differential Equations arising in Wave Propagation and Optical fibers.

Partial differential equations play an important role in modelling and analyzing the nonlinear real life physical phenomena, as there is an abundance of phenomena around us that can be represented by NLPDEs. It is very important to formulate not only the governing PDE of a certain phenomenon but also to find out its exact solutions**.** Since these solutions of the PDE representing a physical phenomenon can be used to simulate and replicate the phenomenon itself in a virtual environment. These PDEs are naturally abstract, so there is no single general solution-recipe that could work on all of them. Usually, each individual equation must be studied as a separate problem. Numerous numerical, analytical, and approximate methods have been proposed and implemented to get the exact solutions of PDEs. However, in this study we would be interested in a particular type of exact solutions known as the solitary wave solutions. For this we will be using such analytical methods that are recently developed and have not been applied to most of the PDEs arising in our field of interest. This project will go further by applying existing methods to fractional nonlinear PDEs. NLFPDEs are generalizations of NLPDEs in which the orders of derivatives involved are fractional. Some of the obtained results have been shown graphically in 3-D, 2-D and contour graphs to study wave dynamics.

Thesis outline:

This thesis includes 6 chapters,

Chapter-1: This chapter includes literature review including general introduction and preliminaries that provides significance of PDEs and FPDEs in different field of sciences along with history and background of solitons, which helps readers to understand the context of this research. This chapter also comprise basic definitions and brief description of methods used. Then it moves to motivation of this study, research objective, significance, and contribution to knowledge.

Chapter-2: The main objective of this chapter is to explore soliton solutions to some nonlinear PDEs by employing a very straightforward and robust technique called, modified extended tanh expansion method [1]. We have solved the Dodd-Bullough-Mikhailov equation, Sinh-Gordan equation, Liouville equation and (3+1)-dimensional Wazwaz-Benjamin-Bona-Mahony equation. We succeed in securing solitary and periodic wave solutions. Which provides deep insights into nonlinear phenomena and is helpful in different fields of sciences. Some of the derived solutions have been discussed in the form of 2-,3-dimensional graphs and contour plots to exhibit the power of proposed method graphically. The results generated by this technique are new and prove that it is a very strong and effective method to generate a variety of solutions and can be applied on different nonlinear models. All the graphs and solutions obtained in this chapter have been solved using computational software Maple.

Chapter -3: In this chapter, variety of exact wave solutions for recently developed $(3+1)$-dimensional Boiti-Leon-Manna-Pempinelli equation and fourth order Ablowitz-Kaup-Newell-Segur water wave (AKNS) equation has been investigated by using the innovative and efficient method called improved tanh$(\frac{\varphi(\xi)}{2})$-expansion method (IThEM). The exact solutions obtained for these equations are in the form of hyperbolic, trigonometric, exponential, logarithmic functions which are completely new and distant from previously derived solutions. Their solutions help scientists to investigate the dynamics of nonlinear fluids with higher dimensional effects. To understand the dynamical physical behavior of this equation some important solutions have been discussed graphically in the form of two and three-dimensional along with contour plots by selecting suitable parameters with the aid of Maple program. The achieved outcomes exhibit that this new method is efficient, direct, and provides different classes of families. This technique can solve many nonlinear differential equations having importance in different fields of sciences. All the graphs and solutions obtained in this chapter have been solved using computational software Maple.

Chapter-4: In this chapter, a very effective technique called generalized auxiliary equation mapping method has been employed to investigate some very important nonlinear equations in optical fibers such as Fokas system and $(2 + 1)$ Davey-Stewartson (DS) system. Under different situations, the obtained solutions exhibit various wave pattern like bright and dark soliton, kink soliton, periodic wave soliton and singular solitons. Solutions of both equations provide valuable insights of wave propagation, signal processing in optical fibers, imaging techniques and have applications in many areas such as mathematical physics, biology, and oceanography. These solutions are novel and interesting and prove the efficiency of the method. The accuracy of the obtained results provides the efficiency of the method and ensures that it can be used for other mathematical models involved in optical fibers. Graphical simulation of some reported results has been discussed here to visualize and support the mathematical results in terms of 3-D, 2-D and contour plots. All the graphs and solutions obtained in this chapter have been solved using computational software Maple.

Chapter-5: In this chapter, improved generalized Riccati equation mapping method has been used to find some new exact travelling wave solutions to space-time fractional non-liner double dispersive equation (DDE), space-time fractional non-liner Telegraph equation for transmission lines, space-time fractional (2+1) dimensional Heisenberg ferromagnetic spin chain equation. Riccati equation mapping method proves to be very effective tool to find a variety of soliton solutions. As a result, we found dark, combined dark-bright, singular periodic wave, combined singular periodic wave solutions and rational solutions. These newly discovered solutions would help a large community of scientists to understand the phenomenon such as earth sciences and shock physics in a more depth also interpretation of these exact solutions can help scientists to develop new technologies such as soliton-based communication devices. We have simulated the solitons, to check their types, with the help of graphs and all the solutions obtained in this article have been verified by back substitution in original equation by using Maple 17.

Chapter-6: This chapter includes the summary of previous chapters, highlights the significance of this research, contribution to the knowledge and conclusions. It also includes limitations of our work and scope of further work in this field.

All the references will be stated in the end of our work.

Chapter 1. Introduction, Preliminaries and Literature Review

1.1 Introduction:

A lot of physical phenomena happening around us can be represented by nonlinear partial differential equations. The NPDEs arising in optical fibers, plasma and biological sciences will be of great interest. It is very important to formulate the governing NPDE of a certain phenomenon as well as to find out its exact solutions. We shall be dealing with nonlinear dispersive PDEs. They are the ones where we could expect to have solitary wave solutions. First ever discovery of solitons, not termed as solitons then, was made in 1834 when the Victorian Engineer John Scott Russell observed a solitary wave, travelling along the Scottish canal [2]. The wave was travelling along the channel of water for a long period of time while still retaining its original identity. He reproduced the phenomenon in a wave tank and named it the "Wave of Translation". Unfortunately, his great observation could not get much attention from the scientists of the nineteenth and early twentieth century era. In the mid-1960's his work got attention when scientists started to use modern digital computers to analyze wave propagation. Nowadays, his ideas are used to formulate abstract dynamical behaviors of wave systems in different branches of science and engineering. The presence of so-called waves of translation has been already noticed in hydrodynamics, nonlinear optics, tornadoes, shock waves, plasma, and the Great Red Spot of Jupiter etc.

A soliton is a nonlinear solitary wave which has an additional property of retaining its permanent visual appearance, even if it interacts with another soliton. The difference between solitary waves and solitons is not much highlighted in the literature and had been blurred. We may define solitary waves to be the soliton like solutions of NPDEs describing the wave processes in dispersive and dissipative media. A single soliton solution is commonly referred to as a solitary wave. However, when two or more soliton like solutions occur, they are termed as solitons [3] . Since solitons have been proved to be the exact solutions of a large class of PDEs that are well accepted as the governing equations of many real-life phenomena, it is very important to understand them well. Their complete understanding would lead us to a broad understanding of the real-life phenomena themselves. Solitons are developed by the balance between nonlinearity and linear dispersion, nonlinearity tends to localize the wave while dispersion spreads it out. If we can create this balance, then we could expect to have soliton solution of a PDE. Well known example having solitons is, Korteweg-de Vries (KdV) equation [4] used to model the shallow water waves. The applications of shallow water equations are very vast in the field of ocean modelling and Coriolis forces in atmosphere. Shallow water wave equation is also introduced to examine the characteristic of moist convection in atmospheric dynamics [5].

Another well-known nonlinear model is Schrödinger equation, is very important equation in Physics for some obvious reasons as it describes nonlinear wave propagation in optics, nonlinear fluids, rouge ocean waves, it generates exact solitary waves called solitons. Zakharov and Shabat solved this equation first time in 1972 [6].

Neither the KDV and Schrödinger equation are the only equations, nor shallow water wave or optical fibers are the only phenomenon which involves solitary waves and their beneficial uses. The applicability of solitary wave solutions covers a broad range of practical problems.

As the solution of the NPDE representing a physical phenomenon is used to simulate and replicate the phenomenon itself in a virtual environment, therefore, the challenges of solving NPDEs have been a subject of interest of many mathematicians. Exact solutions play a very important role in the proper understanding of the physical phenomena they correspond to.

As NPDEs are naturally abstract, there is no single general technique to find out the solution that could work on all of them. Usually, each individual equation must be studied as a separate problem. Several scientists dedicated their bright minds to working out such methods that could be used to find the solutions to NPDEs and FNPDEs (nonlinear fractional partial differential equations). Numerous methods have been proposed and implemented to get the exact solutions of NPDEs. Such as tanh method [7], this is a powerful technique developed by Willy Malfiet in 1992 to compute exact solitary wave solutions in the form of tangent hyperbolic functions. In past many modifications had been done on this technique but Fan [8] extended this method using Riccati equation to generate different type of solutions along with hyperbolic function solutions. The Sine-Cosine method [9] developed by A. M Wazwaz, the pioneer of G'/G expansion method was Wang et al. [10], introduced this method to solve variety of nonlinear evolution equations, Ansatz method [11], R. Hirota introduced new form of Backlund transformation method [12], Painlevé expansion [13] was developed to provide unified approach for both nonlinear ordinary and partial differential equation, Auxiliary equation method [14] was introduced by Sirendaoreji, Functional variable method [15], Hirota method [16] was introduced by R. Hirota as a direct method to generate exact solutions and Backlund transformations of certain nonlinear models. Lie symmetry approach[17], Generalized Riccati equation mapping method [18] , Variational iteration method [19] to find approximate solutions of nonlinear problems, tanh–coth method [20] derived by A. M Wazwaz and many more methods. In the recent past, many techniques have modified, extended to improve the shortcomings of old methods to get more generalized types of exact solutions of nonlinear Partial differential equations of high order such as, double auxiliary equation method [21], modified extended Fan sub-equation method [22], Extended Jacobi's elliptic function method [23], the sardar sub-equation method [24], the generalized G'/G expansion method [25], Extended trial equation method [26], improved tanh $(\frac{\phi}{2})$-expansion method [27], improved generalized Riccati equation mapping method [18], Modified extended Tanh Method [1], generalized auxiliary equation method [28] and generalized Kudryashov method [29] etc.

1.2 Research Objectives:

Prime objective of this study is to investigate and procure novel exact solutions known as the solitary wave solutions for some nonlinear PDEs which are prominent in different fields of sciences and have numerous applications using few well known analytical methods. It is very important to formulate not only the governing PDE of a certain phenomenon but also to find out its exact solutions**.** I will be dealing with integrable nonlinear dispersive PDEs. They are the ones where we could expect to have solitary wave solutions. I handpicked the models that have importance in physics, fluid dynamics, plasma physics. Oceanography, biology and many more. Having the knowledge of the physical behavior of these nonlinear wave solutions helps scientists and Engineers to analyze, predict and control nonlinear phenomena such as rough waves in oceans, signal transmission in optical fibers, seismic waves, neural waves in brain, blood pressure, population dynamics, fluid flow in pipes, heat transfer. Finding these types of solutions is a momentous achievement by the researchers as they provide valuable insights about the behavior of nonlinear systems. All the models that are NLPDEs of order integer and fractional have been considered in this study and are selected wisely due to having significance in their respective field.

- My questions of interest would be:

1. Does a PDE have solitary wave solutions?

2. What types of solitons we may get after finding the solutions?
3. Are obtained solutions exact and novel?
4. What implications would these new solutions have for our understanding of the problem?
5. Are the obtained results accurate?
6. What mathematical tools can be developed/modified to get more and new exact solutions?
7. Where tools for exact solutions fail, can we use alternative approach to find exact solutions such as approximate analytical techniques?
8. Can the existing methods be extended to handle nonlinear partial differential equations of non-integer order?

1.3 Research Methodology:

This project is very intriguing as well as difficult at the same time. For an organized research effort, I had divided my research plan into different stages. Each one of them had its own importance and a timeline. The work structure that I followed can be divided in the following parts:

1.4 Phase 1:

During the literature review, I selected such PDEs that contain both linear dispersive and nonlinear terms. These NLPDEs of order integer and fractional are related to the fields of wave propagation and optical fibers. I reviewed several equations and selected those which have some practical interest and pointed out the possibility of totally novel solutions to those NPDEs. Some basic steps that I followed to get these solutions, and make them presentable to the research community, are explained in the subsections as follows.

1.4.1 Identification of suitable mathematical methods:

This stage involved the identification of "right tool for the right job". There are several ways to find soliton solutions to a given nonlinear dispersive partial differential equation. Many scientists have proposed various effective algorithms and techniques (some of which I have discussed in the section Literature Review). Each one of these techniques has its own advantages and disadvantages. Some are more generalized than the others and some are only suitable for some specific types of NPDEs. So, the selection of appropriate mathematical techniques is very important. I have used modified and generalized analytical methods which are new and robust in deriving new families of solutions. These methods have not applied previously to these models, signifying the importance of this research study.

1.4.2 Software used:

As in other fields such as Computer science and Information technology, mathematics and its related sciences took a great advantage of the modern technologies and their computational capabilities. It really boosted the research both in terms of quality and quantity. Many abstract equations and problems nowadays are only a matter of some built-in commands. However, their use is not limitless. For the PDEs I solved are of abstract nature and their ready-to-use recipes are still a dream. Since the obtainment of exact solutions to these PDEs is not an easy task, calculations just by hand is certainly not a good choice. Fortunately, we are blessed with several modern mathematical software which would help me to perform certain computational tasks and the visualization of the results. I used software named MAPLE and MATHEMATICA. There are many others but the reason for choosing these two is my previous familiarity with them.

1.4.3 Solutions of Equations:

To find exact solutions, I have taken help from the above-mentioned mathematical software. Although the software cannot solve the PDEs directly, they can however be used to perform certain tasks which were not possible, or extremely difficult, without them. Major steps to find the solutions of these equations is to transform PDEs to required ODEs by using complex wave transformation. Then by following the main steps of selected analytical technique I convert nonlinear ODE into system of algebraic system. We solve this system which leads me to the families of solutions of our PDE. The coding helped me to get through this stage with ease and at a rapid pace.

1.4.4 Verification of the solutions:

The next important step in my research was the verification of the obtained solutions. Exact solutions are the solutions that satisfy the PDE exactly. Without the verification, we cannot say for sure that the obtained mathematical expressions are in fact the solutions of the considered NPDE. The verification process again lies on the codes and manual verification is not possible most of the time. For verification, I directly substituted expected solutions into the NPDE and if it satisfies the differential equation, I considered those expressions to be the exact solutions of the NPDE.

1.4.5 Graphical simulation of the solution:

Next step of my research was graphical simulation of solutions. This is the last, but not the least, consequent substage of my research. By their simulations we can judge the type of a solitary wave (such as kink, periodic, singular, compactons, peakon, dark soliton and bright soliton etc.) and its journey across the domain of interest. Again, the illustration of these results is only possible with modern computer technology. Without the graphical simulation, it is very hard to explain the solutions and their practical uses. These solutions would help engineers and computer scientists to make simulators that can directly simulate the waves for practical uses with having too much information about the solutions themselves. It will also save them from going too deep into mathematical aspects of the equations.

1.4.6 Write up of the findings:

Writing the results is a very important part of all the phases throughout my research project. Collecting facts is one thing and presenting them in an interesting and self-explanatory way is another. Along with my writing skills, and guidance from my supervisors, I used a couple of software for the said purpose and performed my write-up in MS-Word and Scientific Workplace (a Latex compiler).

1.5 Phase II:

The second phase of my research proposal was to extend my project to the nonlinear PDEs of fractional integer. Fractional calculus is a branch of mathematical analysis that studies the real or complex number order differential or integral operators. It is currently a very active research issue among the researchers as a lot of physical phenomena can be modelled by means of fractional derivatives in many fields of science and engineering. This phase was conducted simultaneous to Phase I. As I mentioned earlier, this field is more open and even some of the very basic methods have not yet been extended to these types of PDEs. I hope in future I shall be able to extend several already existing methods to make them able to solve NFPDEs.

Equations studied:

The PDEs studied in this thesis are, Dodd-Bullough-Mikhailov equation, Sinh-Gordan equation, Liouville equation and (3+1)-dimensional Wazwaz-Benjamin-Bona-Mahony that plays significant role in problems arising in fluid flows, solid state physics, nonlinear optics, quantum field theory and chemical kinetics [30]. The $(3+1)$-dimensional Boiti-Leon-Manna-Pempinelli (BLMP) equation has imperative impact and significance in the wave propagation in incompressible fluids, moreover when $z = 0$, it describes the interaction of Riemann wave propagation [31]. Fourth order Ablowitz-Kaup-Newell-Segur water wave equation is significant because it can be reduced into some very famous nonlinear equations such as KdV equation, mKdV $(2+1)$ dimensional Boussinesq wave equation, sine-Gordan equation and nonlinear Schrödinger equation has wide range of applications in optical physics, quantum mechanics and many more [32]. Fokas system is the extension of nonlinear Schrodinger equation in $(2+1)$-dimension. Davey-Stewartson (DSS) equation is the generalization of Schrodinger equation. The doubly dispersive equation is an important nonlinear physical model describing the nonlinear wave propagation in the elastic inhomogeneous circular cylinder Murnaghan's rod. Nonlinear Telegraph equation is important mathematical model to study nonlinear wave propagation in electrical transmission lines, Heisenberg ferromagnet model (HFM) is an interesting nonlinear model that exhibits magnetic solitons and, also very important to study magnetic behavior in magnetic materials [33].

1.6 Significance and contribution to knowledge:

This project has two aspects of its significance. One is purely mathematical, and the other is its applications to the other fields of science and technology. The mathematical aspects involve the challenges of solving nonlinear PDEs which has been a subject of interest to many great mathematicians. This interest is due to the reason that behind almost every nonlinear PDE there lies a real-life phenomenon. As the solution of the PDE representing a physical phenomenon is used to simulate and replicate the phenomenon itself in a virtual environment, therefore, the exact solutions play a pivotal role in the proper understanding of that phenomena. New families of solutions for these PDEs provide more valuable information to researchers and scientists in expanding their scientific knowledge, studying insights of practical problems and provides new directions of research. Researchers working in labs can tally their findings with the exact solutions of the models. That would lead us all to more realistic and implementable models. Motivated by the significance of these models we are hopeful that our results which are new is a great contribution to the knowledge as these results will be beneficial to understand how nonlinearity of different models work and changes over time under certain conditions.

1.7 Definitions and properties:

1.7.1 Partial Differential Equation:

A partial differential equation is an equation that contains the dependent (the unknown function), and its partial derivatives. It is known that in the ordinary equations (ODE) the dependent variable $u = u(x)$, depends on only one independent variable x. Whereas, in PDEs the dependent variable $u = u(x,t), or\ u = u(x,y,t)$, must depend on more than one independent variable. Such as if $u = u(x,t)$, than it depends to independent variable x and on the time variable t.

Partial differential equations are classified as **linear** and **nonlinear** [34].

1.7.2 Definition and properties of modified Riemann-Liouville derivative:

Let us consider continuous (but not necessarily differential) function, $f: R \rightarrow R, w \rightarrow f(w)$, then its Jumarie's modified Riemann-Liouville fractional derivative of order α is defined as follows [35]:

$$f^{(\alpha)}(w) = \frac{1}{\Gamma(-\alpha)} \int_0^w (\mathrm{w} - \xi)^{-\alpha-1} [f(\xi) - f(0)]\, d\xi, \alpha < 0.$$

For $\alpha > 0$, we have,

$$f^{(\alpha)}(w) = (f^{(\alpha-1)}(w))',$$

$$= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dw} \int_0^w (\mathrm{w} - \xi)^{-\alpha} [f(\xi) - f(0)]\, d\xi, 0 < \alpha < 1. \tag{1.1}$$

And,

$$f^{(\alpha)}(w) = (f^{(q)}(w))^{\alpha-q}, q \le \alpha < q + 1, q \ge 1,$$

where $\Gamma(.)$ is gamma function defined as:

$$\Gamma(\alpha) = \lim_{q\to\infty} \frac{q!\, q^{\alpha}}{\alpha(\alpha+1)(\alpha+2) \ldots .(\alpha+q)}, \tag{1.2}$$

some characteristics of modified Riemann–Liouville derivative [35] are given below:

$$f_w^{(\alpha)}(w^\delta) = D_w{}^\alpha w^\delta = \frac{\Gamma(1+\delta)}{\Gamma(1+\delta-\alpha)} w^{\delta-\alpha}, \delta > 0. \tag{1.3}$$

The Jumarie's modified fractional differentiation is a linear operation:

$$D_w{}^\alpha\big(Af(w) + Bg(w)\big) = A\, D_w{}^\alpha f(w) + B\, D_w{}^\alpha g(w), \; A \; and \; B \text{ are constants.} \tag{1.4}$$

$$D_w{}^\alpha C = 0, \; C \text{ is constant.} \tag{1.5}$$

$$D_w{}^\alpha [f(w)\, g(w)] = g(w)\; D_w{}^\alpha f(w) + f(w)\, D_w{}^\alpha g(w), \tag{1.6}$$

$$D_w{}^\alpha f(\, g(w)) = f_g'(\, g(w))\, D_w{}^\alpha g(w) = \; D_g{}^\alpha f(\, g(w))(g'(w))^\alpha. \tag{1.7}$$

1.7.3 Properties and Definition of Caputo Derivative:

Let, m to be a smallest integer that is greater than α , the Caputo time fractional derivative operator of order $\alpha > 0$ of the function $u(t,\tau)$ is defined as follows [36].

$$D_t^\alpha f(t) = \frac{\partial^\alpha u(t,\tau)}{\partial t^\alpha},$$
$$= \begin{cases} \dfrac{1}{\Gamma(\mathrm{m}-\alpha)} \displaystyle\int_0^t (\mathrm{t}-\mathrm{s})^{\mathrm{m}-\alpha-1} \frac{\partial^m f(s)}{\partial s^m} ds, & m-1 < \alpha \le m, \\ \dfrac{\partial^m u(t,\tau)}{\partial t^m}, & \alpha = m \in N, \end{cases} \tag{1.8}$$

some characteristics of Caputo fractional derivative are given below [36].

For $\alpha\epsilon(m, m+1]$, the Caputo fractional derivative of the power function $t^\delta, \delta > 1$ is given by,

$$\mathrm{D}^\alpha t^\delta = \begin{cases} 0, & \alpha > \delta, \\ \dfrac{\Gamma(1+\delta)}{\Gamma(1+\delta-\alpha)} t^{\delta-\alpha}, & \alpha \le \delta. \end{cases} \tag{1.9}$$

$$D^\alpha C = 0, \; C \text{ is constant.} \tag{1.10}$$

Caputo derivative is linear.

$$D^\alpha\big(Af(t) + Bg(t)\big) = AD^\alpha f(t) + BD^\alpha g(t), \; A \text{ and } B \text{ are constants.} \tag{1.11}$$

If $f(t)$ is continuous in [0,1] and $g(t)$ has $\mathrm{n}+1$ continuous derivatives in [0, t].
If $f(t)$ is continuous function in $[a, b]$, then,

$$\frac{d}{dt} I^{\beta}\mathrm{f(t)} = I^{\beta}\frac{d}{dt}\mathrm{f(t)} + \frac{t^{\beta-1}}{\Gamma(\beta)}\mathrm{f(0)}. \qquad (1.12)$$

If $f(t)$ is continuous function in $[a, b]$, $f''(t)$ exists and $f'(0) = 0$, then,

$$D^{\alpha}D^{\beta}\mathrm{f(t)} = D^{\alpha+\beta}\mathrm{f(t)}, \qquad (1.13)$$

where $\alpha + \beta\epsilon(1,2)$.

Lemma 1: If $\mathrm{m} - 1 < \alpha < \mathrm{m}, m \in N$, then,

$$D^{\alpha}I^{\alpha}f(t) = f(t) \text{ and} \qquad (1.14)$$

$$I^{\alpha}D^{\alpha}f(t) = f(t) - \sum_{n=0}^{m-1}\frac{t^n}{n!}f^{(n)}(0), t > 0. \qquad (1.15)$$

Now we are going to address recently derived derivative called the conformable derivative in [37, 38] and it satisfies all the conditions of the standard derivative. Here, we shall present the definition and some properties of this new derivative.

1.7.4 Definitions and properties of Conformable derivative

Consider a function $f: [0, \infty) \to R$. The conformable derivative of the function $f\ (t)$ of α^{th} order is defined as[39]

$$D_t{}^{\alpha}(f)(t) = \lim_{\varepsilon\to 0}\frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}, \forall t > 0, \alpha\ \in (0,1). \qquad (1.16)$$

If $f(t)$ is $\alpha -$ differentiable in some $(0, a), a\ >\ 0,$ and $\lim_{t\to 0^+} f^{\alpha}(t)$ exists, then,

$$f^{\alpha}(0) = \lim_{t\to 0^+} f^{\alpha}(t).$$

Properties of differentiable Conformable derivatives that satisfy following properties:

i. Conformable derivative is linear.

$$D_t{}^{\alpha}\big(Af(t) + Bg(t)\big) = A\big(D^{\alpha}f(t)\big) + B\big(D^{\alpha}g(t)\big),\ \ A \textit{ and } B \text{ are constants.} \qquad (1.17)$$

ii. $D_t{}^{\alpha}(t^r) = rt^{r-1},\ \ r \in R.$ (1.18)

iii. $D_t{}^{\alpha}C = 0,\ \ C$ is constant. (1.19)

iv. Leibniz Rule, $D_t{}^{\alpha}(f(t).g(t)) = g(t)D_t{}^{\alpha}f(t) + f(t)D_t{}^{\alpha}g(t).$ (1.20)

v. $D_t{}^{\alpha}\left(\frac{f(t)}{g(t)}\right) = \frac{g(t)D_t{}^{\alpha}f(t)-f(t)D_t{}^{\alpha}g(t)}{g^2(t)}$. (1.21)

vi. If f is differentiable, $D_t{}^{\alpha}f(t) = (t)^{1-\alpha}\frac{df(t)}{dt}$. (1.22)

vii. Chain rule: Let f be an α −differentiable function and g is also differentiable defined in the range of f,

$$D_t{}^{\alpha}\big(gof(t)\big) = f'(t)D_t{}^{\alpha}g\big(f(t)\big). \tag{1.23}$$

1.7.5 Definitions and properties of Atangana's derivative

Here we review the definition of Atangana's conformable derivative and its various properties. The Atangana's conformable is defined as [40]

$${}_0^A D_t^{\alpha} f(t) = \lim_{\epsilon\to 0}\frac{f\left(t+\epsilon\left(t+\frac{1}{\Gamma(\alpha)}\right)^{1-\alpha}\right)-f(t)}{\epsilon}. \tag{1.24}$$

Properties of Atangana's derivative:

i. Let $f\colon[\alpha,\infty)\to R$ is a differentiable function which is also α −differentiable then,

$${}_0^A D_z^{\beta}\big(gof(z)\big) = f'(z)\,{}_0^A D_z^{\beta} g\big(f(z)\big), \tag{1.25}$$

where the function g is also differentiable and is defined in the range of f.

ii. Assume f and g, are functions, and both are β −differentiable then,

$${}_0^A D_z^{\alpha}\big(\acute{a}f(z)+\acute{b}g(z)\big) = \acute{a}\,{}_0^A D_z^{\alpha} f(z) + \acute{b}\,{}_0^A D_z^{\alpha} g(z), \tag{1.26}$$

$\acute{a}, \acute{b}$ are real numbers and $\beta \in (0,1]$.

iii. ${}_0^A D_z^{\alpha}(\mathrm{C}) = 0$, C is constant. (1.27)

iv. Leibniz Rule, ${}_0^A D_z^{\alpha}(f(z).g(z)) = g(z)\,{}_0^A D_z^{\alpha} f(z) + f(z)\,{}_0^A D_z^{\alpha} g(z)$, (1.28)

v. ${}_0^A D_z^{\alpha}\left(\frac{f(z)}{g(z)}\right) = \frac{g(z)\,{}_0^A D_z^{\alpha} f(z) - f(z)\,{}_0^A D_z^{\alpha} g(z)}{g^2(z)}$, provided $g \neq 0$, (1.29)

vi. Let us consider eq. (1.24) where $\epsilon = (z + \frac{1}{\Gamma(\alpha)})^{1-\alpha} h$, and $h \to 0$, when $\epsilon \to 0$, [41] therefore we get,

$$ {}_{0}^{A}D_{z}^{\alpha} f(z) = (z + \frac{1}{\Gamma(\alpha)})^{1-\alpha} \frac{df(z)}{dz}, \tag{1.30} $$

$$ \xi = \frac{\chi}{\alpha}(z + \frac{1}{\Gamma(\alpha)})^{\alpha}, \tag{1.31} $$

where χ is a constant. Hence, we obtain,

$$ {}_{0}^{A}D_{z}^{\alpha} f(\xi) = \chi \frac{df(\xi)}{d\xi}. \tag{1.32} $$

Remark: It is worth mentioning here that Ji-Huan He and Zheng-Biao Li [42] proposed an easy approach, namely the fractional complex transform[43, 44] which converts the fractional differential equations into ordinary differential equations. In Chapter 5 of this thesis, the fractional complex transforms and chain rule[45, 46] have been used with Caputo fractional derivative and conformable derivative to convert fractional-order partial differential equations, into integer order differential equations. The resulting equations are relatively easier to handle. They can be solved with different methods to obtain their exact solitary wave solutions. Moreover, we have used a recent definition of the comfortable fractional derivative called Atangana's conformable derivative[41, 47]. They have also proposed a transformation which converts the conformable fractional differential equation with Atangana's conformable derivative into a nonlinear conformable ordinary differential equation.

1.8 Modified extended tanh expansion method:

Let us consider the nonlinear partial differential equation with independent variables x, t and some dependent function $\dot{u}$:

$$ \text{Å}\left(\dot{u}, \frac{\partial}{\partial x}\dot{u}, \frac{\partial}{\partial \mathrm{t}}\dot{u}, \frac{\partial^2}{\partial x^2}\dot{u}, \frac{\partial^2}{\partial t^2}\dot{u}, \dots\right) = 0, \tag{1.33} $$

where Å is a polynomial in $\dot{u}$ with its various orders of nonlinear partial derivatives.

Step1. Let

$$\dot{u}(x,t) = \dot{u}(\xi\ ), \tag{1.34}$$

where,

$$\xi = kx + \nu t, \tag{1.35}$$

is a wave transformation which can convert nonlinear differential Eq.(1.33) into nonlinear ordinary differential equation,

$$\mathcal{H}(\dot{u}, k\dot{u}', \nu\dot{u}', k^2\dot{u}'', \nu^2\dot{u}'', \dots) = 0, \tag{1.36}$$

where k, ν are nonzero.

Step2. We suppose that the following series expansion is the solution of Eq.(1.36).

$$\dot{u}(\xi) = S = a_0 + \sum_{i=1}^{N}\left(a_i(\Phi(\xi))^i + b_i(\Phi(\xi))^{-i}\right), \tag{1.37}$$

where $\mathrm{a}_0, \mathrm{a}_i, b_i (1 \leq i \leq N)$ are constants, which are to be determined provided $\mathrm{a}_N, b_N \neq 0$. The function $\Phi = \Phi(\xi\ )$ satisfies the following ordinary differential equation.

$$\Phi'(\xi\ ) = \Omega + \Phi(\xi)^2, \quad \text{where } \Omega \text{ is real constant.} \tag{1.38}$$

The parameter N can be found by balancing highest order derivative with nonlinear term. Substituting (1.67) and (1.68) into the ordinary differential equation (1.29) will yield a system of algebraic equations in terms of a_0, a_i, b_i and Ω (where $1 \leq i \leq N$). Solving the resulting system of coefficients, we can then determine a_0, a_i, b_i and Ω. General solutions of Riccati differential equation (1.68) are as follows:

If $\Omega < 0$, we have

$$\phi(\xi) = -\sqrt{-b}\tanh\left(\left(\sqrt{-b}\xi'\right)\right),$$

or,

$$\phi(\xi) = -\sqrt{-b}\coth\left(\left(\sqrt{-b}\xi'\right)\right).$$

If $\Omega > 0$, we have

$$\phi(\xi) = \sqrt{b}\tan\left(\left(\sqrt{b}\xi'\right)\right),$$

or

$$\phi(\xi) = -\sqrt{b}\cot\left(\left(\sqrt{-b}\xi'\right)\right).$$

If $\Omega = 0$, we have

$$\phi(\xi) = -\frac{1}{\xi'}.$$

Using these general solutions of Riccati equation along with the values of a_0, a_i, b_i and Ω in to Eq (1.67), we have obtained the solutions of Eq (1.33).

1.9 Improved tanh $(\frac{\phi}{2})$-expansion method:

Let us consider the nonlinear partial differential equation with independent variables x, t and some dependent function $\dot{u}$:

$$\text{Å}(\dot{u}, \dot{u}_x, \dot{u}_t, \dot{u}_{xx}, \dot{u}_{tt}, \dots) = 0, \tag{1.39}$$

Where Å is a polynomial in $\dot{u}$ with its various orders of nonlinear partial derivatives.

Step1. Let

$$\dot{u}(x,t) = \dot{u}(\xi), \tag{1.40}$$

where,

$$\xi = kx + \nu t, \tag{1.41}$$

is a wave transformation which can convert nonlinear differential Eq. (1.63) into nonlinear ordinary differential equation,

$$\mathcal{H}(\dot{u}, k\dot{u}', \nu\dot{u}', k^2\dot{u}'', \nu^2\dot{u}'', \dots) = 0, \tag{1.42}$$

where k, ν are nonzero.

Step2. We suppose that the following series expansion is the solution of Eq. (1.42)

$$\dot{u}(\xi) = \Lambda(\phi) = \sum_{k=-N}^{N} \mathrm{A}_k [p + \tanh{(\phi/2)}]^k, \tag{1.43}$$

where $\mathrm{A}_k (0 \le k \le N)$ and $\mathrm{A}_{-k} (1 \le k \le N)$ are constants, which are to be determined provided $\mathrm{A}_N \neq 0, \mathrm{A}_{-N} \neq 0$. The function $\phi = \phi(\xi\ )$ satisfies the following ordinary differential equation.

$$\phi'(\xi\ ) = a \sinh\big(\phi(\xi\ )\big) + b\cos\mathrm{h}\big(\phi(\xi\ )\big) + c, \quad \text{where a, b, c are real constants.} \tag{1.44}$$

Eq. (1.44) has following special type of solutions:

Family 1: When $a^2 + c^2 - b^2 < 0, b - c \neq 0$ then

$$\phi(\xi) = 2\,\mathrm{arctanh}\left[-\frac{a}{b-c} + \frac{\sqrt{b^2 - a^2 - c^2}}{b-c} \tan\left(\frac{\sqrt{b^2 - a^2 - c^2}}{2} (\xi')\right)\right].$$

Family 2: When $a^2 + c^2 - b^2 > 0$ and $b - c \neq 0$, then

$$\phi(\xi) = 2\,\mathrm{arctanh}\left[-\frac{a}{b-c} - \frac{\sqrt{a^2 + c^2 - b^2}}{b-c} \tanh\left(\frac{\sqrt{a^2 + c^2 - b^2}}{2} (\xi')\right)\right].$$

Family 3: When $a^2 + c^2 - b^2 < 0$, b≠0 and c=0, then

$$\phi(\xi) = 2\,\mathrm{arctanh}\left[-\frac{a}{b} + \frac{\sqrt{b^2 - a^2}}{b} \tan\left(\frac{\sqrt{b^2 - a^2}}{2} (\xi')\right)\right].$$

Family 4: When $a^2 + c^2 - b^2 > 0$, c≠0 and b=0, then

$$\phi(\xi) = 2\,\mathrm{arctanh}\left[\frac{a}{c} + \frac{\sqrt{a^2 + c^2}}{c} \tan\left(\frac{\sqrt{a^2 + c^2}}{2} (\xi')\right)\right].$$

Family 5: When $a^2 + c^2 - b^2 < 0$, b-c≠0 and a=0, then

$$\phi(\xi) = 2\,\mathrm{arctanh}\left[\sqrt{\frac{b+c}{b-c}} \tan\left(\frac{\sqrt{b^2 - c^2}}{2} (\xi')\right)\right].$$

Family 6: When a =0 and c=0, then

$$\phi(\xi) = \ln\left[\tan\left(\frac{\mathrm{b}}{2}(\xi')\right)\right].$$

Family 7: When b=0 and c=0, then

$$\phi(\xi) = \ln\left[-\tanh\left(\frac{\mathrm{a}}{2}(\xi')\right)\right].$$

Family 8: When $a^2 + b^2 = c^2$, then

$$\phi(\xi) = 2\,\mathrm{arctanh}\left[\frac{a}{-b+\sqrt{a^2+b^2}} + \frac{\sqrt{2a}}{-b+\sqrt{a^2+b^2}}\tanh\left(\frac{\sqrt{2a}}{2}(\xi')\right)\right].$$

Family 9: When a=b=c=ka, then

$$\phi(\xi) = 2\mathrm{a\ \ rctanh}\left[e^{ka(\xi\prime)} - 1\right].$$

Family 10: When $a = c = ka$ and $b = -ka$, then

$$\phi(\xi) = 2\mathrm{arctanh}\left[\frac{e^{ka(\xi\prime)}}{-1 + e^{ka(\xi\prime)}}\right].$$

Family 11: When $b = a$, then

$$\phi(\xi) = -2\,\mathrm{arctanh}\left[\frac{(a+c)\,e^{b(\xi\prime)} - 1}{(a-c)\,e^{b(\xi\prime)} - 1}\right].$$

Family 12: When $b = c$, then

$$\phi(\xi) = 2\,\mathrm{arctanh}\left[\frac{e^{b(\xi')} - c}{a}\right].$$

Family 13: When $a = -c$, and $b = c$ then

$$\phi(\xi) = 2\,\mathrm{arctanh}\left[1 + e^{-c(\xi')}\right].$$

Family 14: When $b = -b$, and $c = -b$ then

$$\phi(\xi) = 2\,\text{arctanh}\left[\frac{b + e^{a(\xi')}}{a}\right].$$

Family 15: When $b = -b$, $a = -b$ and $c = b$ then

$$\phi(\xi) = 2\,\text{arctanh}\left[\frac{1}{e^{b(\xi')} - 1}\right].$$

Family 16: When $b = -c$, then

$$\phi(\xi) = 2\,\text{arctanh}\left[\frac{ae^{a(\xi')}}{ce^{a(\xi')} - 1}\right].$$

Family 17: When $a = 0$ and $b = c$, then

$$\phi(\xi) = 2\,\text{arctanh}\,[\mathrm{c}(\xi')]$$

Family 18: When $a = 0$, and $b = -c$, then

$$\phi(\xi) = 2\,\text{arctanh}\left[\frac{1}{c(\xi')}\right].$$

Family 19: When $b = 0$,and $a = c$ then

$$\phi(\xi) = 2\,\text{arctanh}\left[1 + \sqrt{2}\tanh\left(\frac{\sqrt{2}c}{2}(\xi')\right)\right].$$

Family 20: When $a = 0$,and $b = 0$ then

$$\phi(\xi) = c\xi + C,$$

where $\xi' = \xi + C$, $\mathrm{A}_k, \mathrm{A}_{-k} (k = 1,2, \dots, N), a, b, c$ are constants to be determined later. Positive integer N in Eq. (1.43) can be found by using homogeneous balance principle between the derivatives of highest order and the highest power of nonlinear terms in Eq. (1.43)

Step4. Substituting Eq. (1.43) along with Eq. (1.44) into Eq. (1.42). We get the polynomial equations. Equalizing coefficients of the resulting polynomial to zero, we get over-determined system of algebraic equations for $\mathrm{A}_i$where $i = 0, \pm 1, \pm 2, \dots . \pm N.$

Step5. With the help of Maple, we solve the system described in step 4, provides the values of $\mathrm{A}_0, \mathrm{A}_k, \mathrm{A}_{-k}$ where, $i = 1,2, \dots . N, a, b, c$. We substitute these values in Eq. (1.43) coupled with solutions of Eq. (1.44) and applying the transformation in Eq. (1.42), we construct several exact solutions of Eq. (1.39) , establishing twenty families [27].

1.10 Generalized Auxiliary Equation mapping Method:

It is now evident that NLPDEs have some amazing applications in different fields of sciences. To understand the physical phenomena of these equations some powerful methods are required to generate exact solutions. Finding suitable method for its application on PDEs and its interpretation is very critical for this research. For this reason, many useful methods have been introduced as each PDE is abstract in nature so there is no unified method that can be applicable on all type of PDEs. Some well-known methods in literature are Tanh expansion method [48], modified extended tanh expansion method [49] , Adomian's decomposition method [50], Backlund transformation method [51], Painlevé expansion [52], Fractional Homotopy analysis method [53], Kudryashov's method [54, 55], Exponential Rational function method [56]., $(\frac{G'}{G^2})$-expansion method [57], Khater method[58], Improved generalized Riccati equation mapping method [24]. Here we are rewriting famous method called generalized Auxiliary equation mapping method developed by Sirendaoreji [61]. By using an appropriate auxiliary equation not only makes calculations easy but also, we can find different types of exact solutions.

To describe the leading steps of the auxiliary equation method [61]. we consider the following NLPDE for an unknown function $\varphi(x,t)$.

$$\mathrm{M}\big(\varphi, \varphi_x, \varphi_y, \varphi_z, \varphi_{xx}, \dots\big) = 0. \tag{1.45}$$

Step 1. We assume Eq (1.45) has the following wave transformation $\xi = x - \rho t$. Substituting this wave transformation into Eq (1.45) turns into following ODE:

$$\mathrm{K}(v, v', v'', v''', \dots) = 0. \tag{1.46}$$

Step 2. AEM assumes the solution of Eq. (1.46) is of the form,

$$v(\xi) = \dot{a}_0 + \dot{a}_1 \mathbb{Q}(\xi) + \cdots + \dot{a}_{\aleph}\, \mathbb{Q}^{\aleph}(\xi), \tag{1.47}$$

in which $\dot{a}_\iota (i = 1, 2, \dots, \aleph)$ are all constants to be found.

Step 3. $\aleph$ is a positive integer which can be computed from the homogeneous balance principle.

$\mathbb{Q}(\xi)$ follows the auxiliary ODE as:

$$\left(\frac{d\mathbb{Q}}{d\xi}\right)^2 = a\mathbb{Q}^2(\xi) + b\mathbb{Q}^3(\xi) + c\mathbb{Q}^4(\xi), \tag{1.48}$$

here $a, b,$ and c are real valued parameters. The exact solutions of Eq. (1.48) are as follows.

Family 1: When $a > 0$, then

$$\mathbb{Q}(\xi) = \frac{-absech^2\left(\frac{\sqrt{a}}{2}\xi\right)}{b^2 - ac\left(1 + \varepsilon \tanh\left(\frac{\sqrt{a}}{2}\xi\right)\right)^2}. \tag{1.49}$$

Family 2: When $a > 0$, then

$$\mathbb{Q}(\xi) = \frac{abcsch^2\left(\frac{\sqrt{a}}{2}\xi\right)}{b^2 - ac\left(1 + \varepsilon \coth\left(\frac{\sqrt{a}}{2}\xi\right)\right)^2}. \tag{1.50}$$

Family 3: When $a > 0$ and $\Delta > 0$ then

$$\mathbb{Q}(\xi) = \frac{2asech(\sqrt{a}\xi)}{\varepsilon\sqrt{\Delta} - bsech(\sqrt{a}\xi)}. \tag{1.51}$$

Family 4: When $a < 0$ and $\Delta > 0$ then

$$\mathbb{Q}(\xi) = \frac{2asec(\sqrt{-a}\xi)}{\varepsilon\sqrt{\Delta} - bsec(\sqrt{-a}\xi)}. \tag{1.52}$$

Family 5: When $a > 0$ and $\Delta < 0$ then

$$\mathbb{Q}(\xi) = \frac{2acsch(\sqrt{a}\xi)}{\varepsilon\sqrt{-\Delta} - bcsch(\sqrt{a}\xi)}. \tag{1.53}$$

Family 6: When $a < 0$ and $\Delta > 0$ then

$$\mathbb{Q}(\xi) = \frac{2acsc(\sqrt{-a}\xi)}{\varepsilon\sqrt{\Delta} - bcsc(\sqrt{-a}\xi)}. \tag{1.54}$$

Family 7: When $a > 0$ and $c > 0$ then

$$\mathbb{Q}(\xi) = \frac{-\operatorname{asech}^2\left(\frac{\sqrt{a}}{2}\,\xi\right)}{b + 2\varepsilon\sqrt{ac}\tanh\left(\frac{\sqrt{a}}{2}\,\xi\right)}. \tag{1.55}$$

Family 8: When $c > 0$ and $a < 0$ then

$$\mathbb{Q}(\xi) = \frac{-\operatorname{asec}^2\left(\frac{\sqrt{-a}}{2}\xi\right)}{b + 2\varepsilon\sqrt{-ac}\tan\left(\frac{\sqrt{-a}}{2}\xi\right)}. \tag{1.56}$$

Family 9: When $c > 0$ and $a > 0$ then

$$\mathbb{Q}(\xi) = \frac{\operatorname{acsch}^2\left(\frac{\sqrt{a}}{2}\xi\right)}{b + 2\varepsilon\sqrt{ac}\coth\left(\frac{\sqrt{a}}{2}\xi\right)}. \tag{1.57}$$

Family 10: When $a < 0$ and $c > 0$ then

$$\mathbb{Q}(\xi) = \frac{-\operatorname{acsc}^2\left(\frac{\sqrt{-a}}{2}\xi\right)}{b + 2\varepsilon\sqrt{-ac}\cot\left(\frac{\sqrt{-a}}{2}\xi\right)}. \tag{1.58}$$

Family 11: When $a > 0$ and $\Delta = 0$ then

$$\mathbb{Q}(\xi) = -\frac{a}{b}\left(1 + \varepsilon\tanh\left(\frac{\sqrt{a}}{2}\xi\right)\right). \tag{1.59}$$

Family 12: When $a > 0$ and $\Delta = 0$ then

$$\mathbb{Q}(\xi) = -\frac{a}{b}\left(1 + \varepsilon\coth\left(\frac{\sqrt{a}}{2}\xi\right)\right). \tag{1.60}$$

Family 13: When $a > 0$ then

$$\mathbb{Q}(\xi) = \frac{4ae^{\varepsilon\sqrt{a}\xi}}{(e^{\varepsilon\sqrt{a}\xi} - b)^2 - 4ac}. \tag{1.61}$$

Family 14: When $a > 0$ and $b = 0$ then

$$\mathbb{Q}(\xi) = \frac{\pm 4a\varepsilon e^{\varepsilon\sqrt{a}\xi}}{1 - 4ace^{2\varepsilon\sqrt{a}\xi}}. \tag{1.62}$$

Step 4. We then substitute Eq. (1.47) and Eq. (1.48) into Eq. (1.46) and gathering all the coefficients of $(\mathbb{Q}(\xi))^I(\mathbb{Q}'(\xi))^J (I = 0,1,2 \ldots.)$ and $(J = 0,1)$ and equating them to zero yields a set of algebraic equations for unknowns $\dot{a}_\iota(i = 0, 1, \ldots, \aleph), a, b, c$. We solve this system with the aid of computational software Maple. In the end we plug the obtained solutions of the system along with the solutions of Eq. (1.48), we get solutions of Eq. (1.45).

1.11 Improved Generalized Riccati Equation Mapping Method:

The improved generalized Riccati equation method (IGREM) is one of the methods to get exact traveling wave solutions to the PDEs having both steepening and spreading effects. It is a straight-forward and easy-to-use method that, by symbolic computation, can generate many different types of exact traveling wave solutions. S. Zhu [18] introduced this method with the extended tanh-function method to solve (2+1) dimensional Boiti-Leon-Pempinelle equation. Cevikel et al. [62] used Riccati equation combined with tanh-coth method to solve nonlinear coupled equation in mathematical physics. Li et al. [63] used this method to find exact solutions of (3+1)-dimensional Jimbo-Miwa equation. Tala-Tebue et al. [64] used this method to solve discrete nonlinear electrical transmission lines in (2+1) dimension. Salathiel et al. [65] utilized generalized Riccati equation mapping method to construct soliton and travelling wave solutions for discrete electrical lattice. Koonprasert et al. [27], implemented this method to find more explicit solitary solutions to the space-time fractional fifth order nonlinear Sawada-Kotera equation. Most recently, Bibi. et.al [66] has used this method on Caudrey-Dodd-Gibsson equation. Their work shows that the improved generalized Riccati equation method has a great protentional for solving partial differential equations of integer and fractional order.

Let us consider the following differential equation with independent variables x, t and some dependent function u:

$$M(u, D_t u, D_x u, D_{xx} u, D_{xxx} u, \ldots.\ ) = 0, \tag{1.63}$$

where M is a polynomial in u with its various orders of nonlinear partial derivatives.

Step1. Let

$$u(x,t) = U(\xi\ ), \tag{1.64}$$

$$\xi = (x - \lambda t), \tag{1.65}$$

is a complex transformation which can convert nonlinear differential Eq. (1.63) into nonlinear ordinary differential equation, where λ is a constant which is to be determined, this complex transform is an easy transform to convert nonlinear differential equation into ordinary differential equation. Hence, we get.

$$Q' = Q'(U(\xi), U'(\xi), U''(\xi), \ldots.,) = 0, \tag{1.66}$$

where, $U'(\xi) = \frac{dU(\xi)}{d\xi}$ indicates derivative in term of ξ. We integrate Eq. (1.66) as many times as we get at least one term without derivative.

Step2. We suppose that the following series expansion is the solution of Eq. (1.66).

$$U(\xi) = \sum_{i=-N}^{N} a_i \phi(\xi)^i, \tag{1.67}$$

where $a_i (i = 0, \pm 1, \pm 2, \ldots. \pm N)$ being constants, which are to be determined provided $a_i \neq 0$. The function $\phi = \phi(\xi\ )$ satisfies the Riccati differential equation.

$$\phi'(\xi\ ) = r + p\phi(\xi\ ) + q\phi(\xi\ )^2, \quad \text{where } r, p, q \text{ are constants.} \tag{1.68}$$

Step3. Positive integer N in Eq. (1.67) can be found by using homogeneous balance between the derivatives of highest order and the nonlinear terms in Eq. (1.66) by the following formula.

Step4. Substituting Eq. (1.67) along with Eq. (1.68) into Eq. (1.66) followed by collecting all the same order terms ϕ^i together. We get the polynomial equation in ϕ^i and ϕ^{-i}, where $(i = 0,1,2, \ldots\ldots)$. Equalizing coefficients of the resulting polynomial to zero, we get over-determined system of algebraic equations for $a_i$where $i = 0, \pm 1, \pm 2, \ldots. \pm N$.

Step5. With the help of Maple, we solve the system described in step 4, and obtain a_i, where, $i = 0, \pm1, \pm2, \dots. \pm N$. We substitute these values in Eq. (1.67) coupled with solutions of Eq. (1.68) and applying the transformation in Eq. (1.66) we construct several exact solutions of Eq.(1.63), establishing four families [27].

Family 1: When $\Delta > 0$ and $pq \neq 0$ or $qr \neq 0$, the hyperbolic function solutions of Eq. (1.68) are,

$$\phi_1(\xi) = -\frac{1}{2q}\left[p + \sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{2}\xi\right)\right],$$

$$\phi_2(\xi) = -\frac{1}{2q}\left[p + \sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}}{2}\xi\right)\right],$$

$$\phi_3(\xi) = -\frac{1}{2q}\left[p + \sqrt{\Delta}(\tanh(\sqrt{\Delta}\xi) \pm i\,\mathrm{sech}(\sqrt{\Delta}\xi))\right],$$

$$\phi_4(\xi) = -\frac{1}{2q}\left[p + \sqrt{\Delta}(\coth(\sqrt{\Delta}\xi) \pm \mathrm{csch}(\sqrt{\Delta}\xi))\right],$$

$$\phi_5(\xi) = -\frac{1}{4q}\left[2\,p + \sqrt{\Delta}\left(\tanh\left(\frac{\sqrt{\Delta}}{4}\xi\right) + \coth\left(\frac{\sqrt{\Delta}}{4}\xi\right)\right)\right],$$

$$\phi_6(\xi) = \frac{1}{2q}\left[-p + \frac{\pm\sqrt{(A^2 + B^2)(\Delta)} - A\sqrt{\Delta}\cosh(\sqrt{\Delta}\xi)}{A\ \sinh(\sqrt{\Delta}\xi) + B}\right],$$

$$\phi_7(\xi) = \frac{1}{2q}\left[-p - \frac{\pm\sqrt{(B^2 - A^2)(\Delta)} + A\sqrt{\Delta}\sinh(\sqrt{\Delta}\xi)}{A\ \cosh(\sqrt{\Delta}\xi) + B}\right],$$

where two non-zero real constants A and B satisfies $B^2 - A^2 > 0$.

$$\phi_8(\xi) = \frac{2r\cosh\left(\frac{\sqrt{\Delta}}{2}\xi\right)}{\sqrt{\Delta}\sinh\left(\frac{\sqrt{\Delta}}{2}\xi\right) - p\cosh\left(\frac{\sqrt{\Delta}}{2}\xi\right)},$$

$$\phi_9(\xi) = \frac{-2r\sinh\left(\frac{\Delta}{2}\xi\right)}{p\sinh\left(\frac{\sqrt{\Delta}}{2}\xi\right) - \sqrt{\Delta}\cosh\left(\frac{\sqrt{\Delta}}{2}\xi\right)},$$

$$\phi_{10}(\xi) = \frac{2r\cosh(\sqrt{\Delta}\xi)}{\sqrt{\Delta}\sinh(\sqrt{\Delta}\xi) - p\cosh(\sqrt{\Delta}\xi) \pm i\sqrt{\Delta}},$$

$$\phi_{11}(\xi) = \frac{2r\sinh(\sqrt{\Delta}\xi)}{-p\sinh(\sqrt{\Delta}\xi) + \sqrt{\Delta}\cosh(\sqrt{\Delta}\xi) \pm \sqrt{\Delta}},$$

$$\phi_{12}(\xi) = \frac{4r\sinh\left(\frac{\sqrt{\Delta}}{4}\xi\right)\cosh\left(\frac{\sqrt{\Delta}}{4}\xi\right)}{\begin{pmatrix}-2p\sinh\left(\frac{\sqrt{\Delta}}{4}\xi\right)\cosh\left(\frac{\sqrt{\Delta}}{4}\xi\right)\\ +2\sqrt{\Delta}\cosh^2\left(\frac{\sqrt{\Delta}}{4}\xi\right) - \sqrt{\Delta}\end{pmatrix}}.$$

Family 2: When $\Delta < 0$ and $pq \neq 0$ or $qr \neq 0$, the trigonometric solutions of Eq. (1.68) are.

$$\phi_{13}(\xi) = \frac{1}{2q}\left[-p + \sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}}{2}\xi\right)\right],$$

$$\phi_{14}(\xi) = -\frac{1}{2q}\left[p + \sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}}{2}\xi\right)\right],$$

$$\phi_{15}(\xi) = \frac{1}{2q}\left[-p + \sqrt{-\Delta}(\tan(\sqrt{-\Delta}\xi) \pm \sec(\sqrt{-\Delta}\xi))\right],$$

$$\phi_{16}(\xi) = -\frac{1}{2q}\left[p + \sqrt{-\Delta}(\cot(\sqrt{-\Delta}\xi) \pm \csc(\sqrt{-\Delta}\xi))\right],$$

$$\phi_{17}(\xi) = \frac{1}{4q}\left[-2p + \sqrt{-\Delta}\left(\tan\left(\frac{\sqrt{-\Delta}}{4}\xi\right) - \cot\left(\frac{\sqrt{-\Delta}}{4}\xi\right)\right)\right],$$

$$\phi_{18}(\xi) = \frac{1}{2q}\left[-p + \frac{\pm\sqrt{(A^2 - B^2)(-\Delta)} - A\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)}{A\ \sin\left(\sqrt{4qr - p^2}\xi\right) + B}\right],$$

$$\phi_{19}(\xi) = \frac{1}{2q}\left[-p - \frac{\pm\sqrt{(A^2 - B^2)(-\Delta)} + A\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)}{A\ \sin\left(\sqrt{-\Delta}\xi\right) + B}\right],$$

where two non-zero real constants A and B satisfies $A^2 - B^2 > 0$.

$$\phi_{20}(\xi) = \frac{-2r\cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}{\sqrt{-\Delta}\sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right) + p\cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right)},$$

$$\phi_{21}(\xi) = \frac{2r\sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}{-p\sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right) + \sqrt{-\Delta}\cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right)},$$

$$\phi_{22}(\xi) = \frac{-2r\cos\left(\sqrt{-\Delta}\xi\right)}{\sqrt{-\Delta}\sin\left(\sqrt{-\Delta}\xi\right) + p\cos\left(\sqrt{-\Delta}\xi\right) \pm \sqrt{-\Delta}},$$

$$\phi_{23}(\xi) = \frac{2r\sin\left(\sqrt{-\Delta}\xi\right)}{-p\sin\left(\sqrt{-\Delta}\xi\right) + \sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right) \pm \sqrt{-\Delta}},$$

$$\phi_{24}(\xi) = \frac{4r\sin\left(\frac{\sqrt{-\Delta}}{4}\xi\right)\cos\left(\frac{\sqrt{-\Delta}}{4}\xi\right)}{\left(\begin{array}{c} -2p\sin\left(\frac{\sqrt{-\Delta}}{4}\xi\right)\cos\left(\frac{\sqrt{-\Delta}}{4}\xi\right) \\ +2\sqrt{-\Delta}\cos^2\left(\frac{\sqrt{-\Delta}}{4}\xi\right) - \sqrt{-\Delta} \end{array}\right)},$$

Family 3: When $r = 0$ and $pq \neq 0$ the solutions of Eq. (1.68) are,

$$\phi_{25}(\xi) = -\frac{pd}{q(d + \cosh(p\xi) - \sinh(p\xi))},$$

$$\phi_{26}(\xi) = \frac{-p\,(\cosh(p\xi) + \sinh(p\xi))}{q(d + \cosh(p\xi) + \sinh(p\xi))},$$

where d in the above solution is an arbitrary constant.

Family 4: When $r = p = 0$ and $q \neq 0$ the rational solutions of Eq. (1.68) is

$$\phi_{27}(\xi) = -\frac{1}{q\xi + c},$$

where c' in the above solution is an arbitrary constant.

1.12 Summary:

In this chapter intensive literature review has been done that comprises important definitions and properties that helps reader to get a refresher. It also includes a brief overview, and steps of all the methods used in this thesis, along with some background their significance in the real world together with the contribution to the knowledge.

In chapter 2 we will be finding exact solutions of some well-known equations.

Chapter 2. Abundant travelling wave solutions of some nonlinear equations using modified extended tanh expansion method.

2.1 Introduction:

In recent decades, to describe and analyze non-linear physical phenomena, partial differential equations (PDEs) have been used as the best tool. Seeking exact solutions of partial differential equations has been a hot topic. PDEs are abstract in nature and to find their solutions both numerically and analytically is a tedious task. To find the exact solutions of these PDEs is the main goal of researchers and to achieve their goal they are working hard to develop powerful techniques. There is no unified method to solve these equations, so to cope with this situation researchers are developing new methods and modifying previous methods such as Adomian's decomposition method [50], Backlund transformation method [51], Painlevé expansion [52], Fractional Homotopy analysis method [53], Variational iteration method [67], Sine-Cosine method [68], Homogeneous balance method [69], Fan sub-equation method [70], Modified simple equation method [71], First integral method [72], Extended trial equation method [73], $\exp(-\phi(\varepsilon))$-expansion method [74], Auxiliary equation method [75], Ansatz method [11], Functional variable method [15], improved generalized Riccati equation mapping method [18], tanh expansion method [48], modified extended tanh expansion method [1].

In this chapter, we will investigate the following nonlinear PDEs:

$$\frac{\partial^2}{\partial t\,\partial x}u + A\mathrm{e}^{u} + B\mathrm{e}^{-u} + \mathrm{C}\mathrm{e}^{-2u} = 0\,, \tag{2.1}$$

where A, B, C are arbitrary constants. The above-mentioned equation plays significant role in problems arising in fluid flows, solid state physics, nonlinear optics, quantum field theory and chemical kinetics [30]. For various values of A, B, C we have the following equations:

Dodd–Bullough–Mikhailov equation:

For $A = C = 1, B = 0$, we have

$$\frac{\partial^2}{\partial t\,\partial x}u + \mathrm{e}^{u} + \mathrm{e}^{-2u} = 0\,, \tag{2.2}$$

Dodd-Bullough-Mikhailov equation has significance in fluid flow and quantum field theory.

Sinh-Gordon equation:

For $A = 1, B = -1, C = 0$, we have

$$\frac{\partial^2}{\partial t\,\partial x}u + e^{u} - e^{-u} = 0\,, \tag{2.3}$$

The sine-Gordon equation has various applications and been discussed in literature in detail [76], some of them mentioned here such as, in one-dimensional crystal dislocation theory, magnetic flux propagation in Josephson junctions (gaps between two superconductors), wave propagation in ferromagnetic materials such as the motion of rigid pendula attached to a stretched wire, solid state physic , nonlinear optics, and dislocations in metals [30] and propagation of deformation along the DNA double helix [77] Exact solutions of considered equation has been obtained in terms of hyperbolic and trigonometric solutions using modified tanh method by mean of symbolic software Maple. One of the powerful features of this method comes from the fact that it is the generalization of many known methods, developed by Malfiet [48] and has been used and modified by many renowned researchers.

- **Liouville equation:**

For $A = 1, B = C = 0$, we have Liouville equation [78]

$$\frac{\partial^2}{\partial t\,\partial x}u + e^{u} = 0\,, \tag{2.4}$$

The motivation of this work is to boost the research related to these equations using powerful variation of modified extended tanh function method to provide more precise exact solutions. Tanh method was firstly presented by [48], where he introduced tanh as a new variable. This method is straight forward, simple, and reliable that has ability to find solutions of variety of NPFDEs without reproducing many different forms of the same solution. A lot of work has been done by this method with variations discussed in [79].

2.2 Illustrative Applications:

2.3 Dodd–Bullough–Mikhailov equation:

To use improved tanh expansion method on equation (2.2), first we will use Painlev transformation,

$v = e^u$, so that $u = \ln v$, this transformation will change equation (2.2) into the following ODE,

$$v(\frac{\partial^2}{\partial x\,\partial t}v) - (\frac{\partial}{\partial x}v)(\frac{\partial}{\partial t}v) + v^3 + 1 = 0. \quad (2.5)$$

Now using the following wave transformation,

$\xi = x - ct$,

in equation (2.5), converts the equation into the ODE,

$$-v(\frac{\partial^2}{\partial \xi^2}v)c + \left(\frac{\partial}{\partial \xi}v\right)^2 c + v^3 + 1 = 0. \quad (2.6)$$

Balancing the highest order of linear term with the nonlinear term in equation (2.6) we usually determine the value of N. Here $3N = 2(N+1) \Rightarrow N = 2$. This gives solution of the form,

$$v(\xi) = S = a_0 + a_1\Phi(\xi) + \frac{b_1}{\Phi(\xi)} + a_2\Phi(\xi)^2 + \frac{b_2}{\Phi(\xi)^2}. \quad (2.7)$$

Replacing equation (2.7) into equation (2.6) along with equation (1.38), we get algebraic system and by equating this system to 0 we get values of coefficients $a_0, a_1, a_2, b_1, b_2, c, \Omega$, as follows.

Set 1 :

$\Omega = \frac{3}{4c}, c = c, a_0 = \frac{1}{2}, a_1 = 0, a_2 = 0, b_1 = 0, b_2 = \frac{9}{8c}.$

Substituting these coefficients into equation (2.7) along with the Riccati equation solutions we get solutions of equation (2.6) as follows.

For $\Omega < 0$, we have

$$v_1 = \frac{-\tanh\left(\sqrt{3}/2\sqrt{-c^{-1}}\xi\right)^2 + 3}{2\tanh\left(\sqrt{3}/2\sqrt{-c^{-1}}\xi\right)^2}, \tag{2.8}$$

in addition, substituting $u = \ln v$ we determine the solution of equation (2.2) as

$$u_1 = \ln\frac{-\tanh\left(\sqrt{3}/2\sqrt{-c^{-1}}\xi\right)^2 + 3}{2\tanh\left(\sqrt{3}/2\sqrt{-c^{-1}}\xi\right)^2}. \tag{2.9}$$

Similarly, as done previously in equations (2.8)and (2.9) we get remaining solutions of equation (2.2) as

$$u_2 = \ln\frac{\coth\left(\sqrt{3}/2\sqrt{-c^{-1}}\xi\right)^2 - 3}{2\coth\left(\sqrt{3}/2\sqrt{-c^{-1}}\xi\right)^2}. \tag{2.10}$$

For $\Omega > 0$, we have

$$u_3 = \ln\frac{\tan\left(\frac{1}{2}(\sqrt{3}\sqrt{c^{-1}}\,\xi\right)^2 + 3}{2\tan\left(\frac{1}{2}(\sqrt{3}\sqrt{c^{-1}}\,\xi\right)^2}, \tag{2.11}$$

$$u_4 = \ln\frac{\cot\left(\frac{1}{2}(\sqrt{3}\sqrt{c^{-1}}\,\xi\right)^2 + 3}{2\cot\left(\frac{1}{2}(\sqrt{3}\sqrt{c^{-1}}\,\xi\right)^2}, \tag{2.12}$$

Set 2 :

$$\Omega = -\frac{\frac{3}{2} + \frac{3}{2i\sqrt{3}}}{4c}, c = c, a_0 = -\frac{1}{4} - i\frac{\sqrt{3}}{4}, a_1 = 0, a_2 = 0, b_1 = 0,$$

$$b_2 = 9\frac{\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)}{8c}.$$

For $\Omega < 0$, we have

$$u_5 = \ln \frac{1 - i\sqrt{3}}{2\left(1 + i\sqrt{3}\right)} \frac{\left(\left(\tanh\left(\frac{\sqrt{6}\sqrt{\frac{1+i\sqrt{3}}{c}}\xi}{4}\right)\right)^2 - 3\right)}{\tanh\left(\frac{\sqrt{6}\sqrt{\frac{1+i\sqrt{3}}{c}}\xi}{4}\right)^2}, \tag{2.13}$$

$$u_6 = \ln \frac{\left(1 + i\sqrt{3}\right)\left(\coth\left(\xi\sqrt{6}/4\sqrt{\frac{1+i}{c}}\right)^2 - 3\right)}{2\left(1 + i\sqrt{3}\right)\coth\left(\xi\sqrt{6}/4\sqrt{\frac{1+i}{c}}\right)^2}. \tag{2.14}$$

For $\Omega > 0$, we have

$$u_7 = \ln \frac{1 - i\sqrt{3}}{2\left(1 + i\sqrt{3}\right)} \frac{\left(\tan\left(\frac{\sqrt{\frac{-6(i\sqrt{3}+1)}{c}}\xi}{4}\right)^2 + 3\right)}{\tan\left(\frac{\sqrt{\frac{-6(i\sqrt{3}+1)}{c}}\xi}{4}\right)^2}, \tag{2.15}$$

$$u_8 = \ln \frac{1 - i\sqrt{3}}{2\left(1 + i\sqrt{3}\right)} \frac{\left(\cot\left(\frac{\sqrt{\frac{-6(i\sqrt{3}+1)}{c}}\xi}{4}\right)^2 + 3\right)}{\cot\left(\frac{\sqrt{\frac{-6(i\sqrt{3}+1)}{c}}\xi}{4}\right)^2}, \tag{2.16}$$

Set 3 :

$\Omega = \Omega, c = \frac{3}{4\Omega}, a_0 = \frac{1}{2}, a_1 = 0, a_2 = \frac{3}{2\Omega}, b_1 = 0, b_2 = 0.$

If $\Omega < 0$, we have

$$u_9 = \ln \frac{-2\cosh\left(\frac{4x\Omega - 3t}{4\sqrt{-\Omega}}\right)^2 + 3}{\cosh\left(\frac{4x\Omega - 3t}{4\sqrt{-\Omega}}\right)^2}, \tag{2.17}$$

$$u_{10} = \ln\left(\frac{1}{2} - \frac{3\coth\left(\frac{4x\Omega - 3t}{4\sqrt{-\Omega}}\right)^2}{2}\right). \tag{2.18}$$

For $\Omega > 0$, we have

$$u_{11} = \ln \frac{-2\cos\left(\frac{4x\Omega - 3t}{4\sqrt{\Omega}}\right)^2 + 3}{2\cos\left(\frac{4x\Omega - 3t}{4\sqrt{\Omega}}\right)^2}, \tag{2.19}$$

$$u_{12} = \ln\left(\frac{1}{2} + \frac{3\cot\left(\frac{4x\Omega - 3t}{4\sqrt{\Omega}}\right)^2}{2}\right), \tag{2.20}$$

Set 4 :

$\Omega = \Omega, c = \dfrac{3\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)}{4\Omega}, a_0 = -\dfrac{1}{4} + i\dfrac{\sqrt{3}}{4}, a_1 = 0, a_2 = \dfrac{3\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)}{2\Omega},$
$b_1 = 0, b_2 = 0.$

For $\Omega < 0$, we have

$$u_{13} = \ln \frac{\left(1 - i\sqrt{3}\right)\left(\cosh\left(\frac{3i\sqrt{3}t - 8x\Omega - 3t}{8\sqrt{-\Omega}}\right)^2 - \frac{3}{2}\right)}{2\cosh\left(\frac{3i\sqrt{3}t - 8x\Omega - 3t}{8\sqrt{-\Omega}}\right)^2}, \tag{2.21}$$

$$u_{14} = \ln \frac{3\left(1 - i\sqrt{3}\right)\left(\coth\left(\frac{3i\sqrt{3}t - 8x\Omega - 3t}{8\sqrt{-\Omega}}\right)^2 - \frac{1}{2}\right)}{4}. \tag{2.22}$$

If $\Omega > 0$, we have

$$u_{15} = \ln \frac{\left(1 - i\sqrt{3}\right)\left(\cos\left(\frac{3i\sqrt{3}t - 8x\Omega - 3t}{8\sqrt{\Omega}}\right)^2 - \frac{3}{2}\right)}{2\cos\left(\frac{3i\sqrt{3}t - 8x\Omega - 3t}{8\sqrt{\Omega}}\right)^2}, \tag{2.23}$$

$$u_{16} = \ln \frac{\left(-1 + i\sqrt{3}\right)\left(3\cot\left(\frac{3i\sqrt{3}t - 8x\Omega - 3t}{8\sqrt{\Omega}}\right)^2 + 1\right)}{4}, \tag{2.24}$$

Set 5 :

$\Omega = \frac{3}{8a_2}, c = \frac{a_2}{2}, a_0 = -\frac{1}{4}, a_1 = 0, a_2 = a_2,$

$b_1 = 0, b_2 = \frac{9}{64a_2}.$

For $\Omega < 0$, we have

$$u_{17} = \ln \frac{-3\tanh\left(\frac{\sqrt{6}\sqrt{-a_2{}^{-1}}(a_2t - 2x)}{8}\right)^4 - 2\tanh\left(\frac{\sqrt{6}\sqrt{-a_2{}^{-1}}(a_2t - 2x)}{8}\right)^2 - 3}{8\tanh\left(\frac{\sqrt{6}\sqrt{-a_2{}^{-1}}(a_2t - 2x)}{8}\right)^2}, \tag{2.25}$$

$$u_{18} = \ln \frac{-3\coth\left(\frac{\sqrt{6}\sqrt{-a_2{}^{-1}}(a_2t - 2x)}{8}\right)^4 - 2\coth\left(\frac{\sqrt{6}\sqrt{-a_2{}^{-1}}(a_2t - 2x)}{8}\right)^2 - 3}{8\coth\left(\frac{\sqrt{6}\sqrt{-a_2{}^{-1}}(a_2t - 2x)}{8}\right)^2}. \tag{2.26}$$

For $\Omega > 0$, we have

$$u_{19} = \ln \frac{3\tan\left(\frac{(a_2t - 2x)\sqrt{6}\sqrt{a_2{}^{-1}}}{8}\right)^4 - 2\tan\left(\frac{(a_2t - 2x)\sqrt{6}\sqrt{a_2{}^{-1}}}{8}\right)^2 + 3}{8\tan\left(\frac{(a_2t - 2x)\sqrt{6}\sqrt{a_2{}^{-1}}}{8}\right)^2}, \tag{2.27}$$

$$u_{20} = \ln \frac{3\cot\left(\frac{(a_2 t - 2x)\sqrt{6}\sqrt{a_2{}^{-1}}}{8}\right)^4 - 2\cot\left(\frac{(a_2 t - 2x)\sqrt{6}\sqrt{a_2{}^{-1}}}{8}\right)^2 + 3}{8\cot\left(\frac{(a_2 t - 2x)\sqrt{6}\sqrt{a_2{}^{-1}}}{8}\right)^2}, \tag{2.28}$$

Set 6 :

$$\Omega = -\frac{\frac{3}{2} + \frac{3i\sqrt{3}}{2}}{8a_2}, c = \frac{a_2}{2}, a_0 = \frac{1}{8} + i\frac{\sqrt{3}}{8}, a_1 = 0, a_2 = a_2, b_1 = 0,$$

$$b_2 = 9\frac{\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)}{64a_2}.$$

For $\Omega < 0$, we have

$$u_{21} = \ln \frac{(-1 + i\sqrt{3})}{(1 + i\sqrt{3})} \times \frac{\left(\begin{array}{c} 3\tanh\left(\frac{\sqrt{3}}{8}\sqrt{(i\sqrt{3}+1)a_2{}^{-1}}(a_2 t - 2x)\right)^4 \\ +2\tanh\left(\frac{\sqrt{3}}{8}\sqrt{(i\sqrt{3}+1)a_2{}^{-1}}(a_2 t - 2x)\right)^2 + 3 \end{array}\right)}{8\tanh\left(\frac{\sqrt{3}}{8}\sqrt{(i\sqrt{3}+1)a_2{}^{-1}}(a_2 t - 2x)\right)^2}, \tag{2.29}$$

$$u_{22} = \ln \frac{(-1 + i\sqrt{3})}{(1 + i\sqrt{3})} \times \frac{\left(\begin{array}{c} 3\coth\left(\frac{\sqrt{3}}{8}\sqrt{(i\sqrt{3}+1)a_2{}^{-1}}(a_2 t - 2x)\right)^4 \\ +2\coth\left(\frac{\sqrt{3}}{8}\sqrt{(i\sqrt{3}+1)a_2{}^{-1}}(a_2 t - 2x)\right)^2 + 3 \end{array}\right)}{8\coth\left(\frac{\sqrt{3}}{8}\sqrt{(i\sqrt{3}+1)a_2{}^{-1}}(a_2 t - 2x)\right)^2}. \tag{2.30}$$

For $\Omega > 0$, we have

$$u_{23} = \ln \frac{1-i\sqrt{3}}{i\sqrt{3}+1} \frac{\left(\begin{array}{c} 3\tan\left(1/8(a_2t-2x)\sqrt{\left(-3i\sqrt{3}-3\right){a_2}^{-1}}\right)^4 \\ -\frac{2}{3}\tan\left(1/8(a_2t-2x)\sqrt{\left(-3i\sqrt{3}-3\right){a_2}^{-1}}\right)^2+1 \end{array} \right)}{8\tan\left(1/8(a_2t-2x)\sqrt{\left(-3i\sqrt{3}-3\right){a_2}^{-1}}\right)^2}, \tag{2.31}$$

$$u_{24} = \ln \frac{1-i\sqrt{3}}{i\sqrt{3}+1} \frac{\left(\begin{array}{c} \cot\left(1/8(a_2t-2x)\sqrt{\left(-3i\sqrt{3}-3\right){a_2}^{-1}}\right)^4 \\ -\frac{2}{3}\cot\left(1/8(a_2t-2x)\sqrt{\left(-3i\sqrt{3}-3\right){a_2}^{-1}}\right)^2+1 \end{array} \right)}{8\cot\left(1/8(a_2t-2x)\sqrt{\left(-3i\sqrt{3}-3\right){a_2}^{-1}}\right)^2}. \tag{2.32}$$

2.4 Sinh-Gordon equation:

To use improved tanh expansion method on equation (2.3), first we will use Painlevé transformation. $v = e^u$, so that $u = \ln v$, this transformation will change equation (2.3) into the following ODE,

$$v\left(\frac{\partial^2}{\partial x\,\partial t}v\right) - \left(\frac{\partial}{\partial x}v\right)\left(\frac{\partial}{\partial t}v\right) + v^3 - v = 0. \tag{2.33}$$

By using the following wave transformation,

$$\xi = x - ct\,,$$

in equation (2.33), it converts the equation into the following ODE,

$$-v\left(\frac{\partial^2}{\partial \xi^2}v\right)c + \left(\frac{\partial}{\partial \xi}v\right)^2 c + v^3 - v = 0, \tag{2.34}$$

balancing the highest order of linear term with the nonlinear term in equation (2.34), we usually determine the value of N. Here $3N = 2(N+1) \Rightarrow N = 2$. This gives solution of the form,

$$v(\xi) = S = a_0 + a_1\Phi(\xi) + \frac{b_1}{\Phi(\xi)} + a_2\Phi(\xi)^2 + \frac{b_2}{\Phi(\xi)^2}. \tag{2.35}$$

Replacing equation (2.35) into equation (2.34) along with equation (1.38), we get algebraic system and by equating this system to 0 we get values of coefficients $a_0, a_1, a_2, b_1, b_2, c, \Omega$, as follows:

Set 1 :

$$\Omega = \frac{1}{2c}, c = \mathrm{c}, a_0 = 0, a_1 = 0, a_2 = 0, b_1 = 0, b_2 = \frac{1}{2c}.$$

Substituting above mentioned coefficients into equation (2.35) along with the Riccati equation solutions we get solutions of equation (2.33) as follows:

For $\Omega < 0$, we have

$$v_1 = -\tanh\left(\frac{1}{\sqrt{2}}\sqrt{\frac{-1}{c}}\xi\right)^{-2}, \tag{2.36}$$

moreover, substituting $u = \ln v$ we determine the solution of equation (2.3) as

$$w_1 = \ln\left(-\tanh\left(\frac{1}{\sqrt{2}}\sqrt{\frac{-1}{c}}\xi\right)^{-2}\right). \tag{2.37}$$

Adopting the same procedure, we will retrieve the remaining solutions of equation (2.3) as follows,

$$w_2 = \ln\left(-\coth\left(\frac{1}{\sqrt{2}}\sqrt{\frac{-1}{c}}\xi\right)^{-2}\right). \tag{2.38}$$

For $\Omega > 0$, we have

$$w_3 = \ln\left(\tan\left(\frac{1}{\sqrt{2}}\sqrt{\frac{1}{c}}\xi\right)^{-2}\right), \tag{2.39}$$

$$w_4 = \ln\left(\cot\left(\frac{1}{\sqrt{2}}\sqrt{\frac{1}{c}}\xi\right)^{-2}\right), \tag{2.40}$$

Set 2 :

$\Omega = -\frac{1}{2\text{c}}, c = \text{c}, a_0 = 0, a_1 = 0, a_2 = 0, b_1 = 0, b_2 = \frac{1}{2\text{c}}$.

For $\Omega < 0$, we have

$$w_5 = \ln\left(\tanh\left(\frac{1}{\sqrt{2}}\sqrt{\frac{1}{c}}\xi\right)^{-2}\right), \tag{2.41}$$

$$w_6 = \ln\left(\coth\left(\frac{1}{\sqrt{2}}\sqrt{\frac{1}{c}}\xi\right)^{-2}\right). \tag{2.42}$$

For $\Omega > 0$, we have

$$w_7 = \ln\left(-\tan\left(\frac{1}{\sqrt{2}}\sqrt{\frac{-1}{c}}\xi\right)^{-2}\right), \tag{2.43}$$

$$w_8 = \ln\left(-\cot\left(\frac{1}{\sqrt{2}}\sqrt{\frac{-1}{c}}\xi\right)^{-2}\right), \tag{2.44}$$

Set 3 :

$\Omega = \Omega, c = \frac{1}{2\Omega}, a_0 = 0, a_1 = 0, a_2 = \frac{1}{\Omega}, b_1 = 0, b_2 = 0.$

For $\Omega < 0$, we have

$$w_9 = \ln\left(-\tanh\left(\frac{-2x\Omega + t}{2\sqrt{-\Omega}}\right)^2\right), \tag{2.45}$$

$$w_{10} = \ln\left(-\coth\left(\frac{-2x\Omega + t}{2\sqrt{-\Omega}}\right)^2\right). \tag{2.46}$$

For $\Omega > 0$, we have

$$w_{11} = \ln\left(\tan\left(\frac{-2x\Omega + t}{2\sqrt{\Omega}}\right)^2\right), \tag{2.47}$$

$$w_{12} = \ln\left(\cot\left(\frac{-2x\Omega + t}{2\sqrt{\Omega}}\right)^2\right). \tag{2.48}$$

Set 4 :

$$\Omega = \Omega, c = -\frac{1}{2\Omega}, a_0 = 0, a_1 = 0, a_2 = -\frac{1}{\Omega}, b_1 = 0, b_2 = 0.$$

For $\Omega < 0$, we have

$$w_{13} = \ln\left(\tanh\left(\frac{2x\Omega + t}{2\sqrt{-\Omega}}\right)^2\right), \tag{2.49}$$

$$w_{14} = \ln\left(\coth\left(\frac{2x\Omega + t}{2\sqrt{-\Omega}}\right)^2\right). \tag{2.50}$$

For $\Omega > 0$, we have

$$w_{15} = \ln\left(-\tan\left(\frac{2x\Omega + t}{2\sqrt{\Omega}}\right)^2\right), \tag{2.51}$$

$$w_{16} = \ln\left(-\cot\left(\frac{2x\Omega + t}{2\sqrt{\Omega}}\right)^2\right). \tag{2.52}$$

Set 5 :

$$\Omega = -\frac{1}{4a_2}, c = \frac{a_2}{2}, a_0 = \frac{1}{2}, a_1 = 0, a_2 = a_2, b_1 = 0, b_2 = \frac{1}{16a_2}.$$

For $\Omega < 0$, we have

$$w_{17} = \ln\frac{\left(\tanh\left(\frac{1}{4}\sqrt{\frac{1}{a_2}}(ta_2 - 2x)\right)^2 + 1\right)^2}{4\tanh\left(\frac{1}{4}\sqrt{\frac{1}{a_2}}(ta_2 - 2x)\right)^2}, \tag{2.53}$$

$$w_{18} = \ln \frac{\left(\coth\left(\frac{1}{4}\sqrt{\frac{1}{a_2}}(ta_2 - 2x)\right)^2 + 1\right)^2}{\coth\left(\frac{1}{4}\sqrt{\frac{1}{a_2}}(ta_2 - 2x)\right)^2}. \tag{2.54}$$

For $\Omega > 0$, we have

$$w_{19} = \ln \frac{-\tan\left(\frac{1}{4}\sqrt{\frac{-1}{a_2}}(ta_2 - 2x)\right)^4 + 2\tan\left(\frac{1}{4}\sqrt{\frac{-1}{a_2}}(ta_2 - 2x)\right)^2 - 1}{4\tan\left(\frac{1}{4}\sqrt{\frac{-1}{a_2}}(ta_2 - 2x)\right)^2}, \tag{2.55}$$

$$w_{20} = \ln \frac{-\cot\left(\frac{1}{4}\sqrt{\frac{-1}{a_2}}(ta_2 - 2x)\right)^4 + 2\cot\left(\frac{1}{4}\sqrt{\frac{-1}{a_2}}(ta_2 - 2x)\right)^2 - 1}{4\cot\left(\frac{1}{4}\sqrt{\frac{-1}{a_2}}(ta_2 - 2x)\right)^2}. \tag{2.56}$$

Set 6 :

$\Omega = \frac{1}{4a_2}, c = \frac{a_2}{2}, a_0 = -\frac{1}{2}, a_1 = 0, a_2 = a_2, b_1 = 0, b_2 = \frac{1}{16a_2}.$

For $\Omega < 0$, we have

$$w_{21} = \ln \frac{\left(-\tanh\left(\frac{1}{4}\sqrt{\frac{-1}{a_2}}(ta_2 - 2x)\right)^2 + 1\right)^2}{4\tanh\left(\frac{1}{4}\sqrt{\frac{-1}{a_2}}(ta_2 - 2x)\right)^2}, \tag{2.57}$$

$$w_{22} = \ln \frac{\left(-\coth\left(\frac{1}{4}\sqrt{\frac{-1}{a_2}}(ta_2 - 2x)\right)^2 + 1\right)^2}{4\coth\left(\frac{1}{4}\sqrt{\frac{-1}{a_2}}(ta_2 - 2x)\right)^2}. \tag{2.58}$$

For $\Omega > 0$, we have

$$w_{23} = \ln \frac{\tan\left(\frac{1}{4}\sqrt{\frac{1}{a_2}}(ta_2 - 2x)\right)^4 - 2\tan\left(\frac{1}{4}\sqrt{\frac{1}{a_2}}(ta_2 - 2x)\right)^2 + 1}{4\tan\left(\frac{1}{4}\sqrt{\frac{1}{a_2}}(ta_2 - 2x)\right)^2}, \tag{2.59}$$

$$w_{24} = \ln \frac{\cot\left(\frac{1}{4}\sqrt{\frac{1}{a_2}}(ta_2 - 2x)\right)^4 - 2\cot\left(\frac{1}{4}\sqrt{\frac{1}{a_2}}(ta_2 - 2x)\right)^2 + 1}{4\cot\left(\frac{1}{4}\sqrt{\frac{1}{a_2}}(ta_2 - 2x)\right)^2}. \tag{2.60}$$

2.5 Liouville equation:

By choosing transformation $u = \ln v$, we get Eq (2.4) in the form as:

$$v(\frac{\partial^2}{\partial x\, \partial t} v) - (\frac{\partial}{\partial x} v)(\frac{\partial}{\partial t} v) + v^3 = 0. \tag{2.61}$$

To investigate the exact solutions of Eq (2.61) we introduce wave transformation $\xi = x - ct$, to get following ODE,

$$-v(\frac{\partial^2}{\partial \xi^2} v)c + (\frac{\partial}{\partial \xi} v)^2 c + v^3 = 0, \tag{2.62}$$

by balancing principle in equation (2.62) we determine the value of $N = 2$. This gives solution of the form,

$$v(\xi) = S = a_0 + a_1\Phi(\xi) + \frac{b_1}{\Phi(\xi)} + a_2\Phi(\xi)^2 + \frac{b_2}{\Phi(\xi)^2}. \tag{2.63}$$

Plugging equation (2.63) into equation (2.62) along with Riccati equation(1.38), we get algebraic system and by equating this system to 0 we get values of coefficients $a_0, a_1, a_2, b_1, b_2, c, \Omega$, as follows:

Set 1 :

$\Omega = \Omega, c = \frac{a_0}{2\Omega}, a_0 = a_0, a_1 = 0, a_2 = \frac{a_0}{\Omega}, b_1 = 0, b_2 = 0.$

Substituting above mentioned coefficients into equation (2.63) along with the Riccati equation solution we get solutions of equation (2.61) as follows:

For $\Omega < 0$, we have

$$v_1 = \frac{a_0}{\cosh\left(\frac{-2x\Omega + a_0 t}{2\sqrt{-\Omega}}\right)^2}, \tag{2.64}$$

moreover, substituting $u = \ln v$ we determine the solution of Eq (2.4) as

$$\tau_1 = \ln \frac{a_0}{\cosh\left(\frac{-2x\Omega + a_0 t}{2\sqrt{-\Omega}}\right)^2}, \tag{2.65}$$

$$\tau_2 = \ln \frac{-a_0}{\sinh\left(\frac{-2x\Omega + a_0 t}{2\sqrt{-\Omega}}\right)^2}, \tag{2.66}$$

$$\tau_3 = \ln \frac{a_0}{\cos\left(\frac{-2x\Omega + a_0 t}{2\sqrt{\Omega}}\right)^2}, \tag{2.67}$$

$$\tau_4 = \ln \frac{a_0}{\sin\left(\frac{-2x\Omega + a_0 t}{2\sqrt{\Omega}}\right)^2}. \tag{2.68}$$

Set 2 :

$$\Omega = \Omega, c = \frac{b_2}{2\Omega^2}, a_0 = \frac{2b_2}{\Omega}, a_1 = 0, a_2 = \frac{b_2}{\Omega^2}, b_1 = 0, b_2 = b_2.$$

$$\tau_5 = \ln \frac{-b_2}{\Omega \cosh\left(\frac{-2x\Omega^2 + b_2 t}{2(-\Omega)^{\frac{3}{2}}}\right)^2 \sinh\left(\frac{-2x\Omega^2 + b_2 t}{2(-\Omega)^{\frac{3}{2}}}\right)^2}, \tag{2.69}$$

$$\tau_6 = \ln \frac{b_2}{\Omega \cos\left(\frac{-2x\Omega^2 + b_2 t}{2(\Omega)^{\frac{3}{2}}}\right)^2 \sin\left(\frac{-2x\Omega^2 + b_2 t}{2(\Omega)^{\frac{3}{2}}}\right)^2}, \tag{2.70}$$

Set 3 :

$$\Omega = \Omega, c = \frac{b_2}{2\Omega^2}, a_0 = \frac{b_2}{\Omega}, a_1 = 0, a_2 = 0, b_1 = 0, b_2 = b_2.$$

$$\tau_7 = \ln \frac{b_2\left(\tanh\left(\frac{-2x\Omega^2 + b_2 t}{2(-\Omega)^{\frac{3}{2}}}\right)^2 - 1\right)}{\tanh\left(\frac{-2x\Omega^2 + b_2 t}{2(-\Omega)^{\frac{3}{2}}}\right)^2 \Omega}, \tag{2.71}$$

$$\tau_8 = \ln \frac{b_2\left(\coth\left(\frac{-2x\Omega^2 + b_2 t}{2(-\Omega)^{\frac{3}{2}}}\right)^2 - 1\right)}{\coth\left(\frac{-2x\Omega^2 + b_2 t}{2(-\Omega)^{\frac{3}{2}}}\right)^2 \Omega}, \tag{2.72}$$

$$\tau_9 = \ln \frac{b_2\left(\tan\left(\frac{-2x\Omega^2 + b_2 t}{2(\Omega)^{\frac{3}{2}}}\right)^2 + 1\right)}{\tan\left(\frac{-2x\Omega^2 + b_2 t}{2(\Omega)^{\frac{3}{2}}}\right)^2 \Omega}, \tag{2.73}$$

$$\tau_{10} = \ln \frac{b_2\left(\cot\left(\frac{-2x\Omega^2 + b_2 t}{2(\Omega)^{\frac{3}{2}}}\right)^2 + 1\right)}{\cot\left(\frac{-2x\Omega^2 + b_2 t}{2(\Omega)^{\frac{3}{2}}}\right)^2 \Omega}. \tag{2.74}$$

2.6 Results and discussion

With the assistance of IThM, along with painleve transformation we accomplished to obtain numerous wave patterns for Dodd–Bullough–Mikhailov equation, Sinh-Gordon equation, Liouville equation. The obtained solutions are in the form of hyperbolic and trigonometric function solutions. All the obtained results are either solitary waves or trigonometric solutions. Different wave patterns can be obtained by giving appropriate values to free parameters. We observe the shape of the soliton depends on free parameters and it changes when we change the value of the parameters. These models include exponential functions terms which indicate the solutions are in logarithmic functions. These answers have not been reported previously, which might be a valuable addition in literature to analyze these models. 3-D, 2-D and contour plots explain divergence and physics of these waves by choosing suitable values of parameters included in solutions.

Graphical profile of Real value of Eq (2.10) expressed as u_2 has been exhibit in Figure 2.1, in the form of 3-dimensional, and 2-dimensional and contour plot which demonstrates W type soliton by choosing parameters, $c = -2, t = 1$.

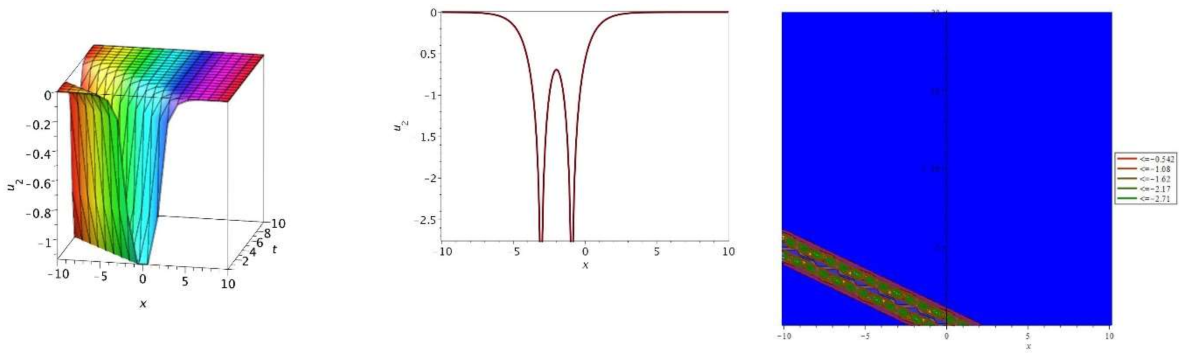


Figure 2.1:-graphs of solitary wave solution $u_2$for $c = -2, t = 1$

Graphical depiction of Real value of Eq (2.14) expressed as u_6 has been exhibit in Figure 2.2, in the form of 3-dimensional , and 2-dimensional and contour plot which demonstrates bright soliton by choosing parameters, $c = 2, t = 1$.

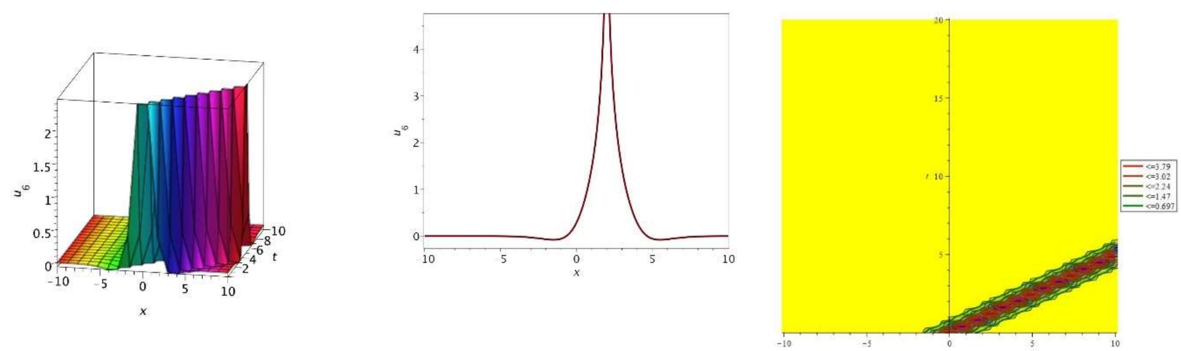


Figure 2.2:graphs of peaked soliton u_6 for $c = 2, t = 1$.

Graphical depiction of Real value of Eq (2.21) expressed as u_{13} has been exhibit in Figure 2.3, in the form of 3-dimensional, and 2-dimensional and contour plot which demonstrates as solitary wave solution by choosing parameters, $\Omega = -0.4$, $t = 2$.

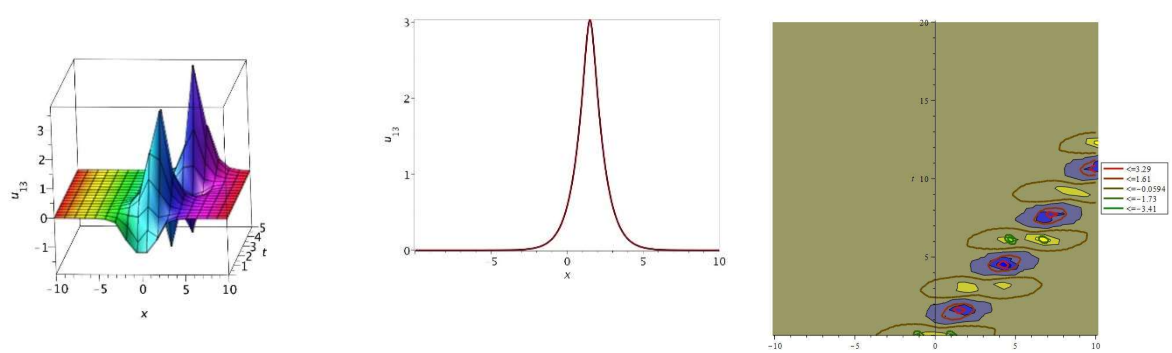


Figure 2.3: graphs of solitary wave solution u_{13}.for $\Omega = -0.4$, $t = 2$

Graphical depiction of Real value of Eq (2.32) expressed as u_{24} has been exhibit in Figure 2.4, in the form of 3-dimensional, 2-dimensional and their contour plot which demonstrates singular periodic wave solution by selecting parameters, $a_2 = -2,\ t = 1$.

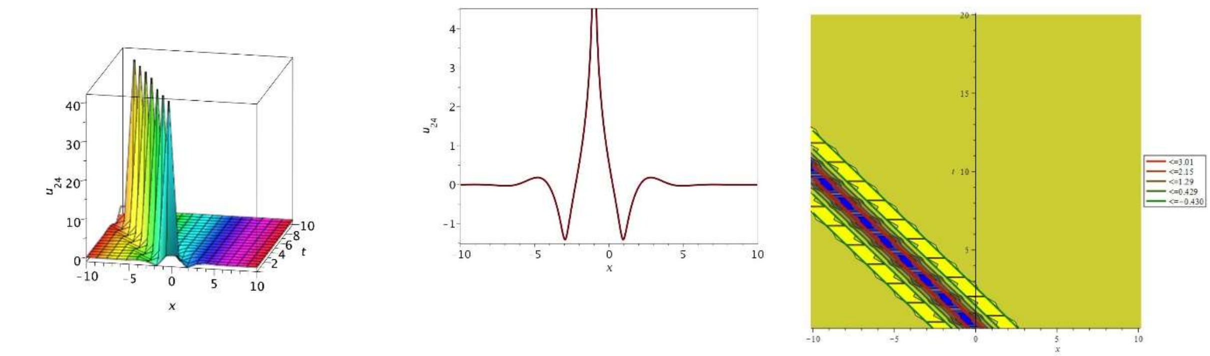


Figure 2.4: -graphs of singular periodic wave solution u_{24} for , $a_2 = -2,\ t = 1$.

Graphical illustration of Real value of Eq (2.37) expressed as w_1 has been exhibit in Figure 2.5, in the form of 3-dimensional, and 2-dimensional and contour plot which demonstrates localized excitation wave pattern as bright soliton by selecting parameters, $c = 0.-0.005,\ t = 1.$ Shape of solitary wave can be change by varying the value of c.

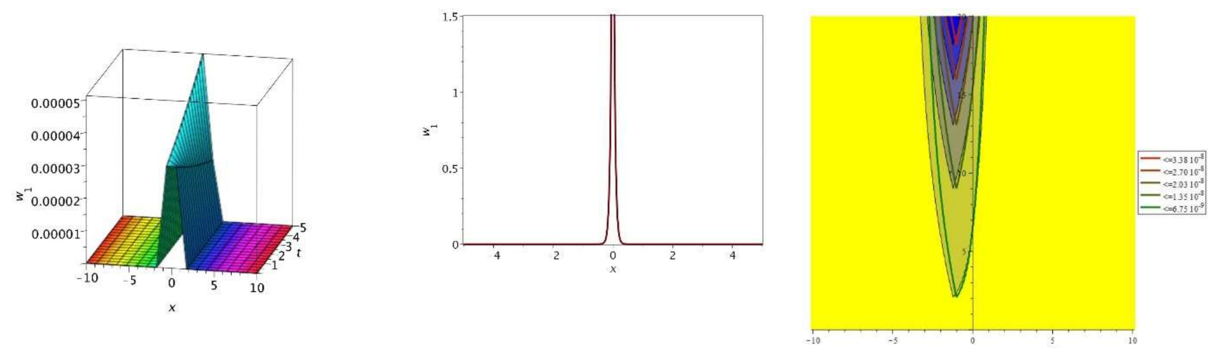


Figure 2.5: -graphical simulation of solitary wave solution $w_1$for $c = -0.005,\ t = 1.$

Graphical illustration of Eq (2.42) expressed as w_6 has been exhibit in Figure 2.6, in the form of 3-dimensional , and 2-dimensional and contour plot which demonstrates localized excitation wave pattern as dark soliton by selecting parameters, $c = 2,\ t = 1$.

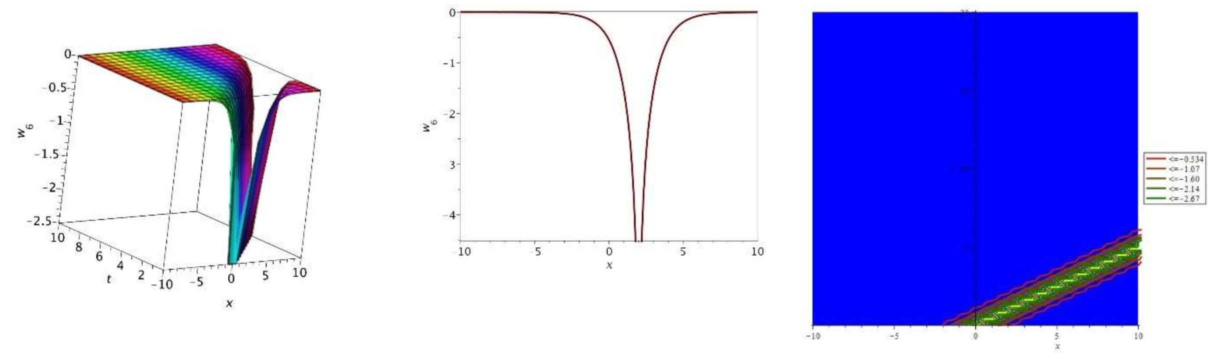


Figure 2.6: -graphical simulation of solitary wave solution w_6 for $c = 2,\ t = 1$.

Graphical illustration of Real value of Eq (2.51) expressed as w_{15} has been exhibit in Figure 2.7, in the form of 3-dimensional, and 2-dimensional and contour plot which demonstrates localized excitation wave pattern as periodic wave solution by selecting parameters, $\Omega = 1.5,\ t = 1$.

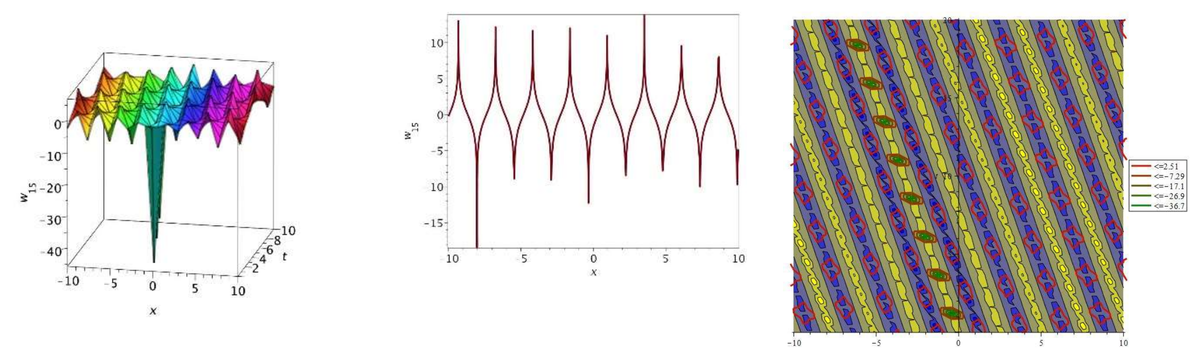


Figure 2.7: -graphical simulation of periodic wave solution w_{15} for $\Omega = 1.5,\ t = 1$.

Graphical illustration Real value of Eq (2.54) expressed as w_{18} has been exhibit in Figure 2.8, in the form of 3-dimensional, 2-dimensional and their contour plot which demonstrates localized excitation wave pattern as bright soliton by selecting parameters, $a_2 = 1,\ t = 1,$

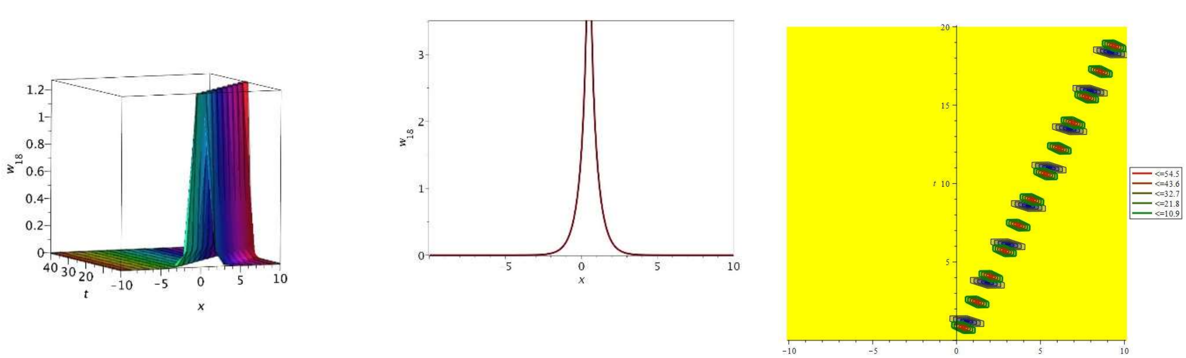


Figure 2.8: -graphical simulation of peaked soliton w_{18} for $a_2 = 1,,\ t = 1$.

Graphical illustration of absolute value of Eq (2.60) expressed as w_{24} has been exhibit in Figure 2.9, in the form of 3 dimensional, and 2 dimensional and contour plot which demonstrates localized excitation wave pattern as periodic wave solution by selecting parameters, $a_2 = 1.5, t = 1$.

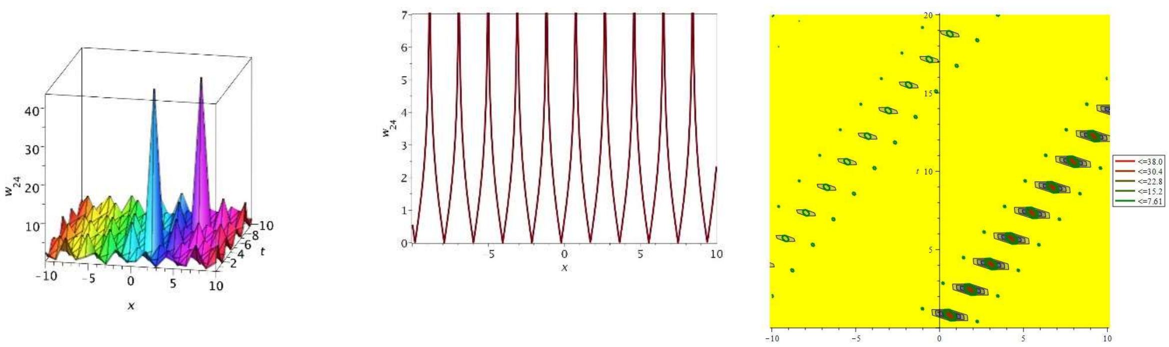


Figure 2.9: -graphical simulation of peaked soliton w_{24} for $a_2 = 1.5,\ t = 1$.

Figure 2.10 exhibits graphical analysis of Compacton for Real value of Eq (2.66) expressed as τ_2. 3-dimensional, 2-dimensional, along with contour plots have been presented with selected parameters, $a_0 = 0.1, \Omega = 0.5, t = 2$. The shape of the wave depends on these parameters.

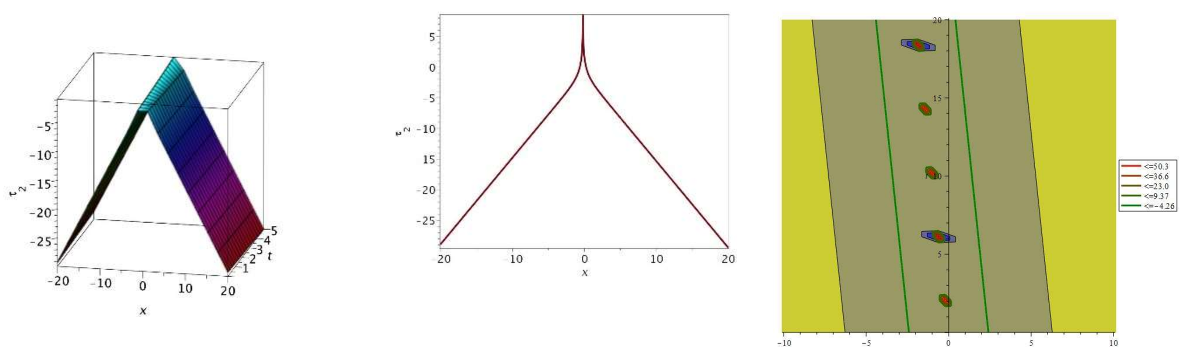


Figure 2.10: -graphical simulation of peaked soliton τ_2 for $a_0 = 0.1, \Omega = 0.5, t = 2$.

Figure 2.11, exhibits graphical analysis of periodic wave solution for Eq (2.74) expressed as τ_{10}. The 3, 2-dimensional along with contour plot have been presented with selected parameters, $\Omega = 1.5, b_2 = 0.5, t = 1$.

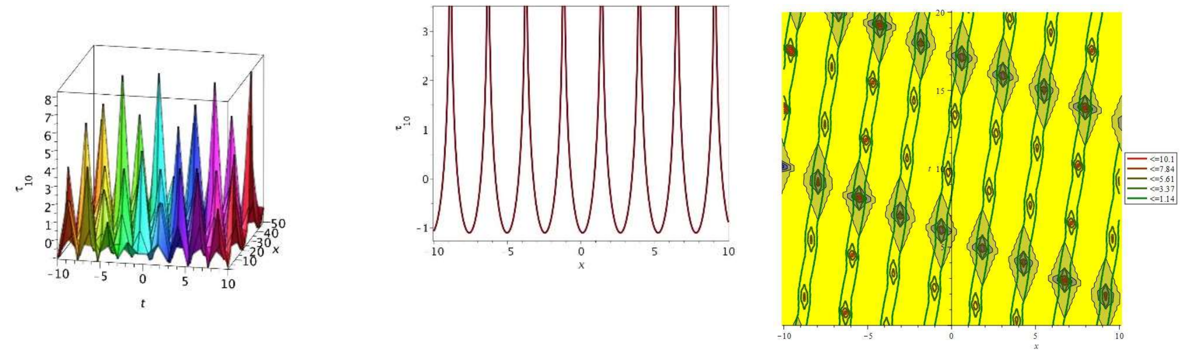


Figure 2.11: -graphical simulation of periodic wave solution τ_{10} for $\Omega = 1.5, b_2 = 0.5, t = 1$..

2.7 Conclusions

Improved tanh expansion method is applied to perceive general solutions of Dodd–Bullough–Mikhailov equation, Sinh-Gordon equation, and Liouville equation. As conclusion of these findings, we succeeded in generating some totally new solutions which are several bright and dark solitary wave solutions obtained in the form of hyperbolic wave solutions and periodic wave solutions. These new solutions may be worthwhile in the field of fluid flows, solid state physics, nonlinear optics, quantum field theory and chemical kinetics. This method is very efficient and straight forward to generate general and abundant solutions. Many researchers have applied this technique to many nonlinear models due to its effectiveness and still they are improving this method to increase its efficiency. The nature of generated solutions has been analyzed physically by 2D and 3D graph and contour plot simulation, and all the solutions obtained in this article have been verified by using mathematical software Maple.

2.8 $(3+1)$-dimensional Wazwaz -Benjamin-Bona-Mahony equations:

Benjamin-Bona-Mahony equation (BBM) was derived by Benjamin, Bona and Mahony in 1972, which is the improved version of Korteweg-de-Vries (KDV) equation for surfaced water waves in uniform channel and regularized version in shallow water waves [80]. A lot of work has been done on this equation due to its importance in surface wave water, in nonlinear dispersive system for long wave lengths, acoustic gravity waves in compressible liquids, hydromagnetic waves in plasma physics and many more. Later in 2017, Wazwaz studied (3+1) dimensional modified BBM equation and derived new equation which he named as Wazwaz-Benjamin-Bona-Mahony equation (WBBM) [81] as follows:

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}u + u^2\left(\frac{\partial}{\partial y}u\right) - \left(\frac{\partial^3}{\partial x\,\partial z\,\partial t}u\right) = 0, \tag{2.75}$$

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial \mathrm{y}}u + u^2\left(\frac{\partial}{\partial \mathrm{z}}u\right) - \left(\frac{\partial^3}{\partial x\,\partial \mathrm{x}\,\partial t}u\right) = 0, \tag{2.76}$$

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial \mathrm{z}}u + u^2\left(\frac{\partial}{\partial \mathrm{x}}u\right) - \left(\frac{\partial^3}{\partial x\,\partial \mathrm{y}\,\partial t}u\right) = 0. \tag{2.77}$$

Wazwaz [81] obtained solitons, periodic wave solutions and kink wave solutions using tanh/sech method. Used sardar sub equation method to obtain generalized hyperbolic and trigonometric function solutions. Based on these ideas we have used modified extended tanh method to derive new generalized solutions of WBBM equation.

Implementation of METEM

Here we study first equation of (3+1)-dimensional Wazwaz-Benjamin-Bona-Mahony equations (2.75)-(2.77), using the following travelling wave transformation,

$u(x, y, z, t) = U(\eta)$, with $\eta = kx + \lambda y + \mu z - ct$,

equations (2.75)-(2.77) reduces to ODEs and after integrating once we get,

$$(k - c)U(\eta) + \frac{\lambda U(\eta)^3}{3} + c\mu k\left(\frac{d^2}{d\eta^2}U(\eta)\right) = 0, \tag{2.78}$$

$$(\mu - c)U(\eta) + \frac{kU(\eta)^3}{3} + c\lambda k\left(\frac{d^2}{d\eta^2}U(\eta)\right) = 0, \tag{2.79}$$

$$(\lambda - c)U(\eta) + \frac{\mu U(\eta)^3}{3} + ck^2\left(\frac{d^2}{d\eta^2}U(\eta)\right) = 0. \tag{2.80}$$

Now applying balancing principle to nonlinear term $U(\eta)^3$ with the order to linear term $\frac{d^2}{d\eta^2}U(\eta)$ in equations (2.78)-(2.80) we get $N = 1$. Therefore we get,

$$U(\eta) = \Lambda(Y) = a_0 + a_1\Phi(\eta) + \frac{b_1}{\Phi(\eta)}, \tag{2.81}$$

now, substituting Eq. (2.81) along with Eq. (1.38) into Eq. (2.78)-(2.80), simultaneously after collecting all terms with the same powers of tanh $\left(\frac{\phi(\xi)}{2}\right)$ and equating each coefficient to 0, we get a system of NL algebraic equations. Solving these equations by using Maple 17, we get the following non-trivial solutions.

2.8.1 Equation 1:

Solving for the first equation of (3+1)-dimensional Wazwaz-Benjamin-Bona-Mahony equation, we have following set of coefficients.

Set 1 :

Substituting these coefficients into equation (2.78) along with the Riccati equation solutions we get solutions of equation (2.75) as follows:

For $\Omega < 0$, we have

$$\Omega = \frac{c-k}{2c\mu k}, a_0 = 0, a_1 = 0, b_1 = \sqrt{-\frac{3}{2\lambda c\mu k}(c-k)},$$

$$u_{1,1} = \frac{-b_1}{\sqrt{-\Omega}\tanh\left(\sqrt{-\Omega}\eta\right)}, \tag{2.82}$$

$$u_{1,2} = \frac{-b_1}{\sqrt{-\Omega}\coth\left(\sqrt{-\Omega}\eta\right)}. \tag{2.83}$$

For $\Omega > 0$, we have

$$u_{1,3} = \frac{b_1\sqrt{2}}{\sqrt{\Omega}\tan\left(\frac{\sqrt{2}\sqrt{\Omega}\eta}{2}\right)}, \tag{2.84}$$

$$u_{1,4} = \frac{-b_1\sqrt{2}}{\sqrt{\Omega}cot\left(\frac{\sqrt{2}\sqrt{\Omega}\eta}{2}\right)}. \tag{2.85}$$

Set 2 :

Substituting these coefficients into equation (2.78) along with the Riccati equation solutions we get solutions of equation (2.75) as follows:

For $\Omega < 0$, we have

$$\Omega = \frac{c-k}{2c\mu k}, a_0 = 0, a_1 = \sqrt{\frac{-6ck\mu}{\lambda}}, b_1 = 0.$$

$$u_{1,5} = -\sqrt{-\frac{6ck\mu}{\lambda}}\sqrt{-\Omega}\tanh\left(\sqrt{-\Omega}\eta\right), \tag{2.86}$$

$$u_{1,6} = -\sqrt{-\frac{6ck\mu}{\lambda}}\sqrt{-\Omega}\coth\left(\sqrt{-\Omega}\eta\right). \tag{2.87}$$

For $\Omega > 0$, we have

$$u_{1,7} = \frac{\sqrt{-\frac{6ck\mu}{\lambda}}\sqrt{2}\sqrt{\Omega}\tan\left(\frac{\sqrt{2}\sqrt{\Omega}\eta}{2}\right)}{2}, \tag{2.88}$$

$$u_{1,8} = -\frac{\sqrt{-\frac{6ck\mu}{\lambda}}\sqrt{2}\sqrt{\Omega}\cot\left(\frac{\sqrt{2}\sqrt{\Omega}\eta}{2}\right)}{2}. \tag{2.89}$$

Set 3 :

Substituting these coefficients into equation (2.78) along with the Riccati equation solutions we get solutions of equation (2.75) as follows:

For $\Omega < 0$, we have

$$a_0 = 0, a_1 = \sqrt{-\frac{6ck\mu}{\lambda}}, b_1 = -\frac{3\left(2c\mu k + \sqrt{-\frac{6ck\mu}{\lambda}}\sqrt{-6\lambda c\mu k}\right)(c-k)}{16ck\mu\sqrt{-6\lambda c\mu k}}.$$

$$u_{1,9} = -\left(\begin{array}{c}\sqrt{-\frac{3ck\mu}{\lambda}}\sqrt{-\Omega}\tanh\left(\frac{\sqrt{-\Omega}\eta}{\sqrt{2}}\right) \\ +3\left(2c\mu k + \sqrt{-\frac{6ck\mu}{\lambda}}\sqrt{-6\lambda c\mu k}\right)(c-k)\sqrt{2}\end{array}\right) \times \left(16ck\mu\sqrt{-6\lambda c\mu k}\left(\sqrt{-\Omega}\tanh\left(\frac{\sqrt{-\Omega}\eta}{\sqrt{2}}\right)\right)\right)^{-1}, \tag{2.90}$$

$$u_{1,10} = -\left(\begin{array}{c}\sqrt{-\frac{3ck\mu}{\lambda}}\sqrt{-\Omega}\coth\left(\frac{\sqrt{-\Omega}\eta}{\sqrt{2}}\right) \\ +3\left(2c\mu k + \sqrt{-\frac{6ck\mu}{\lambda}}\sqrt{-6\lambda c\mu k}\right)(c-k)\sqrt{2}\end{array}\right) \times \left(16ck\mu\sqrt{-6\lambda c\mu k}\left(\sqrt{-\Omega}\coth\left(\frac{\sqrt{-\Omega}\eta}{\sqrt{2}}\right)\right)\right)^{-1}. \tag{2.91}$$

For $\Omega > 0$, we have

$$u_{1,11} = \frac{1}{2}\sqrt{-\frac{6ck\mu}{\lambda}}\sqrt{-\Omega}\tan\left(\frac{\sqrt{-\Omega}\eta}{2}\right) - 3\left(2c\mu k + \sqrt{-\frac{6ck\mu}{\lambda}}\sqrt{-6\lambda c\mu k}\right)(c-k) \times \left(16ck\mu\sqrt{-6\lambda c\mu k}\left(\sqrt{-\Omega}\tan\left(\sqrt{-\Omega}\eta\right)\right)\right)^{-1}, \tag{2.92}$$

$$u_{1,12} = \frac{-1}{2}\sqrt{-\frac{6ck\mu}{\lambda}}\sqrt{-\Omega}\cot\left(\frac{\sqrt{-\Omega}\eta}{2}\right) + 3\left(2c\mu k + \sqrt{-\frac{6ck\mu}{\lambda}}\sqrt{-6\lambda c\mu k}\right)(c-k)$$
$$\times\left(16ck\mu\sqrt{-6\lambda c\mu k}\left(\sqrt{-\Omega}\cot\left(\sqrt{-\Omega}\eta\right)\right)\right)^{-1}, \quad (2.93)$$

2.8.2 Equation 2:

Solving for the second equation of (3+1)-dimensional Wazwaz-Benjamin-Bona-Mahony equation, we have following set of coefficients.

Set 1 :

Substituting these coefficients into equation (2.79) along with the Riccati equation solutions we get solutions of equation (2.76) as follows:

For $\Omega < 0$, we have

$$\Omega = \frac{c-\mu}{2ck\lambda}, a_0 = 0, a_1 = \sqrt{-6c\lambda}, b_1 = 0,$$

$$u_{2,1} = -\sqrt{-6c\lambda}\sqrt{-\Omega}\tanh\left(\sqrt{-\Omega}\eta\right), \quad (2.94)$$

$$u_{2,2} = -\sqrt{-6c\lambda}\sqrt{-\Omega}\coth\left(\sqrt{-\Omega}\eta\right). \quad (2.95)$$

For $\Omega > 0$, we have

$$u_{2,3} = \frac{1}{2}\left(\sqrt{-12c\lambda}\sqrt{\Omega}\tan\left(\frac{\sqrt{2}\sqrt{\Omega}\,\eta}{2}\right)\right), \quad (2.96)$$

$$u_{2,4} = \frac{-1}{2}\left(\sqrt{-12c\lambda}\sqrt{\Omega}\cot\left(\frac{\sqrt{2}\sqrt{\Omega}\,\eta}{2}\right)\right). \quad (2.97)$$

Set 2 :

Substituting these coefficients into equation (2.79) along with the Riccati equation solutions we get solutions of equation (2.76) as follows:

For $\Omega < 0$, we have

$$\Omega = -\frac{-\frac{c}{2}+\frac{\mu}{2}}{4ck\lambda}, a_0 = 0, a_1 = \sqrt{-6c\lambda}, b_1 = \frac{3(c-\mu)}{4\sqrt{-6c\lambda}k}.$$

$$u_{2,5} = -\sqrt{6}\frac{\left(\tanh\left(\frac{\sqrt{\frac{-2c+2\mu}{ck\lambda}}\eta}{4}\right)^2+1\right)(c-\mu)}{2\sqrt{-c\lambda}\sqrt{\frac{-2c+2\mu}{ck\lambda}}k\tanh\left(\frac{\sqrt{\frac{-2c+2\mu}{ck\lambda}}\eta}{4}\right)}, \tag{2.98}$$

$$u_{2,6} = -\sqrt{6}\frac{\left(\coth\left(\frac{\sqrt{\frac{-2c+2\mu}{ck\lambda}}\eta}{4}\right)^2+1\right)(c-\mu)}{2\sqrt{-c\lambda}\sqrt{\frac{-2c+2\mu}{ck\lambda}}k\coth\left(\frac{\sqrt{\frac{-2c+2\mu}{ck\lambda}}\eta}{4}\right)}. \tag{2.99}$$

For $\Omega > 0$, we have

$$u_{2,7} = -\sqrt{3}\frac{\left(\tan\left(\frac{\sqrt{\frac{2c-2\mu}{ck\lambda}}\eta}{4}\right)^2-1\right)(c-\mu)}{2\sqrt{-c\lambda}\sqrt{\frac{c-\mu}{ck\lambda}}k\tan\left(\frac{\sqrt{\frac{-2c+2\mu}{ck\lambda}}\eta}{4}\right)}, \tag{2.100}$$

$$u_{2,8} = \sqrt{3}\frac{\left(\cot\left(\frac{\sqrt{\frac{2c-2\mu}{ck\lambda}}\eta}{4}\right)^2-1\right)(c-\mu)}{2\sqrt{-c\lambda}\sqrt{\frac{c-\mu}{ck\lambda}}k\cot\left(\frac{\sqrt{\frac{-2c+2\mu}{ck\lambda}}\eta}{4}\right)}. \tag{2.101}$$

Set 3 :

Substituting these coefficients into equation (2.79) along with the Riccati equation solutions we get solutions of equation (2.76) as follows:

For $\Omega < 0$, we have

$$\Omega = \frac{c-\mu}{2ck\lambda}, a_0 = 0, a_1 = 0, b_1 = \frac{(c-\mu)}{k}\sqrt{-\frac{3}{2c\lambda}}.$$

$$u_{2,9} = -\frac{\sqrt{-\frac{3}{2c\lambda}}(c-\mu)}{k\sqrt{-\Omega}\tanh\left(\sqrt{-\Omega}\eta\right)}, \tag{2.102}$$

$$u_{2,10} = -\frac{\sqrt{-\frac{3}{2c\lambda}}(c-\mu)}{k\sqrt{-\Omega}\coth\left(\sqrt{-\Omega}\,\eta\right)}. \tag{2.103}$$

For $\Omega > 0$, we have

$$u_{2,11} = \frac{\sqrt{-\frac{3}{2c\lambda}}(c-\mu)\sqrt{2}}{k\sqrt{\Omega}\tan\left(\frac{1}{2}\sqrt{2\Omega}\eta\right)}, \tag{2.104}$$

$$u_{2,12} = -\frac{\sqrt{-\frac{3}{2c\lambda}}(c-\mu)\sqrt{2}}{k\sqrt{\Omega}\cot\left(\frac{1}{2}\sqrt{2\Omega}\eta\right)}. \tag{2.105}$$

2.8.3 Equation 3:

Solving for the third equation of (3+1)-dimensional Wazwaz-Benjamin-Bona-Mahony equation, we have the following set of coefficients.

Set 1 :

Substituting these coefficients into equation (2.80) along with the Riccati equation solutions we get solutions of equation (2.77) as follows:

For $\Omega < 0$, we have

$$\Omega = \frac{c-\lambda}{2ck^2}, a_0 = 0, a_1 = \sqrt{-\frac{6c}{\mu}}k, b_1 = 0$$

$$u_{3,1} = -\frac{\sqrt{-\frac{6c}{\mu}}k\sqrt{-4\Omega}\tanh\left(\sqrt{-4\Omega}\eta\right)}{2}, \quad (2.106)$$

$$u_{3,2} = -\frac{\sqrt{-\frac{6c}{\mu}}k\sqrt{-4\Omega}\coth\left(\sqrt{-4\Omega}\eta\right)}{2}. \quad (2.107)$$

For $\Omega > 0$, we have

$$u_{3,3} = \frac{\sqrt{-\frac{6c}{\mu}}k\sqrt{2}\sqrt{2\Omega}\tan\left(\sqrt{4\Omega}\eta\right)}{2}, \quad (2.108)$$

$$u_{3,4} = -\frac{\sqrt{-\frac{6c}{\mu}}k\sqrt{2}\sqrt{2\Omega}\cot\left(\sqrt{4\Omega}\eta\right)}{2}. \quad (2.109)$$

Set 2 :

Substituting these coefficients into equation (2.80) along with the Riccati equation solutions we get solutions of equation (2.77) as follows:

For $\Omega < 0$, we have

$$\Omega = \frac{c-\lambda}{2ck^2}, a_0 = 0, a_1 = 0, b_1 = \frac{(c-\lambda)}{k}\sqrt{-\frac{3}{2c\mu}},$$

$$u_{3,5} = -\frac{b_1}{k\sqrt{-\Omega}\tanh\left(\sqrt{-\Omega}\eta\right)}, \quad (2.110)$$

$$u_{3,6} = -\frac{b_1}{k\sqrt{-\Omega}\coth\left(\sqrt{-\Omega}\eta\right)}. \quad (2.111)$$

For $\Omega > 0$, we have

$$u_{3,7} = \frac{b_1\sqrt{2}}{k\sqrt{2\Omega}\tan\left(\frac{1}{2}\sqrt{4\Omega}\eta\right)}, \tag{2.112}$$

$$u_{3,8} = -\frac{b_1\sqrt{2}}{k\sqrt{2\Omega}\cot\left(\frac{1}{2}\sqrt{4\Omega}\eta\right)}. \tag{2.113}$$

Set 3 :

Substituting these coefficients into equation (2.80) along with the Riccati equation solutions we get solutions of equation (2.77) as follows:

For $\Omega < 0$, we have

$$\Omega = -\frac{-\frac{3\mu\sqrt{-\frac{6c}{\mu}}\left(\sqrt{-\frac{6c}{\mu}}\sqrt{-6c\mu}+2c\right)}{16\sqrt{-6c\mu}} + \frac{3\mu\sqrt{-\frac{6c}{\mu}}\left(\sqrt{-\frac{6c}{\mu}}\sqrt{-6c\mu}+2c\right)\lambda}{16c\sqrt{-6c\mu}} - c + \lambda}{2ck^2},$$

$$a_0 = 0, a_1 = \sqrt{-\frac{6c}{\mu}}k, b_1 = -\frac{3\left(\sqrt{-\frac{6c}{\mu}}\sqrt{-6c\mu}+2c\right)(c-\lambda)}{16c\sqrt{-6c\mu}k}.$$

$$u_{3,9} = -\left(\sqrt{-\frac{3c}{\mu}}k\sqrt{-\Omega}\tanh\left(\frac{\sqrt{-\Omega}\eta}{\sqrt{2}}\right) + 3\left(2c + \sqrt{-\frac{6c}{\mu}}\sqrt{-6c\mu}\right)(c-\lambda)\sqrt{2}\right) \times \left(16ck\sqrt{-6c\mu}\left(\sqrt{-\Omega}\tanh\left(\frac{\sqrt{-\Omega}\eta}{\sqrt{2}}\right)\right)\right)^{-1}, \tag{2.114}$$

(2.115)

$u_{10} =$

$$\times\left(16ck\sqrt{-6c\mu}\left(\sqrt{-\Omega}\coth\left(\frac{\sqrt{-\Omega}\eta}{\sqrt{2}}\right)\right)\right)^{-1}.$$

For $\Omega > 0$, we have

$$u_{3,11} = \sqrt{-\frac{6c}{\mu}}k\sqrt{-\Omega}\tan\left(\frac{\sqrt{-\Omega}\eta}{2}\right) - 3\left(2c + \sqrt{-\frac{6c}{\mu}}\sqrt{-6c\mu}\right)(c-\lambda)$$
$$\times\left(16ck\sqrt{-6c\mu}(\sqrt{-\Omega}\tan(\sqrt{-\Omega}\eta))\right)^{-1}, \tag{2.116}$$

$$u_{3,12} = -\sqrt{-\frac{6c}{\mu}}k\sqrt{-\Omega}\cot\left(\frac{\sqrt{-\Omega}\eta}{2}\right) + 3\left(2c + \sqrt{-\frac{6c}{\mu}}\sqrt{-6c\mu}\right)(c-\lambda)$$
$$\times\left(16ck\sqrt{-6c\mu}(\sqrt{-\Omega}\cot(\sqrt{-\Omega}\eta))\right)^{-1}. \tag{2.117}$$

2.9 Results and discussion:

In this section we have discussed graphical representation and their physical interpretation of various solutions of (3+1)-dimensional Wazwaz-Benjamin-Bona-Mahony equation. These results have been obtained by using the modified extended tanh method. The physical nature and diversity of these exact solutions can be well explained and analyzed in Figure (2.12) -(2.18) by 3-D, 2-D and contour plots with the appropriate choice of arbitrary constants.

Graphical depiction of imaginary part of $u_{1,1}$ expressed in Eq (2.82) has been shown in Figure 2.12, in the form of 3-dimensional , and 2-dimensional and contour plot which demonstrates propagation of singular kink wave soliton for the values of parameters involved as, $y = 1,\ z = 0.5,\ t = 1, k = 1.5,\ \mu = 0.2,\ c = 1,\ \lambda = 0.1$. This type of wave important in carrying information.

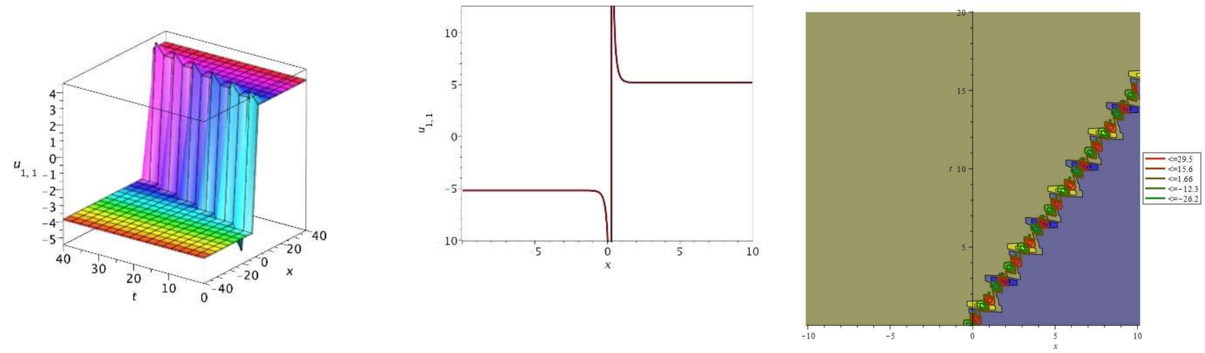


Figure 2.12: -graphs of singular anti kink wave soliton for $u_{1,1}$

Figure 2.13 depict wave propagation of periodic wave solution of imaginary value of $u_{1,8}$ expressed in Eq (2.89), in the form of 3-dimensional , and 2-dimensional and contour plot by selecting arbitrary constant, $y = 1, z = 1,\ t = 0.1,\ k = 1,\ \mu = 1,\ c = 4,\ \lambda = 5.$

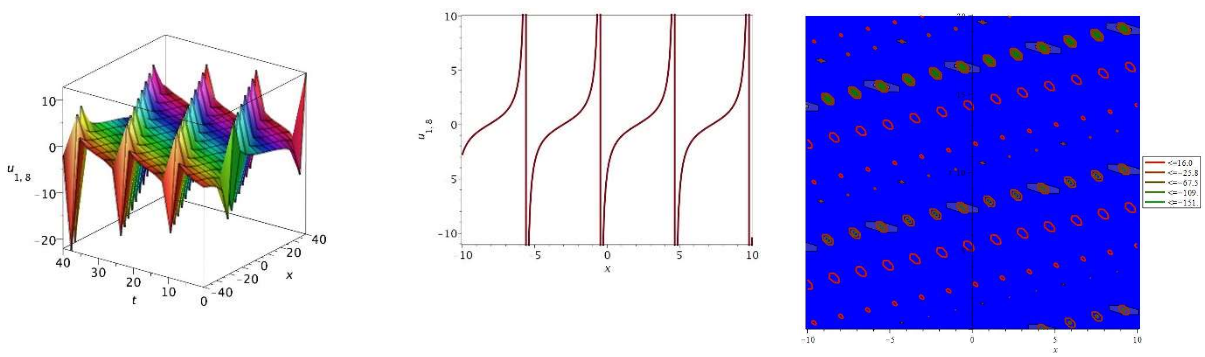


Figure 2.13:-graphs of periodic wave soliton for $u_{1,8}$.

Figure 2.14 depicts bright solitary wave propagation of absolute value of $u_{1,10}$ expressed in Eq (2.91) in the form of 3-dimensional, and 2-dimensional and contour plot by selecting parameters, $y = 1, z = -2,\ t = 0.9,\ k = 0.5,\ \mu = -1,\ c = 1,\ \lambda = 0.5.$

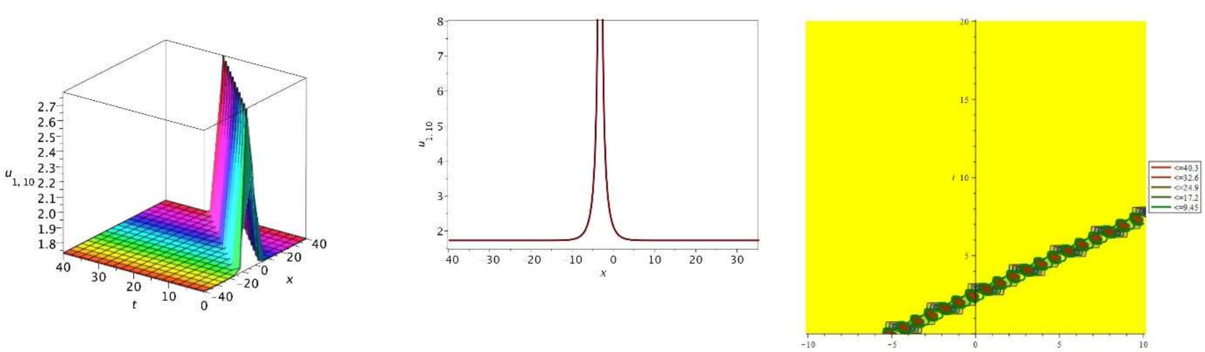


Figure 2.14: graphs of bright solitary wave solution $u_{1,10}$.

Figure 2.15 depicts the wave propagation of periodic wave solution of imaginary value of $u_{2,4}$ expressed in Eq (2.97), in the form of 3-dimensional, 2-dimensional and their contour plot by selecting parameters, $y = 1, z = 1,\ t = 1,\ k = 5,\ \mu = 0.4,\ c = 1.5,\ \lambda = 0.1.$

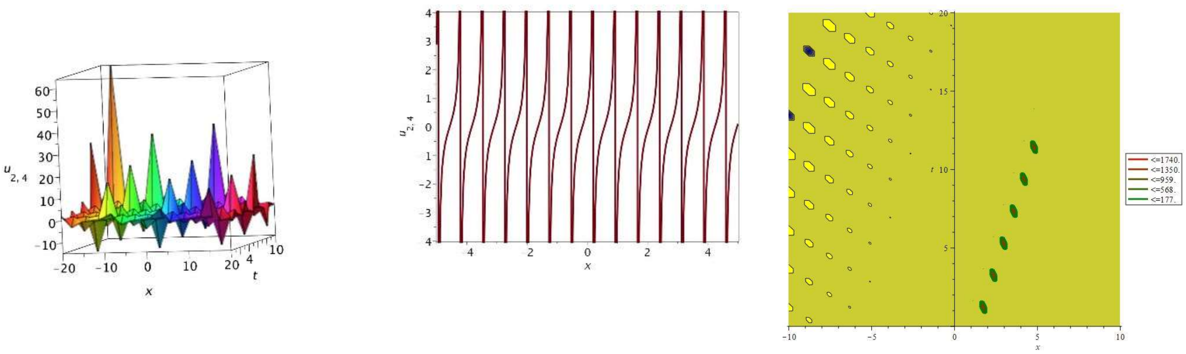


Figure 2.15: graphs of periodic wave solution $u_{2,4}$.

Figure 2.16 depicts wave propagation of singular kink wave soliton of $u_{2,6}$ expressed in Eq (2.99), in the form of 3 dimensional , and 2 dimensional and contour plots by selecting parameters, $y = 1, z = 1,\ t = 2,\ k = 0.5,\ \mu = 3.5,\ c = 2.5,\ \lambda = 0.1.$

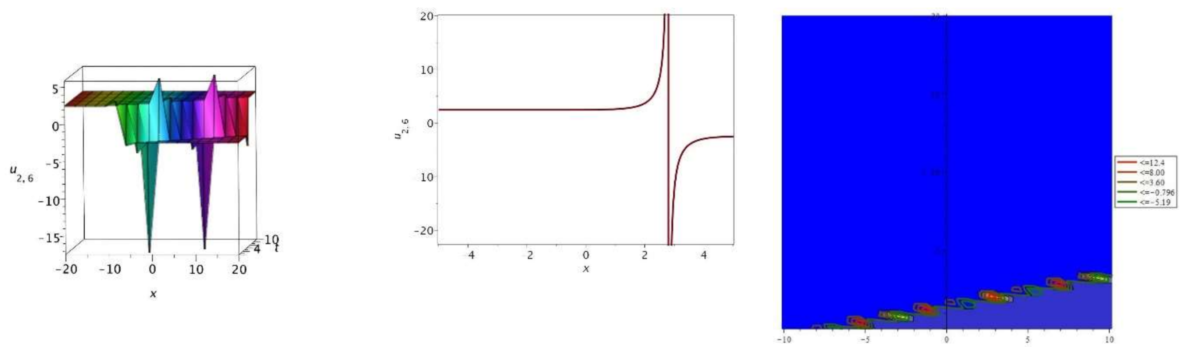


Figure 2.16:graphs of singular kink wave solution $u_{2,6}$.

Figure 2.17 depicts wave propagation of kink wave solution of Real value $u_{2,9}$ expressed in Eq (2.102), in the form of 3 dimensional , and 2 dimensional by selecting parameters, $y = 1, z = -1,$ $t = 2,\ k = 0.5,\ \mu = -1.5,\ c = -0.5,\ \lambda = 0.1.$

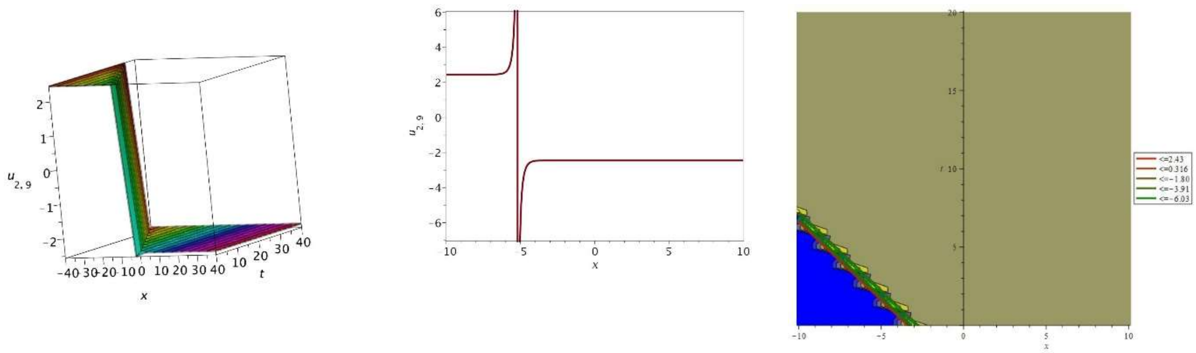


Figure 2.17:graphs of singular kink wave solution $u_{2,9}$.

Figure 2.18 depicts wave propagation of periodic wave of absolute value $u_{3,8}$expressed in Eq (2.113), in the form of 3-dimensional , and 2 -dimensional and contour plot by selecting parameters, $y = 1, z = 1,\ t = 1,\ k = 1,\ \mu = 0.9,\ c = 6,\ \lambda = 1.$

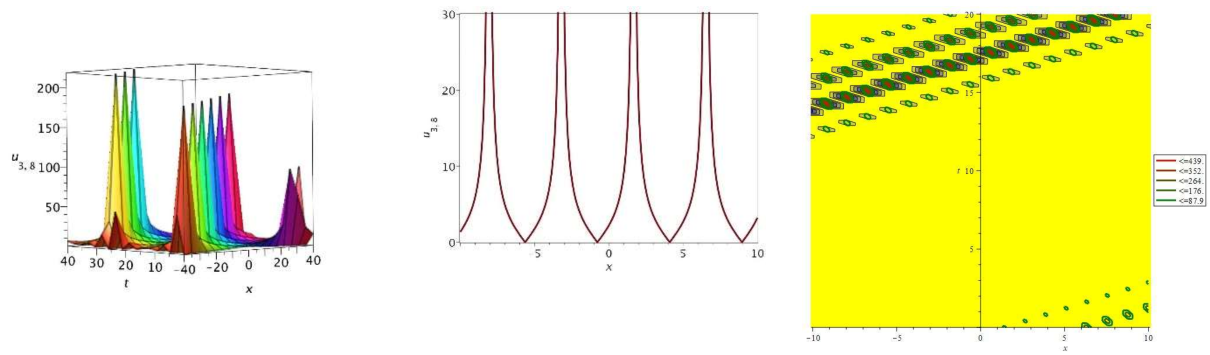


Figure 2.18:-graphs of singular kink wave solution $u_{3,8}$.

2.10 Conclusions:

Modified extended tanh method successfully employed on (3+1)-dimensional Wazwaz-Benjamin-Bona-Mahony equation to perceive new general solutions as an outcome of this technique, we produced some totally new solutions in the form hyperbolic wave solutions and trigonometric wave solutions, which can generate kink, periodic, singular periodic wave, bright solitons by appropriate choice of arbitrary constants involved in solutions. These new solutions may be worthwhile in the field of ocean engineering, astrophysics, and aerodynamics, plasma physics and fluid mechanics to explain wave propagation of incompressible fluids. This technique is very effective in generating exact solutions of almost all nonlinear PDEs arising in wave propagation. Therefore, this method is modifying and evolving continuously. The physical nature and behavior of some of these results has been analyzed by 2D and 3D graph simulation, and contour plots and all the solutions obtained in this article have been verified by using Maple 17.

2.11 Summary:

This chapter demonstrated that modified extended tanh expansion method have been employed successfully on the Dodd-Bullough-Mikhailov equation, Sinh-Gordan equation, Liouville equation and (3+1)-dimensional Wazwaz-Benjamin-Bona-Mahony equation to extract variety of solutions. Main steps of chapter include introduction of governing equations followed by focal steps of methods used and derivation of solutions by proposed method. Finally graphical representation of some results followed by conclusion.

All the obtained results are new and maybe beneficial for researchers who are working on these models. The significance of a few of these solutions has been shown graphically.

In next chapter we will be finding exact solutions of few more NLPEDs by another useful method called improved tanh$(\boldsymbol{\varphi}(\boldsymbol{\xi})/\mathbf{2})$ -expansion method.

Chapter 3. Exact solutions of some nonlinear partial differential equations using improved $\tanh(\varphi(\xi)/2)$ -expansion method

3.1 Introduction:

Nonlinear partial differential equations (NLPDEs) play an indispensable role in numerous fields of mathematics, physical sciences, and engineering. Integrable differential equations gain much attention in the modern era of research for the study of wave propagation especially in plasma physics, ocean and rogue waves, optical fibers, incompressible fluids and many more. Traveling wave solutions in particular solitary wave solutions which are the exact solutions of some NLPEs is the prime objective and most active research area of researchers and scientist to study and understand nonlinear complex physical phenomena [82–89]. It is interesting to point out that with the evolution of soliton theory, many efficient and robust method have been developed and then modified to generate accurate and novel exact solutions of NLPDEs such as Backlund transformation method [51], Painlevé expansion [31], Variational iteration method [67], tanh method [90], Sine-Cosine method [68], improved generalized Riccati equation mapping method [18], Auxiliary equation method [75], Ansatz method [11], Functional variable method [15], G'/G expansion method [91] and many more methods.

3.2 Illustrative Examples:

3.3 $(3+1)$-dimensional Boiti-Leon-Manna-Pempinelli (BLMP) equation:

In the last decade Boiti-Leon-Manna-Pempinelli (BLMP) equation has gained a lot of attraction by researchers due to the uses of this model in plasma physics, fluid dynamics, ocean engineering, astrophysics, and aerodynamics to explain wave propagation of incompressible fluids [31, 88, 92–96]. The $(3+1)$-dimensional Boiti-Leon-Manna-Pempinelli (BLMP) equation has imperative impact and significance in the wave propagation in incompressible fluids, moreover when $z=0$, it describes the interaction of Riemann wave propagation [31].

Boiti-Leon-Manna-Pempinelli (BLMP) model has been introduced in [97, 98]. Later Wazwaz derived new $(3+1)$-dimensional Boiti-Leon-Manna-Pempinelli (BLMP) equation with constant coefficients in [31, 99].

$$\left(u_x+u_y+u_z\right)_t+\alpha\left(u_x+u_y+u_z\right)_{xxx}+\beta\left(u_x\left(u_x+u_y+u_z\right)\right)_x=0, \tag{3.1}$$

where, $u=u(x,y,z,t)$, is unknown analytical function with spatial variables x,y,z and temporal variable t, whereas α and β are no-zero constants.

A lot of work has been done on this model. The stair and step solitons of $(2+1)$ and $(3+1)$ dimensional BLMP has been studied in [97]. Bilinear form, lax pairs and Backlund transformation are constructed by [100]. The authors in [31], [96] secured multiple solitons and complex multi soliton solution by using Painleve test and Hirota's direct method to generate lump solitons, solitary wave solutions and periodic wave solutions and their interactions. New three wave solutions and hyperbolic and trigonometric solutions have been generated for and $(3+1)$ dimensional BLMP in [101, 102]. Moreover, authors in [99] investigated the interaction solutions among lump wave, N-solitons, periodic and breather wave solutions. Solitary wave, periodic wave and trigonometric wave solutions has been obtained in [103] with the aid Sine Gordan expansion method and extended tanh function method. Periodic solitons and periodic type solutions of $(3+1)$ dimensional BLMP has been studied in [104].

The technique, improved tanh $(\frac{\phi}{2})$-expansion method [105], used here is new and direct and very convenient to handle, and no study has not been done so far on this equation by this technique, as both equation and method is new. With the aid of mathematical software, we manage to generate various interesting types of new exact traveling wave solutions.

The prime motive here is to thoroughly study newly derived $(3+1)$-dimensional Boiti-Leon-Manna-Pempinelli (BLMP) equation and concurrently reveals the significance of improved tanh $(\frac{\phi}{2})$-expansion method. It's worth mentioning here that higher dimensional nonlinear models generate large number of exact solutions as compared to lower dimensional equations [31]. We are hopeful that our new abundant exact solutions which are new and have not been reported in literature of this higher dimensional model have great significance for many higher dimensional nonlinear problems in various fields of sciences.

Implementation of IThEM:

To use improved tanh$(\frac{\phi(\xi)}{2})$-expansion method on equation (3.1).

We use following wave transformation,

$u(x,t) = u(\xi)$, with $\xi = k_1 x + k_2 y + k_3 z + \omega t$,

in equation(3.1), substituting $\alpha = \beta = -3$ and after integrating by keeping constant of integration zero, we get the following nonlinear ODE:

$$k_1{}^3(k_3 + k_1 + k_2)\frac{d^3}{d\xi^3}u(\xi) + \omega(k_3 + k_1 + k_2)\frac{d}{d\xi}u(\xi) - \frac{3k_1{}^2(k_3 + k_1 + k_2)}{2}\left(\frac{d}{d\xi}u(\xi)\right)^2 = 0, \quad (3.2)$$

using homogeneous balance principle between $\left(\frac{d^3}{d\xi^3}u(\xi)\right)$ and $\left(\frac{d}{d\xi}u(\xi)\right)^2$ we get $N = 1$. Therefore, the exact series solution has the form,

$$u(\xi) = \Lambda(Y) = \frac{A_{-1}}{p + \tanh\left(\frac{\phi(\xi)}{2}\right)} + A_0 + A_1\left(p + \tanh\left(\frac{\phi(\xi)}{2}\right)\right), \quad (3.3)$$

now, substituting Eq.(3.3) along with Eq.**Error! Reference source not found.** into Eq.(3.2) after collecting all terms with the same powers of $\tanh\left(\frac{\phi(\xi)}{2}\right)$ and equating each coefficient to zero, we obtain a system of nonlinear algebraic equations. Solving these equations by using Maple 17, we get the following non-trivial solutions. All the abbreviations used in the below mentioned solutions have been expressed in table:

$D = a^2 - b^2 + c^2$	$\Omega = xk_1 + zk_3 + yk_2$
$E = (b-c)\big((b-c)p^2 - b - c\big)$	$F = -a^2 + b^2$
$F' = a^2 + c^2$	$G = b^2 - c^2$

Family 1:

Some trigonometric function solutions are formulated for BLMP equation for $a^2 + c^2 - b^2 < 0, b - c \neq 0$:

$a = a, b = c, c = c, \omega = -k_1{}^3\mathrm{D}, p = p,$

$A_{-1} = 2k_1(-(b-c)p^2 + 2pa - b - c), A_1 = 0,$

$$u_1 = \begin{pmatrix} \sqrt{-\mathrm{D}}A_0 \tan\left(\left(tDk_1{}^3 - \Omega\right)\sqrt{-D}\right) + 2k_1(b-c)^2p^2 \\ -4(ak_1 + A_0/4)(b-c)p + 2(b^2-c^2)k_1 + aA_0 \end{pmatrix} \times \left(\sqrt{-\mathrm{D}}\tan\left(\left(tDk_1{}^3 - \Omega\right)\sqrt{-D}\right) + (-b+c)p + a\right)^{-1}, \tag{3.4}$$

$a = a, b = b, c = c, \omega = -Dk_1{}^3, p = p,$

$A_{-1} = 0, A_1 = 2k_1(b-c),$

$$u_2 = \left(-2\sqrt{-D}\tan\left(\left(tDk_1{}^3 - \Omega\right)\sqrt{-D}/2\right) + 2(pb - pc - a)\right)k_1 + A_0, \tag{3.5}$$

$a = p(b - \mathrm{c}), b = b, c = c, \omega = -4Ek_1{}^3,$

$p = p, A_{-1} = 2k_1\left((b-c)p^2 - b - c\right), A_1 = 2k_1(b-c),$

$$u_3 = \begin{pmatrix} A_0/2\sqrt{-E}\tan\left(2\sqrt{-E}\left(tk_1{}^3(2(p^2-1)b^2 - 2bcp^2 + (p^2+1)c^2) - \Omega/4\right)\right) \\ -k_1E\left(\tan\left(2\sqrt{-E}\left(tk_1{}^3(2(p^2-1)b^2 - 2bcp^2 + (p^2+1)c^2) - \Omega/4\right)\right)^2 - 1\right) \end{pmatrix} \times \left(\sqrt{-E}\tan\left(2\sqrt{-E}\left(tk_1{}^3(2(p^2-1)b^2 - 2bcp^2 + (p^2+1)c^2) - \Omega/4\right)\right)\right)^{-1}, \tag{3.6}$$

Family 2:

The hyperbolic function solutions can be derive as using the following conditions:

For $a^2 + c^2 - b^2 > 0$ and $b - c \neq 0$:

$a = a, b = b, c = c, \omega = -Dk_1{}^3, p = p, A_1 = 0,$

$A_{-1} = 2k_1(-(b-c)p^2 + 2pa - b - c),$

$$u_4 = \begin{pmatrix} -\tanh\left(\dfrac{\left(t(D)k_1{}^3 - \Omega\right)\sqrt{D}}{2}\right)\sqrt{D}A_0 + 2k_1(b-c)^2p^2 \\ -(4ak_1 + A_0)(b-c)p + 2(b^2-c^2)k_1 + aA_0 \end{pmatrix} \times \left(-\tanh\left(\left(t(D)k_1{}^3 - \Omega\right)\sqrt{D}/2\right)\sqrt{D} + (-b+c)p + a\right)^{-1}, \tag{3.7}$$

$a = a, b = b, c = c, \omega = -Dk_1{}^3, p = p, A_{-1} = 0,$

$A_1 = 2k_1(b-c),$

$$u_5 = \left(2\tanh\left(\frac{1}{2\left(t(D)k_1{}^3 - \Omega\right)\sqrt{D}}\right)\sqrt{D} + 2pb - 2pc - 2a\right)k_1 + A_0, \tag{3.8}$$

Family 3:

When $a^2 + c^2 - b^2 < 0$, b≠0 and c=0, the trigonometric function solutions generated as:

$$\begin{aligned} &a = a, b = b, c = 0, \omega = 4{k_1}^3 F, p = \frac{a}{b}, \\ &A_{-1} = -2k_1 F/b, A_1 = 2bk_1, \\ &u_6 = \left(-2k_1\sqrt{F}\tan\left(-\sqrt{F}\left(4tF{k_1}^3 + \Omega\right)/2\right) + A_0 + 2k_1\sqrt{F}\right) \\ &\quad \times \left(\tan\left(\sqrt{F}\left(4tF{k_1}^3 - \Omega\right)/2\right)\right)^{-1}, \end{aligned} \tag{3.9}$$

$$a = a, b = b, c = 0, \omega = 4{k_1}^3 F, p = \frac{a}{b}, A_{-1} = -2{k_1}^3 F/b, A_1 = 2bk_1,$$

$$\begin{aligned} &u_7 = \begin{pmatrix} -2F\left(\tan\left(\sqrt{F}\left(-4tF{k_1}^3 - \Omega\right)/2\right)^2 - 1\right)k_1 \\ +A_0\tan\left(\sqrt{F}\left(-4tF{k_1}^3 - \Omega\right)/2\right)\sqrt{F} \end{pmatrix} \\ &\quad \times \left(A_0\tan\left(\sqrt{F}(-4tF{k_1}^3 - \Omega)/2\right)\sqrt{F}\right)^{-1}, \end{aligned} \tag{3.10}$$

$$a = a, b = b, c = 0, \omega = {k_1}^3 F, p = p, A_{-1} = 0, A_1 = 2bk_1,$$

$$u_8 = \left(-2\tan\left(\sqrt{F}(-tF{k_1}^3 - \Omega)/2\right)\sqrt{F} + 2pb - 2a\right)k_1 + A_0, \tag{3.11}$$

Family 4:

Another choice of hyperbolic function solutions for $a^2 + c^2 - b^2 > 0$, c≠0 and b=0:

$$\begin{aligned} &a = a, b = 0, c = c, \omega = -{k_1}^3 F', p = p, A_1 = 0, \\ &A_{-1} = 2k_1(pa - (-p^2 + 1)c, \\ &u_9 = \begin{pmatrix} \left(\tanh\left(\left(t{k_1}^3 F' - \Omega\right)\sqrt{F'}/2\right)\sqrt{F'} - a\right)A_0 + \\ 2(-p^2 + 1)k_1 c^2 - p(4ak_1 + A_0)c \end{pmatrix} \\ &\quad \times \left(\tanh\left(\left(t{k_1}^3 F' - \Omega\right)\sqrt{F'}/2\right)\sqrt{F'} - cp - a\right)^{-1}, \end{aligned} \tag{3.12}$$

$$\begin{aligned} &a = a, b = 0, c = c, \omega = -\mathrm{F}'{k_1}^3, p = p, A_{-1} = 2k_1(2pa - (-p^2 + 1)c), \\ &A_1 = 0, \\ &u_{10} = \begin{pmatrix} \left(\tanh\left(\left(t{k_1}^3 F' - \Omega\right)\sqrt{F'}/2\right)\sqrt{F'} - a\right)A_0 + \\ 2(-p^2 + 1)k_1 c^2 - p(4ak_1 + A_0)c \end{pmatrix} \\ &\quad \times \left(\tanh\left(\left(t{k_1}^3 F' - \Omega\right)\sqrt{F'}/2\right)\sqrt{F'} - cp - a\right)^{-1}, \end{aligned} \tag{3.13}$$

$$a = a, b = 0, c = c, \omega = -{k_1}^3 \mathrm{F}', p = p, A_{-1} = 0, A_1 = -2ck_1,$$

$$u_{11} = \left(\tanh\left((tF'{k_1}^3 - \Omega)\sqrt{F'}\right)\sqrt{F'} - 4pc - 4a\right)k_1 + A_0, \tag{3.14}$$

Family 5:
For $a^2 + c^2 - b^2 < 0$, b-c≠0 and a=0, trigonometric function solutions has been generated as:

$$a = 0, b = b, c = c, \omega = \mathrm{G}{k_1}^3, p = p,$$
$$A_{-1} = 2k_1(-bp^2 + cp^2 - b - c), A_1 = 0,$$
$$u_{12} = \begin{pmatrix} A_0\sqrt{\mathrm{G}}\tan\left((\mathrm{G}t{k_1}^3 + \Omega)\sqrt{\mathrm{G}}/2\right)/2 \\ -2(k_1(b-c)^2p^2 - A_0p/2 + k_1\mathrm{G}) \end{pmatrix} \times \left(\sqrt{\mathrm{G}}\tan\left((\mathrm{G}t{k_1}^3 + \Omega)\sqrt{\mathrm{G}}/2\right) + p(b-c)\right)^{-1}, \tag{3.15}$$

$$a = 0, b = b, c = c, \omega = \mathrm{G}{k_1}^3, p = p, A_{-1} = 0, A_1 = 2bk_1 - 2ck_1,$$

$$u_{13} = 2\tan\left((\mathrm{G}t{k_1}^3 + \Omega)\sqrt{\mathrm{G}}/2\right)k_1\sqrt{\mathrm{G}} + 2p(b-c)k_1 + A_0, \tag{3.16}$$

Family 6:
Mix soliton solution, hyperbolic function solutions have been acquired for a=0 and c=0:

$$b = b, \omega = b^2{k_1}^3, p = p, A_1 = 0, A_{-1} = -2bk_1(p^2 + 1),$$

$$u_{14} = -\frac{2bk_1(p^2+1)}{p + \tanh\left(\frac{1}{2}ln\left(\tan\left(b^2(b^2{k_1}^3t + \Omega)\right)\right)\right)} + A_0, \tag{3.17}$$

$$a = 0, b = b, c = 0, \omega = b^2{k_1}^3, p = p, A_1 = 2bk_1, A_{-1} = 0,$$

$$u_{15} = A_0 + 2bk_1\left(p + \tanh\left(\frac{1}{2}ln\left(\tan\left(b^2(b^2{k_1}^3t + \Omega)\right)\right)\right)\right), \tag{3.18}$$

$$a = 0, b = b, c = 0, \omega = 4b^2{k_1}^3, p = 0, A_1 = 2bk_1, A_{-1} = -2bk_1,$$

$$u_{16} = A_0 + 2bk_1\left(\tanh\left(\frac{1}{2}ln\left(\tan\left(b^2(b^2{k_1}^3t + \Omega)/2\right)\right)\right)\right) - \frac{2bk_1}{\tanh\left(\frac{1}{2}ln\left(\tan\left(b(4b^2{k_1}^3t + \Omega)b/2\right)\right)\right)}, \tag{3.19}$$

Family 7:
The hyperbolic function solution for b=0 and c=0, along with the following conditions:

$$a = a, \omega = -a^2 {k_1}^3, p = p, A_{-1} = 4pak_1, A_1 = 0,$$

$$u_{17} = \frac{4pak_1}{p + \tanh\left(\frac{1}{2} ln\left(\tanh\left(a\left(a^2 {k_1}^3 t - \Omega\right) b/2\right)\right)\right)} + A_0, \tag{3.20}$$

Family 8:
We get mix solutions, trigonometric and hyperbolic function solutions respectively for $a^2 + b^2 = c^2$,

$$a = Ib, b = b, c = 0, \omega = 8b^2 {k_1}^3, p = I, A_{-1} = -4bk_1, A_1 = 2bk_1,$$

$$u_{18} = \frac{\sqrt{2}\begin{pmatrix} 4bk_1 \tan\left(b\sqrt{2}\left(8b^2 {k_1}^3 t + \Omega\right) b/2\right)^2 + \\ A_0 \tan\left(b\sqrt{2}\left(8b^2 {k_1}^3 t + \Omega\right) b/2\right)\sqrt{2} - 4k_1 b \end{pmatrix}}{\tan\left(b\sqrt{2}\left(8b^2 {k_1}^3 t + \Omega\right) b/2\right)\sqrt{2}}, \tag{3.21}$$

Family 11:
Exponential function solutions for $a = b$, we get as:

$$b = b, c = c, \omega = -c^2 {k_1}^3, p = p, A_{-1} = 0, A_1 = 2bk_1 - 2ck_1,$$

$$u_{19} = \begin{pmatrix} 2(b-c)\left(\left((p-1)b - c(p+1)\right)k_1 + A_0/2\right) e^{-c\left(c^2 {k_1}^3 t - \Omega\right)} \\ -2(p-1)(b-c)k_1 - A_0 \end{pmatrix} \times \left(-1 + (b-c) e^{-c\left(c^2 {k_1}^3 t - \Omega\right)}\right)^{-1}, \tag{3.22}$$

$$a = 0, b = 0, c = c, \omega = -4c^2 {k_1}^3, p = 0, A_{-1} = -2ck_1, A_1 = -2ck_1,$$

$$u_{20} = \begin{pmatrix} -4e^{\left(-8c^3 {k_1}^3 t + 2c\Omega\right)} c^3 k_1 - A_0 \\ +e^{\left(-8c^3 {k_1}^3 t + 2c\Omega\right)} c^2 A_0 - 4ck_1 \end{pmatrix} \left(e^{\left(-8c^3 {k_1}^3 t + 2c\Omega\right)} c^2 - 1\right)^{-1}, \tag{3.23}$$

Family 12:
For $b = c$, we get exponential function solution as follows:

$$a = 1/k_1 \sqrt{-\omega/k_1}, c = c, \omega = \omega, p = \frac{4ck_1 + A_{-1}}{4\sqrt{-\omega/k_1}}, A_1 = 0, A_{-1} = A_{-1},$$

$$u_{21} = \frac{4e^{1/k_1 \sqrt{-\omega/k_1}\xi} A_0 k_1 + 4\left(\sqrt{-\omega/k_1} + A_0/4\right) A_{-1}}{4e^{1/k_1 \sqrt{-\omega/k_1}\xi} k_1 + A_{-1}}, \tag{3.24}$$

Family 13:
For $a = -c$, and $b = c$ we get another type of exponential function solution:

$$c = c, \omega = -c^2 {k_1}^3, p = p, A_{-1} = -4pk_1 c - 4k_1 c, A_1 = 0,$$

$$u_{22} = \frac{\left(A_0 \mathrm{e}^{c\left(c^2 {k_1}^3 t - \Omega\right)} - 4(p+1)(k_1 c - A_0/4)\right)}{\left(p + \mathrm{e}^{c\left(c^2 {k_1}^3 t - \Omega\right)} + 1\right)}, \tag{3.25}$$

Family 14:

For $b = -b$, and $c = -b$ we get another type of exponential function solution:

$$a = a, b = 0, c = 0, \omega = -a^2 {k_1}^3, p = p, A_{-1} = 4pak_1, A_1 = 0,$$

$$u_{23} = \frac{\left(A_0 \mathrm{e}^{-a^2\left(a^2 {k_1}^3 t - \Omega\right)} + apA_0 + 4pa^2 k_1\right)}{\left(ap + \mathrm{e}^{-a^2\left(a^2 {k_1}^3 t - \Omega\right)}\right)}, \tag{3.26}$$

Family 16:

For $b = -c$, then we different types of exponential function solutions:

$$a = a, c = c, \omega = -a^2 {k_1}^3, p = p, A_{-1} = 0, A_1 = -4ck_1,$$

$$u_{24} = \frac{\left(4cpk_1 - A_0 - 4c(cpk_1 + ak_1 - A_0/4)\mathrm{e}^{-a^2\left(a^2 {k_1}^3 t - \Omega\right)}\right)}{\left(c\mathrm{e}^{-a^2\left(a^2 {k_1}^3 t - \Omega\right)} - 1\right)}, \tag{3.27}$$

$$a = a, c = c, \omega = -a^2 {k_1}^3, p = p, A_{-1} = 4pak_1 + 4p^2 ck_1, A_1 = 0,$$

$$u_{25} = \begin{pmatrix} -4(cpk_1 + ak_1 + A_0/4)p \\ +4(cpk_1 + A_0/4)(cp + a)\mathrm{e}^{-\mathrm{a}\left(a^2 {k_1}^3 t - \Omega\right)} \end{pmatrix} \times \left((cp + a)\mathrm{e}^{-\mathrm{a}\left(a^2 {k_1}^3 t - \Omega\right)} - p\right)^{-1}, \tag{3.28}$$

$$a = -2cp, c = c, \omega = -16c^2 p^2 {k_1}^3, p = p, A_{-1} = -4p^2 ck_1, A_1 = -4ck_1,$$

$$u_{26} = \begin{pmatrix} -A_0 + 8c^3 pk_1 \mathrm{e}^{4cp\left(16c^2 p^2 {k_1}^3 t - \Omega\right)} \\ +c^2 A_0 \mathrm{e}^{4cp\left(16c^2 p^2 {k_1}^3 t - \Omega\right)} + 8cpk_1 \end{pmatrix} \left(c^2 \mathrm{e}^{4cp\left(16c^2 p^2 {k_1}^3 t - \Omega\right)} - 1\right)^{-1}, \tag{3.29}$$

Family 17:

For $a = 0$ and $b = c$, we get various wave solutions given as follows:

$$c = c, \omega = \omega, p = p, A_{-1} = 0, A_1 = A_1,$$

$$u_{27} = A_0 + A_1(p + c\xi), \tag{3.30}$$

Family 18:
When $a = 0$, and $b = -c$, we get various rational function solutions as follows:

$$c = c, \omega = 0, p = p, A_{-1} = 0, A_1 = -4ck_1,$$

$$u_{28} = \frac{-4cpk_1\Omega + (xA_0 - 4)k_1 + A_0(yk_2 + zk_3)}{\Omega}, \tag{3.31}$$

$$a = 0, b = -c, c = c, \omega = 0, p = p, A_{-1} = 4cp^2k_1, A_1 = 0,$$

$$u_{29} = \frac{4c^2p^2k_1\Omega + pA_0\Omega c + A_0}{pc\Omega + 1}, \tag{3.32}$$

$$a = 0, b = -c, c = c, \omega = \omega, p = 0, A_1 = 0, A_{-1} = \frac{2\omega}{3c{k_1}^2},$$

$$u_{30} = \frac{2\omega^2 t + 2\Omega\omega + 3A_0{k_1}^2}{3{k_1}^2}, \tag{3.33}$$

Family 19:
When $b = 0$, and $a = c$ we get dark solitons:

$$c = c, \omega = -2c^2{k_1}^3, p = p, A_{-1} = 2ck_1(p^2 + 2p - 1), A_1 = 0,$$

$$u_{31} = \frac{\begin{pmatrix} \tanh\left(c\sqrt{2}\left(2c^2t{k_1}^3 - \Omega\right)/2\right)\sqrt{2}A_0 \\ -2cp^2k_1 + (-4ck_1 - A_0)p + 2ck_1 - A_0 \end{pmatrix}}{\tanh\left(c\sqrt{2}\left(2c^2t{k_1}^3 - \Omega\right)/2\right)\sqrt{2} - p - 1}, \tag{3.34}$$

$$c = c, \omega = -2c^2{k_1}^3, p = p, A_{-1} = 0, A_1 = -2ck_1,$$

$$u_{32} = 2\sqrt{2}\tanh\left(c\sqrt{2}\left(2c^2t{k_1}^3 - \Omega\right)/2\right)ck_1 - 2c(p+1)k_1 + A_0, \tag{3.35}$$

Family 20:
we get hyperbolic function solutions for $a = 0$,and $b = 0$,

$$c = c, \omega = -c^2{k_1}^3, p = p, A_{-1} = 2ck_1(p^2 - 1), A_1 = 0,$$

$$u_{33} = \frac{2k_1(p^2 - 1)c}{p - \tanh\left(\left(c^2t{k_1}^3 - \Omega\right)c/2\right)} + A_0, \tag{3.36}$$

$$c = c, \omega = -c^2{k_1}^3, p = p, A_{-1} = 0, A_1 = -2ck_1,$$

$$u_{34} = -2ck_1\left(p - \tanh\left(\left(c^2t{k_1}^3 - \Omega\right)c/2\right)\right) + A_{0},, \tag{3.37}$$

$$c = c, \omega = -4c^2{k_1}^3, p = 0, A_{-1} = -2ck_1, A_1 = -2ck_1,$$

$$u_{35} = \frac{2ck_1}{\tanh\left(\left(4c^2tk_1{}^3 - \Omega\right)c/2\right)} + A_0 + 2ck_1 \tanh\left(\frac{\left(4c^2tk_1{}^3 - \Omega\right)c}{2}\right). \quad (3.38)$$

3.4 Results and discussion:

With the help of IThEM, we secured different wave structures of newly derived equation, $(3 + 1)$-BLMP that includes hyperbolic, trigonometric, exponential, and rational function solutions. All the obtained results are new and generalized solitary waves that comprise kink waves, periodic waves, solitons, singular solitons with suitable choice of free parameters. The uniqueness of our work is evident as we successfully acquired 42 different types of wave solutions. However, keeping in view the length of the article, we only present some selective ones. These solutions are more generalized and novel and had not been reported in literature previously as we compared with published results[103], it is worth mentioning our few solutions have similarity with them but most of the solutions are new, and we were able to derive various periodic wave solutions, singular periodic wave solutions, exponential function solutions and rational solutions other than solitons, kink solitons and singular kink solitons, which have not been explained before. Diverse wave structure of various solutions has been well characterized by 3-D, 2-D and their contour plots and we found out that the existence of periodic wave solutions, kink wave solutions and other solitons depends on free parameters. As these answers have not been reported so far, we are sure our work would be a valuable addition in literature to analyze this new model. The diversity and dynamic characteristics of these exact solutions can be well explained by 3-D, and 2-D and their contour plots with the appropriate choice of parameters. Figure 1- 6 shows 3-D, and 2-D graphs and their contour plots of some obtained results of $(3 + 1)$- BLMP equation to have a good grasp of physical phenomena of these solutions under appropriate choice of free parameters.

Graphical depiction of Eq (3.6) expressed as u_3 has been exhibit in Figure 3.1, in the form of 3-dimensional , and 2-dimensional and contour plot which demonstrates localized excitation wave pattern as singular kink wave soliton by selecting appropriate parameters. The dynamic behavior of singular kink type solution of Eq (3.6) is revealed well by suitable parameters.

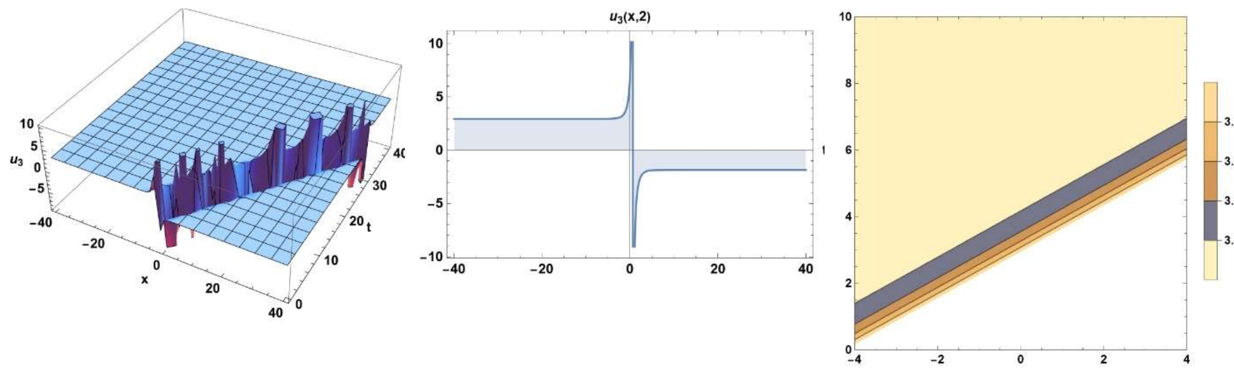


Figure 3.1: Graphical evolution of singular kink wave soliton for u_3 using parameters, $b = 0.9, c = 1.5, p = 0.02, k_1 = 0.5, k_2 = 0.5, k_3 = 0.1, A_0 = 0.55, y = 2, z = 1, t = 2$.

Graphical depiction of Eq (3.9) expressed as u_6 has been exhibit in Figure 3.2, in the form of 3-dimensional, and 2-dimensional and contour plot which demonstrates localized excitation wave pattern as singular kink soliton by selecting suitable parameters.

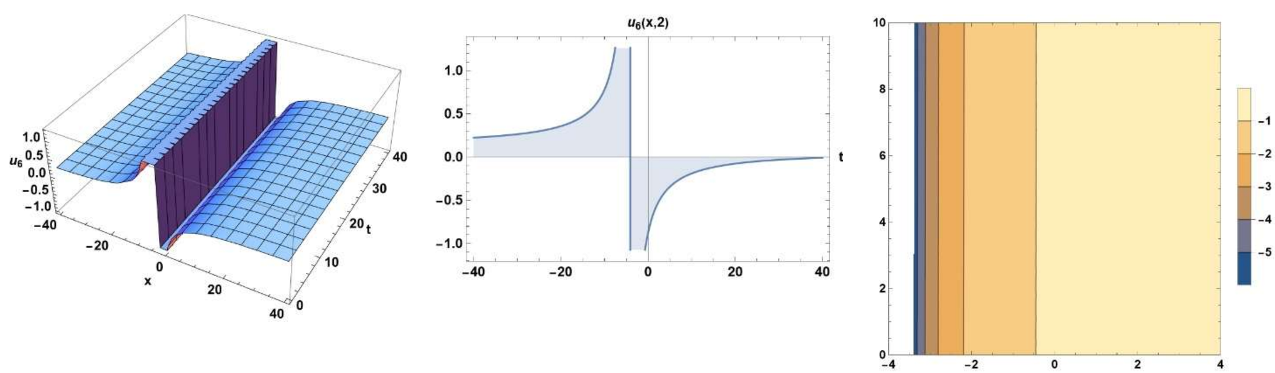


Figure 3.2: Graphical evolution of singular kink wave soliton for u_6.using parameters $a = 0.2, b = 0.1, k_1 = 0.1, k_2 = 0.21, k_3 = 0.2, A_0 = 0.1, y = 1, z = 1, t = 2$.

Graphical depiction of Eq (3.19) expressed as u_{16} has been exhibit in Figure 3.3, in the form of 3-dimensional, and 2-dimensional and their contour plot which demonstrates localized excitation wave pattern as singular periodic wave soliton by selecting appropriate parameters.

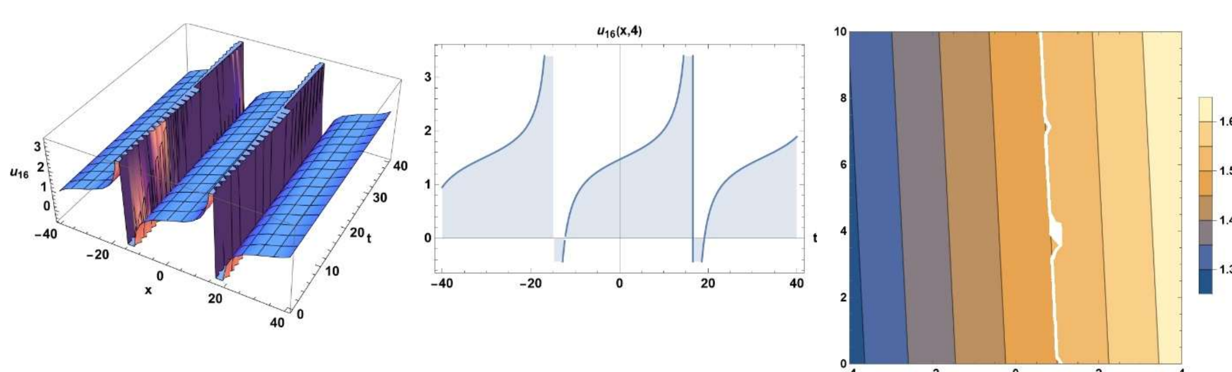


Figure 3.3: Graphical evolution of singular periodic wave soliton for u_{16} using parameters $b = 0.5, k_1 = 0.2, k_2 = -0.1, k_3 = 0.3, A_0 = 1.5, y = -1, z = -1, t = 4$..

Graphical depiction of Eq (3.22) expressed as u_{19} has been exhibit in Figure 3.4, in the form of 3-dimensional, 2-dimensional and their contour plot which demonstrates localized excitation wave pattern as singular kink soliton by selecting suitable parameters.

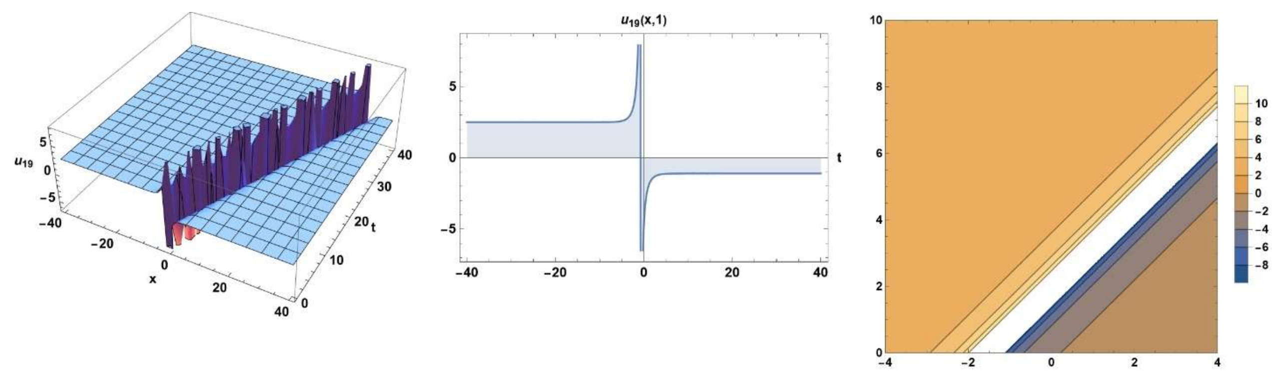


Figure 3.4: Graphical evolution of singular kink wave soliton for u_{19}.using parameters $b = 0.1, c = 0.9, p = 0.2, k_1 = 0.5, k_2 = 0.1, k_3 = 0.8, A_0 = 0.7, y = 1, z = 1, t = 1$.

Graphical depiction of Eq (3.24) expressed as u_{21} has been exhibit in Figure 3.5 , in the form of 3 dimensional , and 2 dimensional and their contour plot which demonstrates localized excitation wave pattern as periodic wave solution by selecting suitable parameters

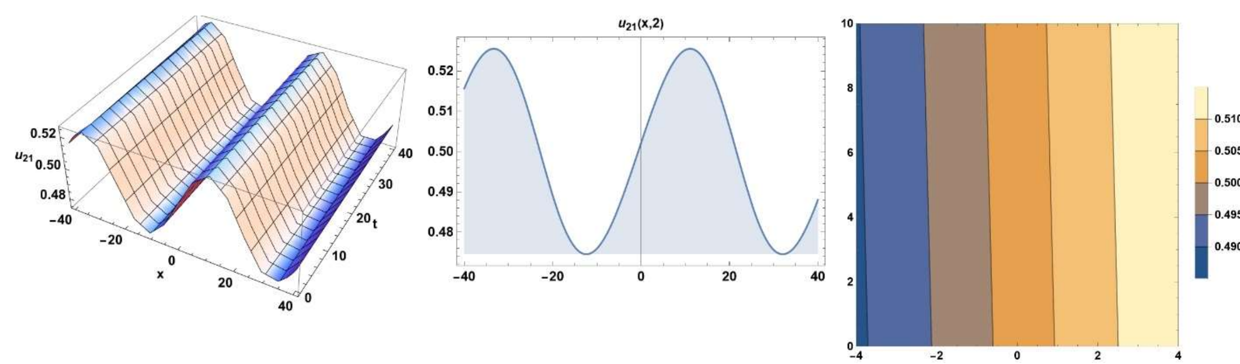


Figure 3.5: 3D and 2D-graphs of periodic wave solution for u_{21}.using parameters . $c = 2, k_1 = 5, k_2 = 1, k_3 = 2, A_0 = 0.5, A_{-1} = 0.9, p = 2, y = 1, z = 1, t = 2$.

Graphical depiction of Eq (3.35) expressed as u_{32} has been exhibit in Figure 3.6 , in the form of 3 dimensional , and 2 dimensional and their contour plot which demonstrates localized excitation wave pattern as kink shape soliton by selecting appropriate parameters.

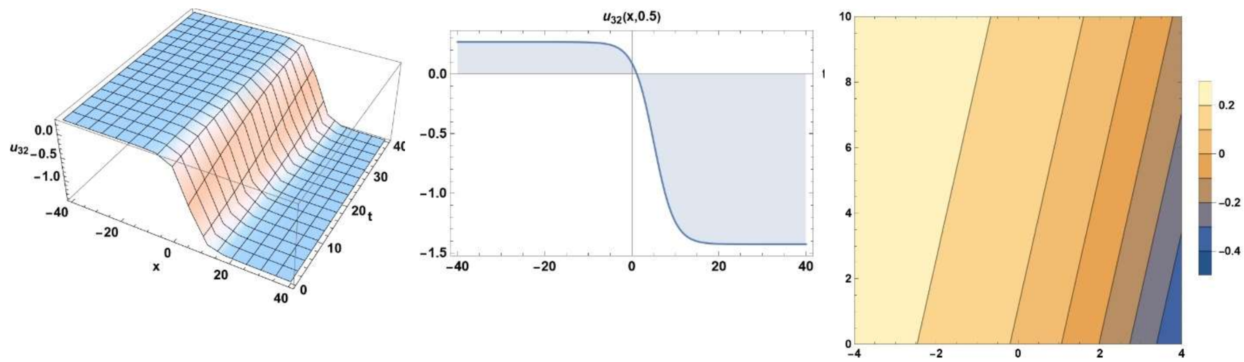


Figure 3.6: graphical evolution of kink wave soliton for u_{32} using parameters $c = 3, k_1 = 0.1, k_2 = 0.5, k_3 = 1, A_0 = 0.5,\ p = 0.8, y = 1, z = -1,\ t = 0.5$.

3.5 Conclusions:

Improved tanh $(\frac{\phi}{2})$-expansion method is applied to perceive general solutions of newly derived $(3+1)$-dimensional Boiti-Leon-Manna-Pempinelli equation. As a result, some totally new solutions have been obtained which are several solitary wave solutions including hyperbolic wave solutions, periodic wave solutions, exponential solutions. These new solutions may be worthwhile in the field of ocean engineering, astrophysics, and aerodynamics, plasma physics and fluid mechanics to explain wave propagation of incompressible fluids. Each type of solitary wave has its importance in nonlinear media such as kink solitons which propagates in nonlinear physical phenomena having high order nonlinearity, high order nonlinear effects and self-steepening. These solitons have been studied extensively due to its perfect propagation through nonlinear media [106]. Singular solitons are also very important types of solitons that appear with singularity. These solitons likely provide information about formation of rouge waves, also another type of solitary waves are periodic wave solutions that plays notable role in the study of chemistry, physics, biology and many more [107]. This newly derived method, IThEM is more effective than many other techniques such as tanh method and extended tanh method [108, 109], sine-cosine method [110], ansatz method [111], Improved $\tan(\frac{\phi}{2})$-expansion method [112] to generate more general and abundant solutions. This technique has developed recently and has not been used much previously, results show that this scheme is robust and effective to find plenty of new solutions of different types. It can be applied to many nonlinear PDEs arising in different fields of sciences to generate new types of solutions. The nature of these results has been analyzed physically by 2D and 3D graph simulation and their corresponding contour plots with the aid of computational software.

3.6 Nonlinear fourth order Ablowitz-Kaup-Newell-Segur Water Wave equation:

Higher order nonlinear PDEs are considered very valuable to describe physical mechanism and a lot of useful work have been done to extract exact solutions of PDEs arising in various fields such as engineering, medicine, plasma physics, nonlinear optics, earth sciences [56, 113–117]. Moreover, fractional calculus has become a compelling field for the study of many important phenomena. Many researchers have worked in this field to exhibit its usefulness [118–122].

To find the solutions of these equations various powerful analytical and numerical methods have been derived over the years some of them are, Homotopy perturbation method (HPM) [123], Lie algebra method [124, 125], Variational iteration method (VIM) [126, 127], tanh method and extended tanh method[108, 109], F-expansion method [128], Exp-function method [129, 130], Fan sub-equation method [131], $\left(\frac{G'}{G}\right)$-expansion method [132], sine-cosine method[110], Improved $\tan(\frac{\phi}{2})$-expansion method [112], Exp $\left(-\phi(\xi)\right)$ method [133], and Kudryashov method [134], auxiliary equation method [135]. The idea of improved $\tanh(\frac{\varphi(\xi)}{2})$-expansion method has been provided by [105] where authors have established exact solutions of some fifth order PDEs. $(3+1)$-dimensional Boiti-Leon-Manna-Pempinelli equation has been solved in[136] by using same scheme. This technique is new and generate different solution from improved $\tan(\frac{\varphi}{2})$ - expansion method.

Motivated by these studies we applied innovative IThEM [105] to construct different wave structures of exact solutions of fourth order nonlinear AKNS water wave equation [137, 138]. This novel approach has been practiced on AKNS equation for the first time. IThEM is a direct and convenient computational method and can handle a wide range of PDEs. This technique generates a variety of exact solutions and hence by applying this procedure we succeed in exploring various interesting families of exact wave solutions for under investigated model. These reported results might help in the study of shock waves, water wave phenomena, especially in ocean waves and other fields of physics and engineering. Accuracy of obtained results have been verified by back substitution. AKNS equations are considered very important in nonlinear physics and have been introduced by Albowitz, Kaup, Newell and Seguer for the first time in [139, 140].

$$4u_{xt} + u_{xxxt} + 8u_x u_{xy} + 4u_{xx} u_y - \gamma u_{xx} = 0, \tag{3.39}$$

these equations are significant because it can be reduce into some very famous nonlinear equations such as KdV equation, mKdV equation which are used for the study of shallow water waves and wave propagation in plasma, $(2+1)$ dimensional Boussinesq wave equation which is used for the investigation of nonlinear wave effect on shallow water, sine-Gordan equation have application in different fields of physics and nonlinear Schrödinger equation has wide range of applications in optical physics, quantum mechanics and many more [32]. Several studies has been done on these equations, [141] studied conformable (2+1)-dimensional AKNS equation by using sine-Gordan expansion method, [142] obtained new hyperbolic solutions, [137] solved AKNS equation by simple equation method and modified simple equation method, [143] construct new solutions of this equation by $(\frac{G'}{G})$ expansion method and [144] solved AKNS equation by modified exponential function method. Recently, [145] have used $\left(\frac{G'}{G}, \frac{1}{G}\right)$-expansion method on fractional AKNS equation to derive various type of solutions. In our research article we are using improved $\tanh(\frac{\varphi(\xi)}{2})$-expansion method to generate contemporary and unique solutions to make addition to already present literature on model.

Implementation of IThEM:

Here, we implement improved $\tanh(\frac{\varphi(\xi)}{2})$-expansion method to extract travelling wave solutions of fourth order Ablowitz-Kaup-Newell-Segur water wave (AKNS) Eq. (3.39).

After applying the following wave transformation,

$u(x,t) = u(\xi)$, with $\xi = x + y + \omega t$,

in Eq. (3.39) and integrating twice by assuming constant of integration zero, we acquire the following nonlinear ordinary differential equation:

$$\left(\frac{d^3}{d\xi^3}u(\xi)\right)\omega+(4\omega-\gamma)\frac{d}{d\xi}u(\xi)+6\left(\frac{d}{d\xi}u(\xi)\right)^2=0, \tag{3.40}$$

using homogeneous balance principle between $\left(\frac{d^3}{d\xi^3}u(\xi)\right)$ and $\left(\frac{d}{d\xi}u(\xi)\right)^2$ we get $n=1$. Hence we get exact series solution in the form,

$$u(\xi)=S(\varphi)=\frac{\tilde{A}_{-1}}{\dot{p}+\tanh\left(\frac{\varphi(\xi)}{2}\right)}+\tilde{A}_0+\tilde{A}_1\left(\dot{p}+\tanh\left(\frac{\varphi(\xi)}{2}\right)\right), \tag{3.41}$$

now, substituting Eq. (3.41) along with Eq. **Error! Reference source not found.** into Eq. (3.40) and by accumulating all terms having the similar powers of $\tanh\left(\frac{\varphi(\xi)}{2}\right)$ and then equate these coefficient to zero, we get a system of NL algebraic equations. Next by solving these equations the help of mathematical software, we get following solutions:

Family 1:

For this family we get periodic and singular periodic wave solitons as follows:

$$a=a,b=b,c=c,\omega=\frac{\gamma}{a^2-b^2+c^2+4},\dot{p}=\dot{p},$$
$$\tilde{A}_{-1}=-\frac{1}{6(b+c)}\left(\frac{\left((-2a^2-b^2+c^2-8)p^2+6(b+c)ap-3(b^2-2bc-c^2)\right)\gamma}{a^2-b^2+c^2+4}+2\gamma p^2\right),$$
$$\tilde{A}_1=0,D=a^2-b^2+c^2,$$

$$u_1=\frac{\left(\begin{array}{c}-2\sqrt{-D}A_0(\mathrm{D}+4)\times\\ \tan\left(\frac{\left((x+y)a^2-(x+y)b^2+(x+y)c^2+\gamma t+4x+4y\right)\sqrt{-\mathrm{D}}}{2(\mathrm{D}+4)}\right)\\ +2(\mathrm{D}+4)\left((-b+c)p+a\right)A_0\\ +((-b+c)p^2+2pa-b-c)\gamma(b-c)\end{array}\right)}{\left(\begin{array}{c}2(\mathrm{D}+4)\times\\ \left\{-\tan\left(\frac{\left((x+y)a^2-(x+y)b^2+(x+y)c^2+\gamma t+4x+4y\right)\sqrt{-\mathrm{D}}}{2(a^2-b^2+c^2+4)}\right)\right.\\ \left.\sqrt{-\mathrm{D}}+(-b+c)p+a\right\}\end{array}\right)}, \tag{3.42}$$

$$\dot{p} = \frac{\sqrt{-\frac{-\omega b^2 + \omega c^2 - \gamma + 4\omega}{\omega}}}{b - c}, \tilde{A}_{-1} = 0,$$

$$\tilde{A}_1 = -\frac{1}{2} b\omega + \frac{1}{2} c\omega, E = \frac{\gamma - 4\omega}{\omega},$$

$$u_2 = \tilde{A}_0 - \frac{1}{2}\omega \tan\left(1/2\sqrt{-E}\xi\right)\sqrt{-\mathrm{E}}, \tag{3.43}$$

$$a = \sqrt{-\frac{-4\omega b^2 + 4\omega c^2 - \gamma + 4\omega}{4\omega}}, b = b, c = c, \omega = \omega,$$

$$\dot{p} = \frac{\sqrt{-\frac{-\omega b^2 + \omega c^2 - \gamma + 4\omega}{\omega}}}{b - c},$$

$$\tilde{A}_{-1} = -\frac{-4\omega + \gamma}{8(b - c)}, \tilde{A}_1 = -\frac{1}{2} b\omega + \frac{1}{2} c\omega, E = \frac{\gamma - 4\omega}{\omega},$$

$$u_3 = \frac{\omega E \tan\left(\frac{\sqrt{-\mathrm{E}}\xi}{4}\right)^2 + 4\tilde{A}_0 \tan\left(\sqrt{-\mathrm{E}}\xi\right)\sqrt{-\mathrm{E}} - \omega E}{4\sqrt{-\mathrm{E}}\tan\left(\sqrt{-\mathrm{E}}\xi\right)}, \tag{3.44}$$

$$a = a, b = b, c = c, \omega = \frac{\gamma}{a^2 - b^2 + c^2 + 4}, \dot{p} = \dot{p}, \tilde{A}_{-1} = 0,$$

$$\tilde{A}_1 = \frac{-\gamma(b - c)}{2(a^2 - b^2 + c^2 + 4)}, D = a^2 - b^2 + c^2,$$

$$u_4 = \frac{\left(-\tan\left(\frac{\begin{array}{c}((x+y)a^2 + (-x-y)b^2 + (x+y)c^2 \\ +\gamma t + 4x + 4y)\sqrt{-\mathrm{D}}\end{array}}{2(\mathrm{D}+4)}\right) \times \sqrt{-\mathrm{D}}\gamma + 2(\mathrm{D}+4)\tilde{A}_0 + \gamma(-pb + pc + a)\right)}{2(\mathrm{D}+4)}. \tag{3.45}$$

Family 2:

The kink and singular kink wave solutions are as follows:

$$a = a, b = b, c = c, \omega = \frac{\gamma}{}, \dot{p} = \dot{p}, \tilde{A}_{-1} = 0$$

$$\tilde{A}_1 = -\frac{\gamma(b-c)}{2(a^2-b^2+c^2+4)}, D = a^2-b^2+c^2,$$

$$u_5 = \frac{\left(\tanh\left(\frac{\begin{array}{c}((x+y)a^2+(-x-y)b^2+(x+y)c^2\\+\gamma t+4x+4y)\sqrt{\mathrm{D}}\end{array}}{2(\mathrm{D}+4)}\right)\times \atop \sqrt{\mathrm{D}}\gamma+2(\mathrm{D}+4)\tilde{A}_0+\gamma(-pb+pc+a)\right)}{2(\mathrm{D}+4)}, \tag{3.46}$$

$$a = a, b = b, c = c, \omega = \frac{\gamma}{a^2-b^2+c^2+4}, \dot{p} = \dot{p}, A_1 = 0,$$

$$\tilde{A}_{-1} = -\frac{\frac{((2a^2-b^2+c^2-8)p^2+(6b+6c)ap-3b^2-6bc-3c^2)\gamma}{(a^2-b^2+c^2+4)}+2\gamma p^2}{6b+6c},$$

$$D = a^2-b^2+c^2,$$

$$u_6 = \frac{\left(\begin{array}{c}\tanh\left(\frac{\begin{array}{c}((x+y)a^2+(-x-y)b^2+(x+y)c^2\\+\gamma t+4x+4y)\sqrt{\mathrm{D}}\end{array}}{2(\mathrm{D}+4)}\right)\times\\ 2\tilde{A}_0(\mathrm{D}+4)\sqrt{\mathrm{D}}+2(\mathrm{D}+4)\big((b-c)p+a\big)A_0+\\ 2\gamma\left(\left(\frac{c}{2}-\frac{b}{2}\right)p^2+pa-\frac{c}{2}-\frac{b}{2}\right)(b-c)\gamma\end{array}\right)}{\left(\begin{array}{c}\tanh\left(\frac{\begin{array}{c}((x+y)a^2+(-x-y)b^2+(x+y)c^2\\+\gamma t+4x+4y)\sqrt{\mathrm{D}}\end{array}}{2(\mathrm{D}+4)}\right)\times\\ \sqrt{\mathrm{D}}+(c-b)p+a\end{array}\right)2(\mathrm{D}+4)}. \tag{3.47}$$

Family 3:

Another set of periodic wave solutions for the following conditions:

$$a = 0, b = b, c = 0, \omega = -\frac{\gamma}{b^2-4}, \dot{p} = 1, \tilde{A}_{-1} = -\frac{\gamma b}{b^2-4}, \tilde{A}_1 = 0,$$

$$u_7=\frac{\left(\begin{array}{c}\tilde{A}_0\left(\tan\left(\frac{b\left((x+y)b^2-\gamma t-4x-4y\right)}{2b^2-8}\right)+1\right)b^2\\-\gamma b-4\tilde{A}_0\left(\tan\left(\frac{b\left((x+y)b^2-\gamma t-4x-4y\right)}{2b^2-8}\right)+1\right)\end{array}\right)}{\left(\tan\left(\frac{b\left((x+y)b^2-\gamma t-4x-4y\right)}{2b^2-8}\right)+1\right)b^2-4}, \tag{3.48}$$

$a=a, b=b, c=0, \omega=\frac{\gamma}{a^2-b^2+4}, \dot{p}=\dot{p}, \tilde{A}_{-1}=-\frac{\gamma(bp^2+2ab-b)}{2(a^2-b^2+4)}, \tilde{A}_1=0,$

$D'=a^2-b^2,$

$$u_8=\frac{\left(\begin{array}{c}-2\tilde{A}_0\sqrt{-D'}(D'+4)\\\tan\left(\frac{\left((y+x)a^2+(-x-y)b^2+\gamma t+4x+4y\right)\sqrt{-D'}}{2(D'+4)}\right)+\\2b^3p\tilde{A}_0+\left(-\gamma p^2-2a\tilde{A}_0-\gamma\right)b^2-2p\left((a^2+4)\tilde{A}_0-a\gamma\right)b\\+2a\tilde{A}_0(a^2+4)\end{array}\right)}{2(D'+4)\left(-\tan\left(\frac{\begin{array}{c}((y+x)a^2+(-x-y)b^2\\+\gamma t+4x+4y)\sqrt{-D'}\end{array}}{2(D'+4)}\right)\sqrt{-D'}-pb+a\right)}, \tag{3.49}$$

$a=0, b=b, c=0, \omega=-\frac{\gamma}{b^2-4}, \dot{p}=-1, \tilde{A}_{-1}=-\frac{\gamma b}{b^2-4}, \tilde{A}_1=0,$

$$u_9=\frac{\left(\begin{array}{c}\tilde{A}_0\left(\tan\left(\frac{b\left((y+x)b^2-\gamma t-4x-4y\right)}{2b^2-8}\right)-1\right)b^2\\-\gamma b-4\tilde{A}_0(\tan(\frac{b((y+x)b^2-\gamma t-4x-4y)}{2b^2-8})-1)\end{array}\right)}{\left(\tan\left(\frac{b\left((y+x)b^2-\gamma t-4x-4y\right)}{\tilde{A}_0}\right)-1\right)(b^2-4)}, \tag{3.50}$$

$a=0, b=b, c=0, \omega=-\frac{\gamma}{b^2-4}, \dot{p}=0, \tilde{A}_1=\frac{\gamma b}{2(b^2-4)}, \tilde{A}_{-1}=0,$

$$u_{10}=\frac{\gamma\tan\left(\frac{b\left((y+x)b^2-\gamma t-4x-4y\right)}{2b^2-8}\right)b+2b^2\tilde{A}_0-8\tilde{A}_0}{2b^2-8}, \tag{3.51}$$

$a=a, b=b, c=0, \omega=\frac{\gamma}{a^2-b^2+4}, \dot{p}=\dot{p}, \tilde{A}_1=-\frac{\gamma b}{2(a^2-b^2+4)}, \tilde{A}_{-1}=0,$

$D'=a^2-b^2,$

$$u_{11} = \frac{-1}{2(D'+4)}\left(\tan\left(\frac{\left((y+x)a^2-(x+y)b^2+\gamma t+4(x+y)\right)\sqrt{-D'}}{2(D'+4)}\right) \times \sqrt{-D'}\gamma + 2(D'+4)\tilde{A}_0 + \gamma(a-pb)\right), \tag{3.52}$$

$$a = 0, b = b, c = 0, \omega = -\frac{\gamma}{4(b^2-1)}, \dot{p} = 0, \tilde{A}_1 = \frac{\gamma b}{8(b^2-1)}, \tilde{A}_{-1} = \frac{-\gamma b}{8(b^2-1)},$$

$$u_{12} = \frac{\left(\begin{array}{c} 8\tilde{A}_0\left(\tan\left(\frac{4b\left((y+x)b^2-\frac{\gamma t}{4}-x-y\right)}{8b^2-8}\right)\right)b^2 + \\ \gamma\left(\tan\left(\frac{4b\left((y+x)b^2-\frac{\gamma t}{4}-x-y\right)}{8b^2-8}\right)^2 - 1\right)b - \\ 8\tilde{A}_0\left(\tan\left(\frac{4b\left((y+x)b^2-\gamma t/4-x-y\right)}{8b^2-8}\right)\right) \end{array}\right)}{\tan\left(\frac{4b\left((y+x)b^2-\gamma t/4-x-y\right)}{2b^2-8}\right)(b^2-1)}, \tag{3.53}$$

$$a = bp, b = b, c = 0, \omega = \frac{\gamma}{4(b^2p^2-b^2+1)}, \dot{p} = \dot{p}, \tilde{A}_{-1} = -\frac{\gamma b(p^2-1)}{8(b^2p^2-b^2+1)},$$
$$\tilde{A}_1 = -\frac{\gamma b}{8(b^2p^2-b^2+1)},$$

$$u_{13} = \frac{\begin{array}{c} \left(8(b^2p^2-b^2+1)\tilde{A}_0\sqrt{(-p^2+1)b^2}\right) \times \\ \tan\left(\frac{4\left((p^2-1)(y+x)b^2+\frac{\gamma t}{4}+x+y\right)\sqrt{(-p^2+1)b^2}}{8+(8p^2-8)b^2}\right) + \\ b^2\gamma(p^2-1)\left(\tan\left(\frac{4\left((p^2-1)(y+x)b^2+\frac{\gamma t}{4}+x+y\right)\sqrt{(-p^2+1)b^2}}{8+(8p^2-8)b^2}\right)^2 - 1\right) \end{array}}{\begin{array}{c} \tan\left(\frac{4\left((p^2-1)(y+x)b^2+\frac{\gamma t}{4}+x+y\right)\sqrt{(-p^2+1)b^2}}{8+(8p^2-8)b^2}\right) \times \\ 8(b^2p^2-b^2+1)\sqrt{(-p^2+1)b^2} \end{array}}. \tag{3.54}$$

Family 4:

We generate more kink wave solutions for the following conditions:

$$a = a, b = 0, c = c, \omega = \frac{\gamma}{a^2 + c^2 + 4}, \dot{p} = \dot{p}, \tilde{A}_1 = 0,$$
$$\tilde{A}_{-1} = -\frac{\gamma(cp^2 + 2ap - c)}{2(a^2 + c^2 + 4)}, F = a^2 + c^2,$$

$$u_{14} = \frac{\left(\begin{array}{c} 2\tilde{A}_0\sqrt{\mathrm{F}}(\mathrm{F}+4)\tanh\left(\frac{\begin{array}{c}((y+x)a^2 + (y+x)c^2 + \gamma t \\ +4x + 4y)\sqrt{\mathrm{F}}\end{array}}{2(\mathrm{F}+4)}\right) + \\ 2c^3p\tilde{A}_0 + \left(-\gamma p^2 + 2a\tilde{A}_0 + \gamma\right)c^2 + 2p\left((a^2+4)\tilde{A}_0 - a\gamma\right)c \end{array}\right) \\ +2aA_0(a^2+4)}{\left(\begin{array}{c} 2(\mathrm{F}+4)\left(\tanh\left(\frac{\left((y+x)a^2 + (y+x)c^2 + \gamma t + 4(x+y)\right)\sqrt{\mathrm{F}}}{2(a^2+c^2+4)}\right)\right. \\ \left.\sqrt{\mathrm{F}} + pc + a)\right) \end{array}\right)}, \tag{3.55}$$

$$a = a, b = 0, c = c, \omega = \frac{\gamma}{a^2 + c^2 + 4}, \dot{p} = I, \tilde{A}_1 = 0,$$
$$\tilde{A}_{-1} = \frac{\gamma(Ia - c)}{a^2 + c^2 + 4}, F = a^2 + c^2,$$

$$u_{15} = \frac{\left(\begin{array}{c} \tilde{A}_0\sqrt{F}(\mathrm{F}+4)\tanh\left(\frac{\begin{array}{c}((y+x)a^2 + (y+x)c^2 \\ +\gamma t + 4x + 4y)\sqrt{\mathrm{F}}\end{array}}{2(\mathrm{F}+4)}\right) \\ +(Ic + a)(\mathrm{F}+4)\tilde{A}_0 - c\gamma(Ia - c) \end{array}\right)}{(\mathrm{F}+4)(Ic + \tanh\left(\frac{\begin{array}{c}((y+x)a^2 + (y+x)c^2 \\ +\gamma t + 4x + 4y)\sqrt{\mathrm{F}}\end{array}}{2(\mathrm{F}+4)}\right)\sqrt{\mathrm{F}} + a)}, \tag{3.56}$$

$$a = -\dot{p}c, b = 0, c = c, \omega = \frac{\gamma}{4(c^2\dot{p}^2 + c^2 + 1)}, \dot{p} = \dot{p}, A_{-1} = \frac{\gamma c(\dot{p}^2 + 1)}{8(c^2\dot{p}^2 + c^2 + 1)},$$
$$A_1 = \frac{c\gamma}{8(c^2\dot{p}^2 + c^2 + 1)}, P = \dot{p}^2 + 1,$$

$$u_{16} = \frac{\begin{pmatrix} c^2\gamma P\tanh\left(\frac{(4P(y+x)c^2+\gamma t+4x+4y)\sqrt{c^2P}}{8+(8\dot{p}^2+8)c^2}\right)^2 \\ +8\sqrt{c^2P}(1+c^2P)\times \\ \tilde{A}_0\tanh\frac{(4P(y+x)c^2+\gamma t+4x+4y)\sqrt{c^2P}}{8+(8\mathrm{P})c^2} \end{pmatrix} +\gamma c^2 P}{8\sqrt{c^2P}(1+c^2P)\left(\tanh\left(\frac{(4P(y+x)c^2+\gamma t+4x+4y)\sqrt{c^2P}}{8+(8\mathrm{P})c^2}\right)\right)}, \tag{3.57}$$

$a = a, b = 0, c = c, \omega = \frac{\gamma}{a^2+c^2+4}, \dot{p} = \dot{p}, \tilde{A}_{-1} = 0,$

$\tilde{A}_1 = \frac{c\gamma}{2(a^2+c^2+4)}, F = a^2+c^2,$

$$u_{17} = \frac{1}{2(\mathrm{F}+4)}\begin{pmatrix} \gamma\tanh\left(\frac{\left((y+x)a^2+(y+x)c^2+\gamma t+4x+4y\right)\sqrt{\mathrm{F}}}{2(\mathrm{F}+4)}\right) \\ \times\sqrt{\mathrm{F}}+(2\mathrm{F}+8)\tilde{A}_0+\gamma(\dot{p}c+a) \end{pmatrix}. \tag{3.58}$$

Family 5:

More periodic wave solutions for the given conditions:

$a = 0, b = b, c = c, \omega = \frac{\gamma}{-c^2+b^2-4}, \dot{p} = 0, \tilde{A}_{-1} = \frac{\gamma(-b-c)}{2(-c^2+b^2-4)}, \tilde{A}_1 = 0,$

$G = -c^2+b^2,$

$$u_{18} = \frac{\begin{pmatrix} \tan\left(\frac{\left((y+x)b^2+(-x-y)c^2-\gamma t-4x-4y\right)\sqrt{\mathrm{G}}}{2(\mathrm{G}-4)}\right) \\ \times(\mathrm{G}-4)\tilde{A}_0\sqrt{\mathrm{G}}-(\gamma b^2)/2+(c^2\gamma)/2 \end{pmatrix}}{\begin{pmatrix} \left(\sqrt{\mathrm{G}}(\mathrm{G}-4)\right)\times \\ \tan\left(\frac{\left((y+x)b^2+(-x-y)c^2-\gamma t-4x-4y\right)\sqrt{\mathrm{G}}}{2(\mathrm{G}-4)}\right) \end{pmatrix}}, \tag{3.59}$$

$a = 0, b = b, c = c, \omega = \frac{\gamma}{4(-c^2+b^2-1)}, \dot{p} = 0, \tilde{A}_{-1} = \frac{\gamma(-b-c)}{8(-c^2+b^2-1)},$

$A_1 = \frac{\gamma(b-c)}{8(-c^2+b^2-1)}, G = -c^2+b^2,$

$$u_{19}=\frac{\left(\begin{array}{c}8(\mathrm{G}-1)\tan\left(\frac{(4(y+x)b^2-4(x+y)c^2-\gamma t-4x-4y)\sqrt{\mathrm{G}}}{8(\mathrm{G}-1)}\right)\\ \tilde{A}_0\sqrt{\mathrm{G}}+\gamma(\mathrm{G})\times\\ \left(\tan\left(\frac{(4(y+x)b^2-4(x+y)c^2-\gamma t-4x-4y)\sqrt{\mathrm{G}}}{8(\mathrm{G}-1)}\right)^2-1\right)\end{array}\right)}{\begin{array}{c}8\sqrt{\mathrm{G}}(\mathrm{G}-1)\times\\ \left(\tan\left(\frac{(4(y+x)b^2+4(-x-y)c^2-\gamma t-4x-4y)\sqrt{\mathrm{G}}}{8(\mathrm{G}-1)}\right)\right)\end{array}}, \tag{3.60}$$

$a=0, b=b, c=c, \omega=\frac{\gamma}{-c^2+b^2-4}, \dot{p}=\dot{p}, \tilde{A}_{-1}=-\frac{(-\dot{p}^2b-\dot{p}^2c+b+c)}{2(-c^2+b^2-4)}$,
$\tilde{A}_1=0, G=-c^2+b^2, P=\dot{p}^2+1$,

$$u_{20}=\frac{\left(\begin{array}{c}2\tan\left(\frac{\left((y+x)b^2+(-x-y)c^2-\gamma t-4x-4y\right)\sqrt{G}}{2(\mathrm{G}-4)}\right)\\ \times(\mathrm{G}-4)\tilde{A}_0\sqrt{\mathrm{G}}+(2\dot{p}\tilde{A}_0b^2-\gamma(\mathrm{P})b\\ -2c^2\dot{p}\tilde{A}_0+2((\gamma\dot{p}^2)/2-\gamma/2)c-8\dot{p}A_0)(b-c)\end{array}\right)}{\left(\begin{array}{c}2\left(\tan\left(\frac{\left((y+x)b^2+(-x-y)c^2-\gamma t-4x-4y\right)\sqrt{\mathrm{G}}}{2(\mathrm{G}-4)}\right)\right.\\ \left.\times\sqrt{\mathrm{G}}+\dot{p}(b-c)\right)(\mathrm{G}-4)\end{array}\right)}, \tag{3.61}$$

$a=0, b=b, c=c, \omega=-\frac{\gamma}{-c^2+b^2-4}, \dot{p}=\dot{p}, A_1=\frac{\gamma(b-c)}{2(-c^2+b^2-4)}$,
$A_{-1}=0, G=-c^2+b^2$,

$$u_{21}=\frac{1}{2(\mathrm{G}-4)}\left(\begin{array}{c}\sqrt{\mathrm{G}}\gamma\tan\left(\frac{\left(\begin{array}{c}(y+x)b^2+(-x-y)c^2\\ -\gamma t-4x-4y\end{array}\right)\sqrt{\mathrm{G}}}{2(\mathrm{G}-4)}\right)\\ +p(-c+b)\gamma+2(\mathrm{G}-4)\tilde{A}_0\end{array}\right). \tag{3.62}$$

Family 6:

Here we get mix soliton under following conditions:

$a=0, b=b, c=0, \omega=-\frac{\gamma}{b^2-4}, \dot{p}=\dot{p}, A_1=0$,
$A_{-1}=-\frac{\gamma b(\dot{p}^2+1)}{2(b^2-4)}, P=\dot{p}^2+1$,

$$u_{22} = -\frac{\gamma P b}{2(b^2-4)} \times$$

$$\left(\dot{p} + \tanh\left(\frac{ln\left(\tan\left(\frac{((y+x)b^2-\gamma t-4x-4y)b}{(b^2-4)}\right)\right)}{2}\right)\right)^{-1} + \tilde{A}_0, \quad (3.63)$$

$a = 0, b = b, c = 0, \omega = -\frac{\gamma}{4(b^2-4)}, \dot{p} = 0, \tilde{A}_1 = \frac{b\gamma}{8(b^2-1)},$

$\tilde{A}_{-1} = -\frac{b\gamma}{8(b^2-1)},$

$$u_{23} = -\frac{b\gamma}{8(b^2-1)\tanh\left(\frac{ln\left(\tan\left(\frac{\left((y+x)b^2-\gamma t/2-x-y\right)b}{2(b^2-1)}\right)\right)}{2}\right)}$$

$$+\tilde{A}_0 + \frac{b\gamma\tanh\left(\frac{ln\left(\tan\left(\frac{\left((y+x)b^2-\gamma t/4-x-y\right)b}{2(b^2-1)}\right)\right)}{2}\right)}{8(b^2-1)}, \quad (3.64)$$

$a = 0, b = b, c = 0, \omega = -\frac{\gamma}{(b^2-4)}, \dot{p} = \dot{p}, \tilde{A}_1 = \frac{b\gamma}{2(b^2-4)},$

$\tilde{A}_{-1} = 0,$

$$u_{24} = \tilde{A}_0 + \frac{\gamma b}{2(b^2-4)} \times$$

$$\left(\dot{p} + \tanh\left(\frac{\ln\left(\tan\left(\frac{((y+x)b^2-\gamma t-4x-4y)b}{2(b^2-4)}\right)\right)}{2}\right)\right). \quad (3.65)$$

Family 7:

We get singular kink soliton:

$a = a, b = 0, c = 0, \omega = \frac{\gamma}{(a^2+4)}, \dot{p} = \dot{p}, \tilde{A}_{-1} = -\frac{\dot{p}\gamma a}{(a^2+4)}, \tilde{A}_1 = 0,$

$$u_{25} = \tilde{A}_0 - \frac{\dot{p}\gamma a}{2(a^2+4)} \times$$

$$\left(\dot{p} + \tanh\left(\frac{\ln\left(-\tanh\left(\frac{((y+x)a^2+\gamma t+4x+4y)a}{2(a^2+4)}\right)\right)}{2}\right)\right)^{-1}. \tag{3.66}$$

Family 8:

Set of mix solitons are as follows:

$$a = Ib, b = b, c = 0, \omega = -\frac{\gamma}{2(b^2-2)}, \dot{p} = I, \tilde{A}_{-1} = \frac{b\gamma}{2(b^2-2)}, \tilde{A}_1 = 0,$$

$$u_{26} = \frac{\sqrt{2}\left((b^2-2)\tilde{A}_0 \tan\left(\frac{2\sqrt{2}\left((x+y)b^2 - \frac{\gamma t}{2} - 2x - 2y\right)b}{4b^2-8}\right)\sqrt{2} - b\gamma/2\right)}{2(b^2-2)\tan\left(\frac{2\sqrt{2}\left((x+y)b^2 - \frac{\gamma t}{2} - 2x - 2y\right)b}{4b^2-8}\right)}, \tag{3.67}$$

$$a = -2\sqrt{-\frac{-\gamma+4\omega}{8\omega}}, b = \frac{\sqrt{-\frac{-\gamma+4\omega}{8\omega}}}{p}(p^2-1), c = \frac{\sqrt{-\frac{-\gamma+4\omega}{8\omega}}}{p}(p^2+1),$$

$$\omega = \omega, p = p, \tilde{A}_{-1} = 0, \tilde{A}_1 = \frac{\omega\sqrt{-\frac{-\gamma+4\omega}{8\omega}}}{p}, E = \frac{\gamma - 4\omega}{\omega}, P = p^2+1,$$

$$u_{27} = \frac{1}{4\sqrt{E}p^2 - 4\sqrt{E\frac{(P)^2}{p^2}}\,p - 4\sqrt{E}} \times$$

$$\left(\begin{array}{c} -p\left(\omega\sqrt{2}\sqrt{E} + 4\tilde{A}_0\right)\sqrt{E\frac{(P)^2}{p^2}} - \\ 4(E\omega)\tanh\left(\frac{1}{2\sqrt{E}(\xi)}\right) + \\ (4p^2\tilde{A}_0 - 4\tilde{A}_0)\sqrt{E} + \sqrt{2}P(E\omega) \end{array}\right), \tag{3.68}$$

$$a = -2\sqrt{-\frac{-\gamma+4\omega}{32\omega}}, b = \frac{\sqrt{-\frac{-\gamma+4\omega}{32\omega}}}{p}(p^2-1), c = \frac{\sqrt{-\frac{-\gamma+4\omega}{32\omega}}}{p}(p^2+1), \omega = \omega,$$

$$p = p, \tilde{A}_{-1} = \frac{p(\gamma - 4\omega)}{16\sqrt{-\frac{-\gamma + 4\omega}{32\omega}}}, \tilde{A}_1 = \frac{\omega\sqrt{-\frac{-\gamma + 4\omega}{32\omega}}}{p}, E = \frac{\gamma - 4\omega}{\omega}, P = p^2 + 1,$$

$$u_{28} = \frac{E\omega\left(\begin{array}{c}\frac{-3p}{8}\left(\begin{array}{c}\left(-\frac{4}{3\sqrt{E}\omega} - \frac{8}{3\sqrt{2}A_0}\right)\tanh\left(\frac{\sqrt{E}\xi}{4}\right)\\ +3\sqrt{2}\omega\left(p^2 - \frac{1}{3}\right)\sqrt{E} + \frac{8}{3p^2\tilde{A}_0}\end{array}\right)\sqrt{E\frac{(P)^2}{p^2}}\\ +\frac{\sqrt{2}}{2}(E\omega)\tanh\left(\frac{\sqrt{E}\xi}{4}\right)^2 - 1/2(p^2+1)(E\omega))\tanh\left(\frac{\sqrt{E}\xi}{4}\right) + \tilde{A}_0 p^2(P)\\ 3/8\sqrt{E}(E\omega)(p^4 + 2/3p^2 + 1)\sqrt{2}\end{array}\right)}{\begin{array}{c}\sqrt{E}\left(-\sqrt{E\frac{(P)^2}{p^2}}p + \sqrt{E}(p^2 - 1)\right)\times\\ \left(-\sqrt{E\frac{(P)^2}{p^2}}p + \sqrt{E}\left(p^2 - 2\tanh\left(\frac{\sqrt{E}\xi}{4}\right)\sqrt{2} + 1\right)\right)\omega\end{array}}, \tag{3.69}$$

$$a = Ib, b = b, c = 0, \omega = -\frac{\gamma}{2(b^2 - 2)}, p = I, \tilde{A}_1 = \frac{b\gamma}{4(b^2 - 2)}, \tilde{A}_{-1} = 0,$$

$$u_{29} = \frac{\sqrt{2}\left(b\gamma\tan\left(\frac{2\sqrt{2}\left((x+y)b^2 - \frac{\gamma t}{2} - 2x - 2y\right)b}{4b^2 - 8}\right) + 4b^2\tilde{A}_0 - 8\tilde{A}_0\right)}{4b^2 - 8}, \tag{3.70}$$

$$a = Ib, b = b, c = 0, \omega = -\frac{\gamma}{4(2b^2 - 1)}, p = I, \tilde{A}_{-1} = -\frac{b\gamma}{4(2b^2 - 1)},$$

$$\tilde{A}_1 = \frac{b\gamma}{8(2b^2 - 1)}$$

$$u_{30}=\sqrt{2}\frac{\left(\begin{array}{c}8\tan\left(\frac{8\sqrt{2}\left((x+y)b^2-\frac{\gamma t}{8}-\frac{x}{2}-\frac{y}{2}\right)b}{16b^2-8}\right)\tilde{A}_0\left(b^2-\frac{1}{2}\right)\sqrt{2}\\ +b\gamma\left(\tan\left(\frac{8\sqrt{2}\left((x+y)b^2-\frac{\gamma t}{8}-\frac{x}{2}-\frac{y}{2}\right)b}{16b^2-8}\right)^2-1\right)\end{array}\right)}{8\left(2b^2-1\right)\tan\left(\frac{8\sqrt{2}\left((x+y)b^2-\frac{\gamma t}{8}-\frac{x}{2}-\frac{y}{2}\right)b}{16b^2-8}\right)}+\tilde{A}_0. \tag{3.71}$$

Family 11:

For this family we get exponential function solutions as:

$$b=a,b=b,c=c,\omega=\frac{\gamma}{c^2+4},p=p,\tilde{A}_{-1}=\frac{\gamma(bp^2-cp^2-2pb+b+c)}{2(c^2+4)},\tilde{A}_1=0,$$

$$u_{31}=\frac{\left(\begin{array}{c}((-p-1)c+(p-1)b)(2c^2\tilde{A}_0-\gamma(p-1)c\\ +b\gamma p-b\gamma+8\tilde{A}_0)\mathrm{e}^{\frac{c\left((x+y)c^2+\gamma t+4x+4y\right)}{c^2+4}}\\ -(2c^2\tilde{A}_0-\gamma(p+1)c+b\gamma p-b\gamma+8\tilde{A}_0)(p-1)\end{array}\right)}{(2((-p-1)c+(p-1)b)\mathrm{e}^{\frac{c((x+y)c^2+\gamma t+4x+4y)}{c^2+4}}-2p+2)(c^2+4)}, \tag{3.72}$$

$$a=b,b=b,c=-b,\omega=\frac{\gamma}{4(b^2+1)},p=\frac{1}{2},\tilde{A}_{-1}=-\frac{\gamma b}{16(b^2+1)},\tilde{A}_1=\frac{\gamma b}{4(b^2+1)},$$

$$u_{32}=\frac{\left((16b^4\tilde{A}_0-4\gamma b^3+16b^2\tilde{A}_0)\mathrm{e}^{-\frac{(4(x+y)b^2+\gamma t+4x+4y)b}{2b^2+2}}-4b^2\tilde{A}_0-b\gamma-4\tilde{A}_0\right)}{(16b^2+16)b^2\mathrm{e}^{-\frac{(4(x+y)b^2+\gamma t+4x+4y)b}{2b^2+2}}-4b^2-4}, \tag{3.73}$$

$$a=b,b=b,c=c,\omega=\frac{\gamma}{(c^2+4)},p=p,\tilde{A}_{-1}=0,\tilde{A}_1=-\frac{\gamma(b-c)}{2(c^2+4)},$$

$$u_{33}=\frac{\left(\begin{array}{c}-(b-c)\left(-2c^2\tilde{A}_0-\gamma(p+1)c+(p-1)b\gamma-8\tilde{A}_0\right)\mathrm{e}^{\frac{c\left((x+y)c^2+\gamma t+4x+4y\right)}{c^2+4}}\\ -2c^2\tilde{A}_0-\gamma(p-1)c+(p-1)b\gamma-8\tilde{A}_0\end{array}\right)}{(-2+2(b-c)\mathrm{e}^{\frac{c((x+y)c^2+\gamma t+4x+4y)}{c^2+4}})(c^2+4)}, \tag{3.74}$$

$$a = b = \frac{cp}{p-1}, c = c, \omega = \frac{\gamma}{4(c^2+1)}, p = p, \tilde{A}_{-1} = -\frac{\gamma c(p-1)}{8(c^2+1)},$$
$$\tilde{A}_1 = -\frac{\gamma c}{8(c^2+1)(p-1)},$$

$$u_{34} = \frac{\left(\begin{array}{c} \left(4c^4\tilde{A}_0 + c^3\gamma + 4c^2\tilde{A}_0\right)\mathrm{e}^{\frac{c\left(4c^2x+4c^2y+\gamma t+4x+4y\right)}{2c^2+2}} \\ -4(p-1)^2(c^2\tilde{A}_0 - c\gamma/4 + \tilde{A}_0) \end{array} \right)}{\begin{array}{c}\left(4c\mathrm{e}^{\frac{c(4c^2x+4c^2y+\gamma t+4x+4y)}{4c^2+4}} - 4p + 4\right)\left(c\mathrm{e}^{\frac{c(4c^2x+4c^2y+\gamma t+4x+4\ )}{4c^2+4}} + p - 1\right) \\ \times (c^2+1)\end{array}}. \tag{3.75}$$

Family 12:
Another exponential function solution:

$$a = a, b = c, c = c, \omega = \frac{\gamma}{a^2+4}, p = p, \tilde{A}_{-1} = -\frac{\gamma(ap-c)}{(a^2+4)}, \tilde{A}_1 = 0,$$

$$u_{35} = \frac{\left(\tilde{A}_0(a^2+4)\mathrm{e}^{\frac{\left((x+y)a^2+\gamma t+4x+4y\right)a}{a^2+4}} + (a^2\tilde{A}_0 - a\gamma + 4\tilde{A}_0)(ap-c)\right)}{(a^2+4)\left(ap + \mathrm{e}^{\frac{((x+y)a^2+\gamma t+4x+4y)a}{a^2+4}} - c\right)}. \tag{3.76}$$

Family 13:
More set of exponential function solutions

$$a = -c, b = c, c = c, \omega = \frac{\gamma}{a^2+4}, p = p, \tilde{A}_{-1} = \frac{\gamma c(p+1)}{(c^2+4)}, \tilde{A}_1 = 0,$$

$$u_{36} = \frac{\left(\tilde{A}_0(c^2+4)\mathrm{e}^{\frac{-c\left((x+y)c^2+\gamma t+4x+4y\right)}{c^2+4}} + ((c^2+4)\tilde{A}_0 + c\gamma)(p+1)\right)}{(c^2+4)\left(p + \mathrm{e}^{-\frac{c((x+y)c^2+\gamma t+4x+4y)}{c^2+4}} + 1\right)}. \tag{3.77}$$

Family 14:
$$a = a, b = 0, c = 0, \omega = \frac{\gamma}{a^2+4}, p = p, \tilde{A}_{-1} = -\frac{p\gamma a}{a^2+4}, \tilde{A}_1 = 0,$$

$$u_{37} = \frac{\left(\tilde{A}_0(a^2+4)\mathrm{e}^{\frac{a\left((x+y)a^2+\gamma t+4x+4y\right)}{a^2+4}} + ap(a^2\tilde{A}_0 - a\gamma + 4\tilde{A}_0\right)}{a^2+4\left(ap+\mathrm{e}^{\frac{a((x+y)a^2+\gamma t+4x+4y)}{a^2+4}}\right)}. \tag{3.78}$$

Family 16:

$$a = a, b = -c, c = c, \omega = \frac{\gamma}{a^2+4}, p = p, \tilde{A}_{-1} = -\frac{p\gamma(cp+a)}{a^2+4}, \tilde{A}_1 = 0,$$

$$u_{38} = \frac{\left(\left(-c\gamma p + \tilde{A}_0(a^2+4)\right)(cp+a)\mathrm{e}^{\frac{\left((x+y)a^2+\gamma t+4x+4y\right)a}{a^2+4}} - p(a^2\tilde{A}_0 - c\gamma p - a\gamma + 4\tilde{A}_0)\right)}{(a^2+4)\left((cp+a)\mathrm{e}^{\frac{((x+y)a^2+\gamma t+4x+4\ \ )a}{a^2+4}} - p\right)}, \tag{3.79}$$

$$a = a, b = -c, c = c, \omega = \frac{\gamma}{a^2+4}, p = p, \tilde{A}_{-1} = 0, \tilde{A}_1 = \frac{c\gamma}{a^2+4},$$

$$u_{39} = \frac{\left(c(a^2\tilde{A}_0 + c\gamma p + a\gamma + 4\tilde{A}_0)\mathrm{e}^{\frac{a((x+y)a^2+\gamma t+4x+4y)}{a^2+4}} - a^2\tilde{A}_0 - c\gamma p - 4\tilde{A}_0\right)}{(a^2+4)\left(-1+\mathrm{e}^{\frac{a((x+y)a^2+\gamma t+4x+4y)}{a^2+4}}c\right)}, \tag{3.80}$$

$$a = a, b = \frac{a}{2p}, c = -\frac{a}{2p}, \omega = \frac{\gamma}{4(a^2+1)}, p = p, \tilde{A}_{-1} = -\frac{pa\gamma}{8(a^2+1)}, \tilde{A}_1 = -\frac{a\gamma}{8p(a^2+1)},$$

$$u_{40} = \frac{\left(\left(4a^4A_0 + \gamma a^3 + 4a^2\tilde{A}_0\right)\mathrm{e}^{\frac{a\left(4a^2x+4a^2y+\gamma t+4x+4y\right)}{2a^2+2}} - 16p^2(a^2\tilde{A}_0 - a\gamma/4 + \tilde{A}_0)\right)}{4(a^2+1)\left(\mathrm{e}^{\frac{a(4a^2x+4a^2y+\gamma t+4x+4\ \ )}{2a^2+2}}a^2 - 4p^2\right)}. \tag{3.81}$$

Family 17:

We get plane wave solutions:

$$a = 0, b = c, c = c, \omega = \omega, p = 0, \tilde{A}_{-1} = 0, \tilde{A}_1 = \frac{\gamma - 4\omega}{6c},$$

$$u_{41} = \tilde{A}_0 + \frac{(\gamma - 4\omega)\xi}{6}, \tag{3.82}$$

$$a = 0, b = c, c = c, \omega = \frac{\gamma}{4}, p = p, \tilde{A}_{-1} = \frac{c\gamma}{4}, \tilde{A}_1 = 0,$$

$$u_{42}=\frac{c\gamma}{(\gamma t+4x+4y)c+4p}+\tilde{A}_0, \tag{3.83}$$

$$a=0, b=c, c=c, \omega=-\frac{3c\tilde{A}_1}{2}+\frac{\gamma}{4}, p=p, \tilde{A}_{-1}=0, \tilde{A}_1=\tilde{A}_1,$$

$$u_{43}=-\frac{3}{2tc^2\tilde{A}_1{}^2}+\frac{\big((\gamma t+4x+4y)c+4p\big)A_1}{4}+\tilde{A}_0. \tag{3.84}$$

Family 18:

$$a=0, b=-c, c=c, \omega=\omega, p=0, \tilde{A}_{-1}=\frac{\gamma-4\omega}{6c}, \tilde{A}_1=0,$$

$$u_{44}=\frac{(\gamma-4\omega)\xi}{6}+\tilde{A}_0, \tag{3.85}$$

$$a=0, b=-c, c=c, \omega=\frac{\gamma}{4}, p=p, \tilde{A}_{-1}=-\frac{c\gamma p^2}{4}, \tilde{A}_1=0,$$

$$u_{45}=-\frac{c^2\gamma p^2(\gamma t+4x+4y)}{16+16p\left(\frac{\gamma t}{4}+x+y\right)c}+\tilde{A}_0, \tag{3.86}$$

$$a=0, b=-c, c=c, \omega=\frac{\gamma}{4}, p=p, \tilde{A}_{-1}=0, \tilde{A}_1=\frac{c\gamma}{4},$$

$$u_{46}=\tilde{A}_0+\frac{c\gamma\left(p+\frac{1}{c\left(\frac{\gamma t}{4}+x+y\right)}\right)}{4}. \tag{3.87}$$

Family 19:

More kink wave type of solutions for these families:

$$a=c, b=0, c=c, \omega=\frac{\gamma}{2(c^2+2)}, p=p, \tilde{A}_{-1}=-\frac{\gamma c(p^2+2p-1)}{4(c^2+2)}, \tilde{A}_1=0,$$

$$u_{47}=\frac{\left(\begin{array}{c}4\tilde{A}_0\sqrt{2}(c^2+2)\tanh\left(\frac{c\sqrt{2}(2c^2x+2c^2y+\gamma t+4x+4y)}{4c^2+8}\right)\\ +4\tilde{A}_0(p+1)c^2-\gamma c(p^2+2p-1)+8\tilde{A}_0(p+1)\end{array}\right)}{(4c^2+8)\left(p+\sqrt{2}\tanh\left(\frac{c\sqrt{2}(2c^2x+2c^2y+\gamma t+4x+4y)}{4c^2+8}\right)+1\right)}, \tag{3.88}$$

$$a=c, b=0, c=c, \omega=\frac{\gamma}{4(2c^2+1)}, p=-1, \tilde{A}_{-1}=\frac{\gamma c}{4(2c^2+1)}, \tilde{A}_1=\frac{\gamma c}{8(2c^2+1)},$$

$$u_{48} = \frac{\left(\begin{array}{c}\gamma c\sqrt{2}\tanh\left(\frac{c\sqrt{2}((8x+8y)c^2+\gamma t+4x+4y)}{16c^2+8}\right)^2 + \\ (16\tilde{A}_0c^2+8\tilde{A}_0)\tanh\left(\frac{c\sqrt{2}((8x+8y)c^2+\gamma t+4x+4y)}{16c^2+8}\right)+\gamma c\sqrt{2}\end{array}\right)}{(16c^2+8)\tanh\left(\frac{c\sqrt{2}((8x+8y)c^2+\gamma t+4x+4y)}{16c^2+8}\right)}, \tag{3.89}$$

$$a = c, b = 0, c = c, \omega = \frac{\gamma}{2(c^2+2)}, p = p, \tilde{A}_{-1} = 0, \tilde{A}_1 = \frac{\gamma c}{4(c^2+2)}$$

$$u_{49} = \frac{1}{4c^2+8}\left(\begin{array}{c}\sqrt{2}\gamma\tanh\left(\frac{c\sqrt{2}(2c^2x+2c^2y+\gamma t+4x+4y)}{4c^2+8}\right)c \\ +4\tilde{A}_0c^2+\gamma(p+1)c+8\tilde{A}_0\end{array}\right). \tag{3.90}$$

Family 20:

$$a = 0, b = 0, c = c, \omega = \frac{\gamma}{c^2+4}, p = p, \tilde{A}_{-1} = -\frac{\gamma c(p^2-1)}{2(c^2+4)}, \tilde{A}_1 = 0,$$

$$u_{50} = -\frac{\gamma c(p^2-1)}{2c^2+8}\left(p+\tanh\left(\frac{((x+y)c^2+\gamma t+4x+4y)c}{2c^2+8}\right)\right)^{-1} + \tilde{A}_0, \tag{3.91}$$

$$a = 0, b = 0, c = c, \omega = \frac{\gamma}{4(c^2+1)}, \tilde{A}_{-1} = \frac{\gamma c}{8(c^2+1)}, \tilde{A}_1 = \frac{\gamma c}{8(c^2+1)},$$

$$u_{51} = \frac{\gamma c}{8c^2+8}\left(\tanh\left(\frac{(4c^2x+4c^2y+\gamma t+4x+4y)c}{8c^2+8}\right)\right)^{-1} + \tilde{A}_0 + \frac{\gamma c}{8c^2+8}\tanh\left(1/2\left(\left(\frac{\gamma t}{4c^2+4}+x+y\right)\right)c\right), \tag{3.92}$$

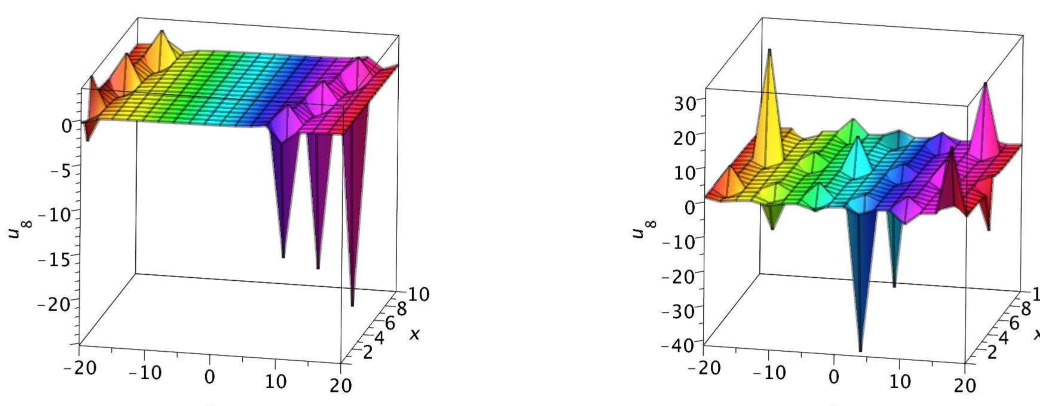
$$a = 0, b = 0, c = c, \omega = \frac{\gamma}{c^2+4}, p = p, \tilde{A}_{-1} = 0, \tilde{A}_1 = \frac{\gamma c}{2(c^2+4)},$$

$$u_{52} = \tilde{A}_0 + \frac{\gamma c}{2c^2+8}\left((p+\tanh\left(1/2\left(\frac{\gamma t}{c^2+4}+x+y\right)c\right)\right). \tag{3.93}$$

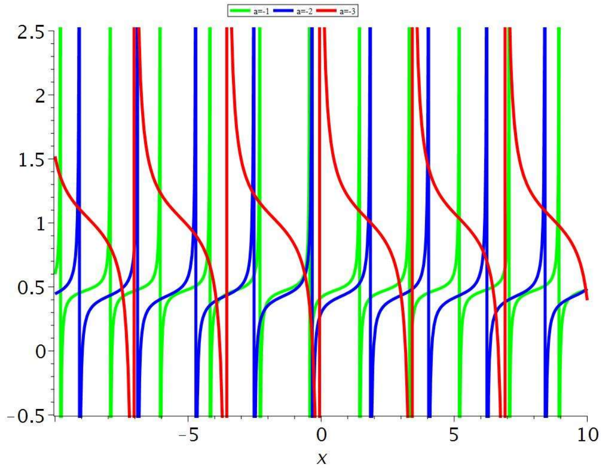
3.7 Results and discussion

In this part of chapter, we derived exact solitary wave solutions of AKNS equation by IThEM with the help of symbolic computation. To understand the physical dynamics of these waves, 3D and 2D graphs and contour plots have been plotted to demonstrate the behavior of acquired solutions by choosing appropriate values of parameters. These results will be beneficial for researchers to acknowledge the application of this model in different fields of sciences as to best of knowledge no study has been done on this equation by using proposed method. Many solutions AKNS of various types have been reported in literature, by comparing our results with recently derived solutions in [144], the authors have used modified exponential function method to derive hyperbolic, periodic, exponential function solutions however, we succeed to generate more than 50 solutions in the form of hyperbolic, trigonometric, and rational solutions, all the results are new and have not been reported before. Similarly, most recently authors in [145] have used $\left(\frac{G'}{G}, \frac{1}{G}\right)$-expansion method on fractional AKNS equation to derive various type of solutions but we found that we established comprehensive results which are distant and novel from others. For the better understanding of these results physical analysis of some of the solutions has been depicted through 3D, 2D and contour plots. Fig (3.7) -(3.12) shows graphical behavior of some solutions of AKNS equation by choosing appropriate parameters.

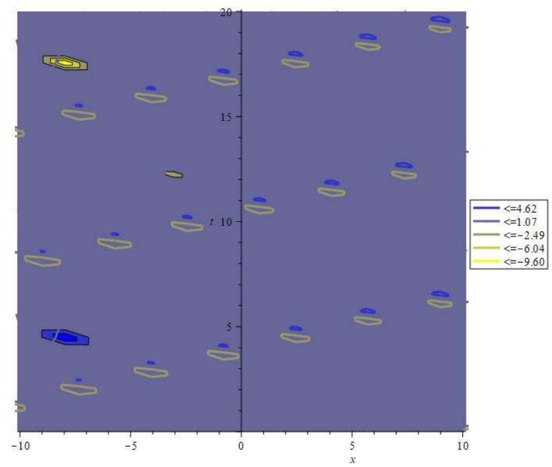
In Figure 3.7: Represents dynamical behavior of singular periodic wave solution of u_8 mentioned in Eq (3.94), for 3D fig (a) and (b) we used $\mathrm{a} = -2, \mathrm{a} = -3$, respectively. For 2 D fig (c) and (d) we used for $a = -1, a = -2, a = -3$, respectively, with $-10 \leq x \leq 10, y = 2, t = 1,$. For contour plot we used parameters $-30 \leq x \leq 30, t = 0..20,$ and $-40 \leq x \leq 40, t = 0..20$ with $a = -3$ $p = 0.3, b = 3.5, A_0 = 0.5, \gamma = 0.2, y = 2$.



(a) **(b)**



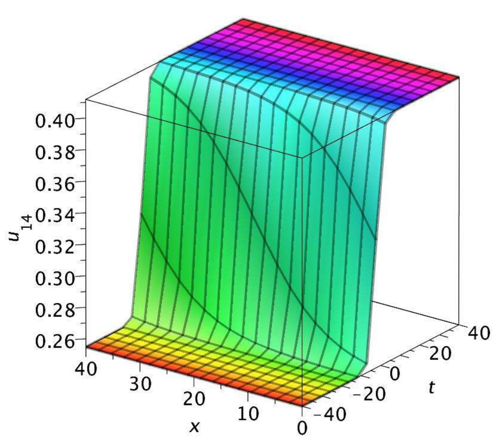
(c)



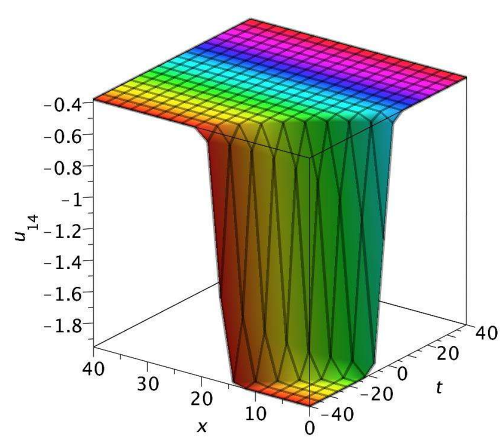
(d)

Figure 3.7:For u_8 graphs exhibits periodic wave solution.

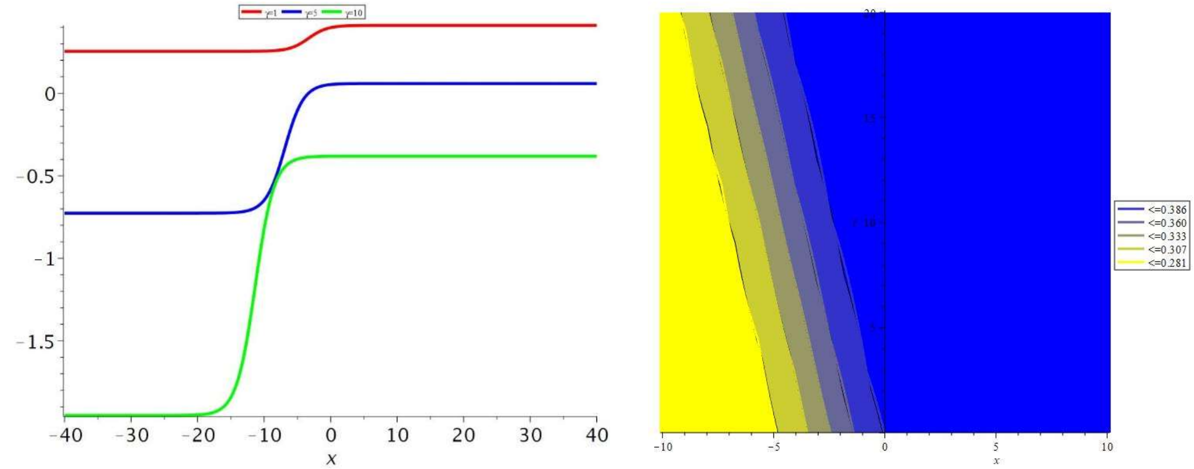
In Figure 3.8: 3D, 2D graphs and contour plot represents kink solitary wave solution of u_{14} mentioned in Eq (3.95), by choosing parameters,$p = 2, a = 0.5, c = 0.5, A_0 = 0.5, y = 1.$ For 3D fig(a) and fig(b) we choose $\gamma = 1. \gamma = 10$, For 2D fig(c) we choose $\gamma = 1,\ \gamma = 5,\ \gamma = 10$, with $-15 \leq \text{x} \leq 15$, and for contour plot fig(d) we have values $-30 \leq x \leq 30, t = 0..20$, and $-40 \leq x \leq 40, t = 0..20$, for $\gamma = 1$.



(a)



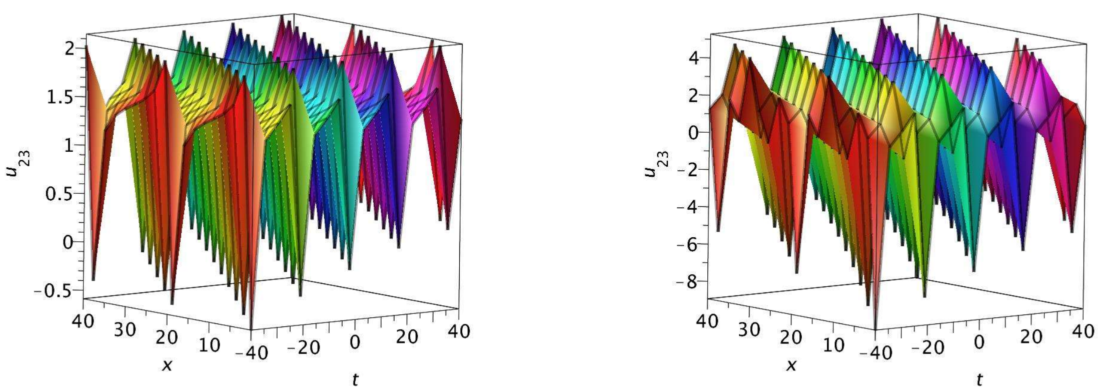
(b)



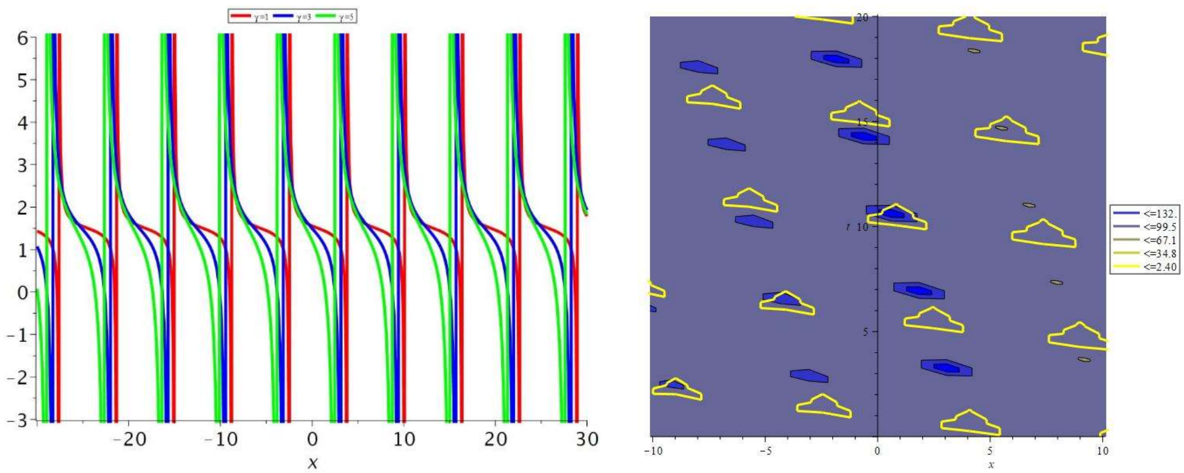
(c) **(d)**

Figure 3.8: :For u_{14} graphs exhibits kink wave solution

In Figure 3.9: shows 3D and 2D graphs and contour plot of periodic solitary wave solution of u_{23} mentioned in equation (3.64) by choosing parameters $b = 0.5, A_0 = 1.5, y = -1$. For 3D fig(a) and fig(b) we choose $\gamma = 1, \gamma = 5,$ and for 2D fig(c) we choose $\gamma = 1$, $\gamma = 3$, $\gamma = 5$, respectively, and for contour plot fig(d) we have values $-30 \leq \mathrm{x} \leq 30, \mathrm{t} = 0..20$, and $-40 \leq \mathrm{x} \leq 40, \mathrm{t} = 0..20$, respectively for $\gamma = 1$.



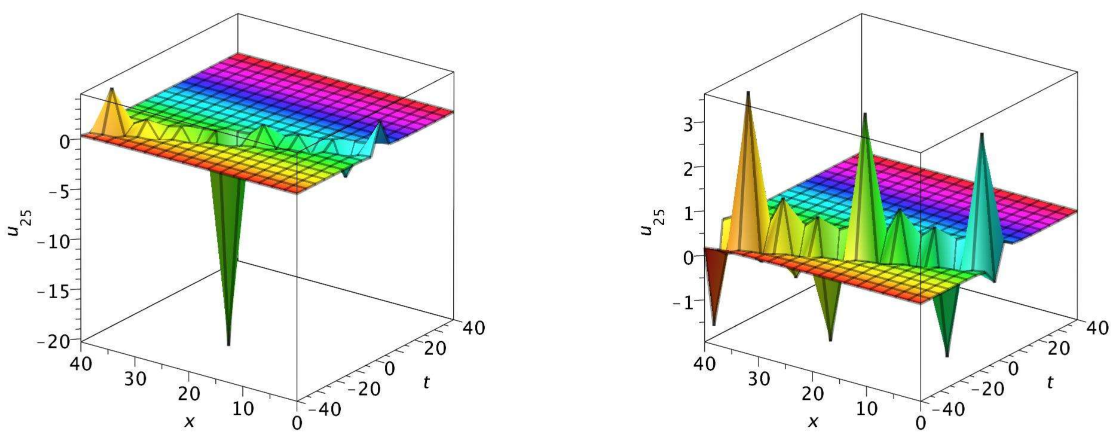
(a) **(b)**



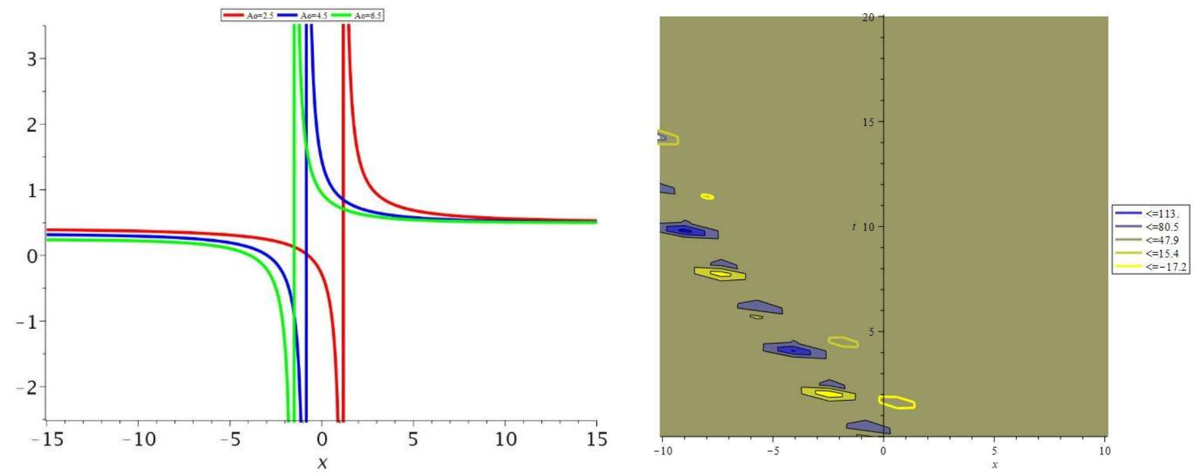
(c) **(d)**

Figure 3.10::For u_{23} graphs exhibits periodic wave solution.

In Figure 3.11, 3D, 2D and contour plot exhibits graphical nature of singular kink solitary wave u_{25} mentioned in equation (3.96) for the values $p = 1.5, A_0 = 0.5, \gamma = 3.5, y = 2.$ For 3D fig(a) and fig(b) we choose $a = 0.1, a = 0.3,$ for 2D fig(c) we choose $a = 0.1, a = 0.2, a = 0.3,$ respectively with parameters $-15 \leq x \leq 15, y = 2, t = 1,$ for contour plot fig(d) we have values$-30 \leq x \leq 30, t = 0..20,$ and $-40 \leq x \leq 40, t = 0..20, a = 0.3$ respectively.



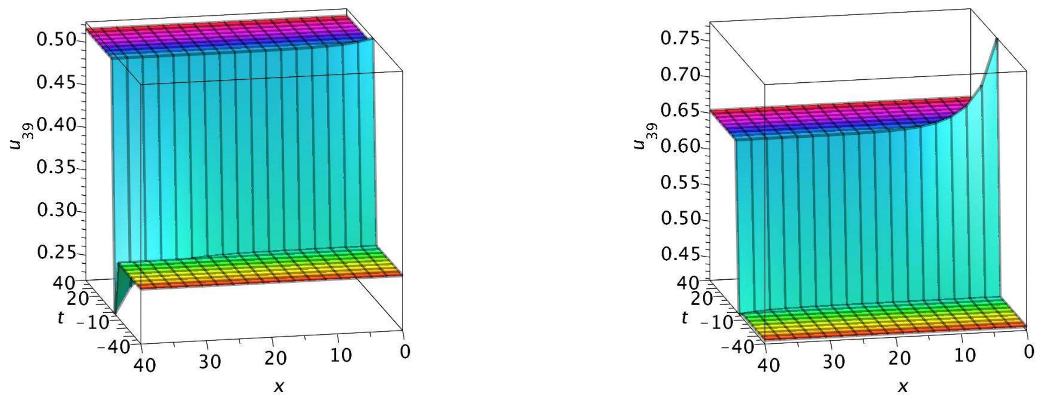
(a) **(b)**



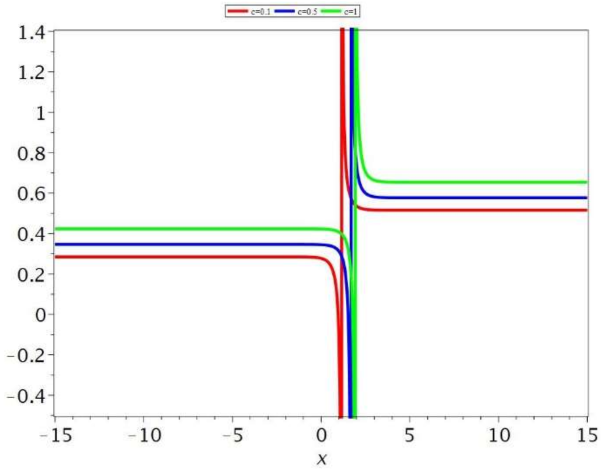
(c) **(d)**

Figure 3.11: :For u_{25} graphs exhibits singular kink wave solution.

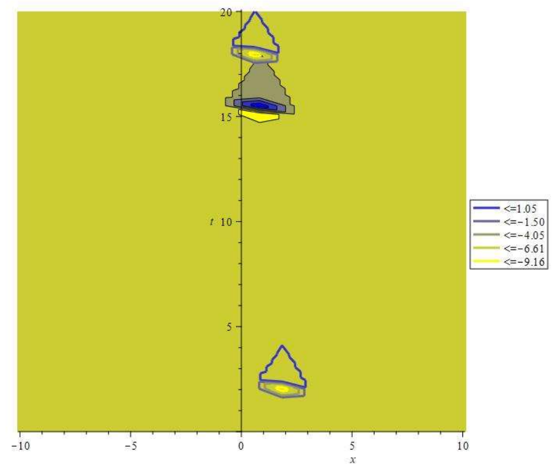
In Figure 3.12, 3D, 2D graphs and contour plot of kink solitary wave, u_{39} mentioned in Eq (3.97) by choosing parameters $p = 2, a = -3, A_0 = 0.5,\ \gamma = 1, y = -2$. For 3D fig(a) and fig(b) we choose $c = 0.1, c = 1$, for 2D fig(c) we choose $c = 01, c = 0.5, c = 1$, respectively with $-15 \leq x \leq 15,\ y = -2,\ t = 1$, and for contour plot fig(d) we have $-30 \leq x \leq 30, t = 0..20$, and $-40 \leq x \leq 40, t = 0..20$ respectively for $c = 1$.



(a) **(b)**



(c)



(d)

Figure 3.12: For u_{39}graphs exhibits kink wave solution.

3.8 Conclusion:

Improved tanh $(\frac{\phi}{2})$-expansion method has been successfully administered to achieve new and general solutions to fourth order Ablowitz-Kaup-Newell-Segur water wave (AKNS) equation. As an outcome of this technique, abundant new solutions have been derived including solitons which can be classified into distinct types specified by their profiles such as, periodic, kink solitons. Each solution has some physical interpretation like kink solitons have permanent profile that it remains same over time while periodic wave solitons show dynamical profile and can depend on time. Kink solitons have applications in almost all nonlinear phenomena as it propagates in high nonlinear media with self-steeping effect such as in nonlinear fibers, singular solitons are one with singularity and have applications in the study of rouge waves whereas periodic waves are also very important and have many applications in various fields. These newly derived solutions may have valuable scope for future study of the shock waves, water wave phenomena especially in ocean waves. IThEM is more effective than tanh method and extended tanh method[108, 109], sine-cosine method[110], ansatz method[111], Improved tan$(\frac{\phi}{2})$-expansion method [112] in producing different types of solutions which are more general and abundant. This is a new method and has not been implemented much recently. The efficiency of this method can be predicted easily by the rich variety of obtained results. This scheme is applicable to a variety of nonlinear PDEs. The concluded wave structures can be helpful to understand the characteristics of nonlinear phenomena that develop in various realms of nonlinear sciences. Moreover, the outcome of this article can predict that this method is suitable to apply on various higher order nonlinear models to produce many interesting solutions involved in engineering, nonlinear optics, physics and other life sciences. In future we will be using this technique to other higher PDEs and on nonlinear fractional PDEs.

3.8.1 Remark:

Since improved $\tan(\frac{\phi}{2})$-expansion method and improved $\tanh(\frac{\varphi(\xi)}{2})$-expansion method looks similar, but their results are totally different. Improved $\tan(\frac{\phi}{2})$-expansion method produces seventeen families whereas improved $\tanh(\frac{\varphi(\xi)}{2})$-expansion method produces twenty families that generate abundant solutions in the form of hyperbolic, periodic, exponential, logarithmic functions.

3.9 Summary:

In this chapter we have solved recently developed $(3+1)$-dimensional Boiti-Leon-Manna-Pempinelli equation and fourth order Ablowitz-Kaup-Newell-Segur water wave (AKNS) equation by using the innovative and efficient method called improved $\tanh(\frac{\varphi(\xi)}{2})$-expansion method (IThEM). A lot of solitary wave solutions have been generated that prove the efficiency of methods. The results are new and had not been reported in literature previously. Important steps of the chapter include introduction of governing equations followed by focal steps of methods used and derivation of solutions by proposed method. Finally graphical representation of some results followed by conclusion.

Chapter 4 investigates two more important models which are the generalization of nonlinear Schrodinger equation using generalized auxiliary mapping method.

Chapter 4. Optical soliton solutions of some nonlinear equations using versatile technique.

4.1 Introduction:

Exact solutions of complex nonlinear differential equations especially solitons have been studied actively by researchers due to its numerous characteristics. Optical solitons have showed significant effect in telecommunication field because of its key role in data transmission through optical fibers over large distances, such passing through oceans and from one continent to other without loss of data [146–149]. Therefore, to find optical solitons and other exact solutions many powerful analytical methods have been developed [127, 133, 150–156].

The prime objective of this chapter is to study certain optical solitons using generalized auxiliary mapping method developed by Sirendaoreji [61]. This method is very effective in extracting a variety of exact solutions with the aid of mathematical symbolic computation. The optical solitons will be studied through a supportive illustration.

4.2 Illustrative Applications:

In this section, optical solitons solutions of two renowned nonlinear partial differential equations will be constructed using the above-mentioned method.

4.3 Fokas System:

We will first investigate the Fokas system for complex valued function ψ and real valued function ϕ representing pulse propagation in monomode optical fibers [157].

$$
\begin{aligned}
& i\frac{\partial}{\partial t}\psi + r_1\frac{\partial^2}{\partial x^2}\psi + r_2\psi\phi = 0\,, \\
& r_3\frac{\partial}{\partial y}\phi - r_4\frac{\partial}{\partial x}(|\phi|^2) = 0.
\end{aligned}
\tag{4.1}
$$

Where the parameters, $r_1, r_2, r_3, r_4 \neq 0$, are arbitrary constants. Fokas system is the extension of nonlinear Schrodinger equation in $(2+1)$-dimension. A S Fokas [158] and Shulman [159] derived this model to study nonlinear Schrodinger equation in multiple dimensions. Chakravarty et.al [159] reduced the dual Yang-Mills equation into Fokas equation. Due to the importance of this model in many fields, researchers are interested in deriving solutions of this model. K. J Wang employed Exp-function to construct exact solutions of Fokas system [160]. S. Tarla et.al. [161]

investigated model via Jacobi elliptic function expansion method. J.Rao et.al. investigated doubly localized rogue waves and lump solitons.

Let us use the following complex transformations to solve Eq. ((4.1).

$\psi(x,y,t) = u(\zeta)e^{i\theta}, \qquad \phi(x,y,t) = V(\zeta),$

where,

$\zeta = (x + y - \eta t), \qquad \theta = \lambda_1 x + \lambda_2 y + \lambda_3 t + \lambda_4.$

Using the above-mentioned wave transformation in Eq. ((4.1) , converts the system into the following nonlinear system of ODE,

$$i(-2r_1\lambda_1 + \nu)\frac{\mathrm{d}}{\mathrm{d}\xi}u(\zeta) + u(\zeta)\lambda_3 - r_1\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}u(\zeta) + r_1 u(\zeta){\lambda_1}^2 - r_2 u(\zeta)V(\xi) = 0,$$
$$r_3\frac{\mathrm{d}}{\mathrm{d}\xi}V(\zeta) - 2r_4 u(\zeta)\frac{\mathrm{d}}{\mathrm{d}\xi}u(\zeta) = 0, \tag{4.2}$$

separating real and imaginary parts of first equation of Eq. (4.2) we get,

$$u(\zeta)\lambda_3 - r_1\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}u(\zeta) + r_1 u(\zeta){\lambda_1}^2 - r_2 u(\zeta)V(\zeta) = 0,$$
$$\nu = 2r_1\lambda_1. \tag{4.3}$$

Integrating second equation in Eq. (4.2) we get,

$$V(\xi) = \frac{r_4 u^2(\zeta)}{r_3}, \tag{4.4}$$

substituting equation (4.3) in the first equation of Eq. (4.2) we get,

$$u(\zeta)\lambda_3 - r_1\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}u(\zeta) + r_1 u(\zeta){\lambda_1}^2 - \frac{r_2 r_4 u^3(\zeta)}{r_3} = 0. \tag{4.5}$$

Balancing the highest order of linear term $\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}u(\zeta)$ with the nonlinear term $u^3(\zeta)$ in Eq. (4.5) determine the value of N. Here $3N = N + 2 \Rightarrow N = 1$. This gives solution of the form.

$$u(\zeta) = S = a_0 + a_1\mathbb{Q}(\zeta) + \frac{b_1}{\mathbb{Q}(\zeta)} + \frac{d_1\left(\frac{d}{d\zeta}\mathbb{Q}(\zeta)\right)}{\mathbb{Q}(\zeta)^2}. \tag{4.6}$$

Replacing Eq. (4.6) into Eq. (4.5) along with Eq. (1.48), we get algebraic system and by equating this system to 0 we get values of coefficients $a_0, a_1, b_1, d_1, \lambda_1, \lambda_2, \lambda_3$ as follows:

To make this manuscript nice and simple we are assuming,

$$\Delta = \sqrt{\beta_2{}^2 - 4\beta_1\beta_3}, \quad E = (\tanh(f))^2 + \frac{(\Delta)\beta_2}{2\beta_1\beta_3} + 1 - \frac{\beta_2{}^2}{2\beta_1\beta_3},$$

$$T = \sqrt{(\Delta\beta_2 + 2\beta_1\beta_3 - \beta_2{}^2)\beta_1\beta_3},$$

$$J = \Delta\beta_2 + 2\beta_1\beta_3 - \beta_2{}^2, \qquad H = (\tan(f))^2 - \frac{2\Delta\tan(f)}{\beta_2} + 1,$$

$$F = \left(\frac{\sqrt{\beta_1}\zeta}{2}\right), F' = \left(\frac{\sqrt{-\beta_1}\zeta}{2}\right), f = \left(\frac{\sqrt{\beta_1}\zeta}{4}\right),$$

$$\boldsymbol{G} = \sqrt{\frac{\beta_1}{2}}(\zeta).$$

Set 1 :

$$a_0 = 0, \quad a_1 = 0, \quad d_1 = \sqrt{2}\sqrt{-\frac{r_1r_3}{r_2r_4}}, \quad b_1 = 0, \quad \beta_1 = \beta_1, \quad \beta_2 = 0,$$

$$\beta_3 = \beta_3, \ \lambda_3 = -\lambda_1{}^2 r_1 - 2\beta_1 r_1, \ \lambda_1 = \lambda_1.$$

Substituting these coefficients along with the auxiliary solutions of Eq. (1.48), we get solutions of Eq. ((4.1) as follows.

$$\psi_j(x,y,t) = u_j(\zeta)e^{i\theta}, \qquad \phi_j(x,y,t) = V_j(\zeta) = \frac{r_4 u_j{}^2(\zeta)}{r_3},$$

$$u(\zeta) = S = \frac{d_1\left(\frac{d}{d\zeta}\mathbb{Q}(\zeta)\right)}{\mathbb{Q}(\zeta)}.$$

For $\beta_1 > 0$,

$$\psi_1 = \left(-\sqrt{\beta_1}d_1\right)e^{i\theta}, \tag{4.7}$$

$$\phi_1 = \frac{r_4}{r_3}\left(-\sqrt{\beta_1}d_1\right)^2. \tag{4.8}$$

For $\beta_1 > 0, \Delta > 0$,

$$\psi_2 = \left(-d_1\sqrt{\beta_1}\tanh\left(\sqrt{\beta_1}\zeta\right)\right)e^{i\theta}, \tag{4.9}$$

$$\phi_2 = \frac{r_4}{r_3}\left(-d_1\sqrt{\beta_1}\tanh\left(\sqrt{\beta_1}\zeta\right)\right)^2, \tag{4.10}$$

$$\psi_3 = \left(\frac{-\sqrt{\beta_1}\cosh\left(\sqrt{\beta_1}\zeta\right)}{\sinh(\sqrt{\beta_1}\zeta)}d_1\right)e^{i\theta}, \tag{4.11}$$

$$\phi_3 = \frac{r_4}{r_3}\left(\frac{-\sqrt{\beta_1}\cosh\left(\sqrt{\beta_1}\zeta\right)}{\sinh(\sqrt{\beta_1}\zeta)}d_1\right)^2. \tag{4.12}$$

For, $\beta_1 > 0, \beta_3 > 0$,

$$\psi_4 = \left(\frac{-\sqrt{\beta_1}\left(\sinh(2G)\sinh(F) + \sqrt{2}\cosh(F)\right)d_1}{\sinh(2G)\cosh(F)}\right)e^{i\theta}, \tag{4.13}$$

$$\phi_4 = \frac{r_4}{r_3}\left(\frac{-\sqrt{\beta_1}\left(\sinh(2G)\sinh(F) + \sqrt{2}\cosh(F)\right)d_1}{\sinh(2G)\cosh(F)}\right)^2, \tag{4.14}$$

$$\psi_5 = \left(\frac{\sqrt{\beta_1}\left(-2\cosh(F)\sinh(2G) + \sqrt{2}\sinh(F)\right) d_1}{\sinh(2G)\sinh(F)} \right) e^{i\theta}, \tag{4.15}$$

$$\phi_5 = \frac{r_4}{r_3}\left(\frac{\sqrt{\beta_1}\left(-2\cosh(F)\sinh(2G) + \sqrt{2}\sinh(F)\right) d_1}{\sinh(2G)\sinh(F)} \right)^2 . \tag{4.16}$$

For $\beta_1 > 0, \Delta = 0$,

$$\psi_6 = e^{i\theta}\left(d_1 \frac{\sqrt{\beta_1}}{2}\left(1 - \tanh(F)\right) \right), \tag{4.17}$$

$$\phi_6 = \frac{r_4}{r_3}\left(d_1 \frac{\sqrt{\beta_1}}{2}\left(1 - \tanh(F)\right) \right)^2, \tag{4.18}$$

$$\psi_7 = e^{i\theta}\left(d_1 \frac{\sqrt{\beta_1}}{2}\left(1 - \coth(F)\right) \right), \tag{4.19}$$

$$\phi_7 = \frac{r_4}{r_3}\left(d_1 \frac{\sqrt{\beta_1}}{2}\left(1 - \coth(F)\right) \right)^2 . \tag{4.20}$$

For $\beta_1 < 0, \Delta > 0$,

$$\psi_8 = e^{i\theta}\left(d_1 \frac{\sqrt{-\beta_1}}{2}\left(\tan\left(\sqrt{\beta_1}\zeta\right)\right) \right), \tag{4.21}$$

$$\phi_8 = \frac{r_4}{r_3}\left(d_1\frac{\sqrt{-\beta_1}}{2}\left(\tan\left(\sqrt{\beta_1}\zeta\right)\right)\right)^2, \tag{4.22}$$

$$\psi_9 = e^{i\theta}\left(-d_1\sqrt{-\beta_1}\left(\cot\left(\sqrt{-\beta_1}\zeta\right)\right)\right), \tag{4.23}$$

$$\phi_9 = \frac{r_4}{r_3}\left(-d_1\sqrt{-\beta_1}\left(\cot\left(\sqrt{-\beta_1}\zeta\right)\right)\right)^2. \tag{4.24}$$

For $\beta_1 < 0, \beta_3 > 0$

$$\psi_{10} = \left(-\frac{\left(\cos(F')^2 - \frac{1}{2}\right)\sqrt{-\beta_1}}{\sin(F')\cos(F')}d_1\right)e^{i\theta}, \tag{4.25}$$

$$\phi_{10} = \frac{r_4}{r_3}\left(-\frac{\left(\cos(F')^2 - \frac{1}{2}\right)\sqrt{-\beta_1}}{\sin(F')\cos(F')}d_1\right)^2, \tag{4.26}$$

$$\psi_{11} = \left(\frac{\left(\frac{1}{2} - \cos(F')^2\right)\sqrt{-\beta_1}}{\sin(F')\cos(F')}d_1\right)e^{i\theta}, \tag{4.27}$$

$$\phi_{11} = \frac{r_4}{r_3}\left(\frac{\left(\frac{1}{2} - \cos(F')^2\right)\sqrt{-\beta_1}}{\sin(F')\cos(F')}d_1\right)^2. \tag{4.28}$$

For $\beta_1 > 0$,

$$\psi_{12} = \left(\frac{-(4\beta_1\beta_3 + \mathrm{e}^{2\sqrt{\beta_1}(\zeta)})\sqrt{\beta_1}}{\mathrm{e}^{2\sqrt{\beta_1}(\zeta)} - 4\beta_1\beta_3}d_1\right)e^{i\theta}, \tag{4.29}$$

$$\phi_{12} = \frac{r_4}{r_3}\left(\frac{-(4\beta_1\beta_3 + \mathrm{e}^{2\sqrt{\beta_1}(\zeta)})\sqrt{\beta_1}}{\mathrm{e}^{2\sqrt{\beta_1}(\zeta)} - 4\beta_1\beta_3}d_1\right)^2. \tag{4.30}$$

For $\beta_1 > 0, \beta_2 = 0$,

$$\psi_{13} = \left(\frac{-(4\beta_1\beta_3\mathrm{e}^{2\sqrt{\beta_1}(\zeta)} + 1)\sqrt{\beta_1}}{4\beta_1\beta_3\mathrm{e}^{2\sqrt{\beta_1}(\zeta)} - 1}d_1\right)e^{i\theta}, \tag{4.31}$$

$$\phi_{13} = \frac{r_4}{r_3}\left(\frac{-(4\beta_1\beta_3 e^{2\sqrt{\beta_1}(\zeta)} + 1)\sqrt{\beta_1}}{4\beta_1\beta_3 e^{2\sqrt{\beta_1}(\zeta)} - 1} d_1\right)^2, \tag{4.32}$$

Set 2 :

$a_0 = a_0, \quad a_1 = 0, \quad d_1 = 0, \quad b_1 = 0, \quad \beta_1 = \beta_1, \quad \beta_2 = \beta_2,$

$\beta_3 = \beta_3,\ \lambda_3 = \frac{{a_0}^2 r_2 r_4 - {\lambda_1}^2 r_1 r_3}{r_3},\ \lambda_1 = \lambda_1.$

Substituting these coefficients along with the auxiliary solutions of Eq. (1.48), we get solutions of Eq. ((4.1) as follows.

$$u(\zeta) = S = a_0. \tag{4.33}$$

For $\beta_1 > 0$,

$$\psi_{14} = a_0 e^{i\theta}, \tag{4.34}$$

$$\phi_{14} = \frac{r_4}{r_3}\left(a_0 e^{i\theta}\right)^2, \tag{4.35}$$

Set 3 :

$a_0 = a_0, \quad a_1 = 0, \quad d_1 = d_1, \quad b_1 = 0, \quad \beta_1 = \frac{{a_0}^2}{{d_1}^2}, \quad \beta_2 = 0,$

$\beta_3 = 0,\ \lambda_3 = \frac{4{a_0}^2 r_2 r_4 - {\lambda_1}^2 r_1 r_3}{r_3},\ \lambda_1 = \lambda_1.$

Substituting these coefficients along with the auxiliary solutions of Eq. (1.48), we get solutions of Eq. ((4.1) as follows.

$$u(\zeta) = S = a_0 + \frac{d_1\left(\frac{d}{d\zeta}\mathbb{Q}(\zeta)\right)}{\mathbb{Q}(\zeta)}. \tag{4.36}$$

For $\beta_1 > 0, \Delta = 0$,

$$\psi_{15} = \left(a_0 - \frac{1}{2} d_1\sqrt{\beta_1}\tanh(F) + \frac{1}{2} d_1\sqrt{\beta_1}\right) e^{i\theta}, \tag{4.37}$$

$$\phi_{15} = \frac{r_4}{r_3}\left(a_0 - \frac{1}{2} d_1\sqrt{\beta_1}\tanh(F) + \frac{1}{2} d_1\sqrt{\beta_1}\right)^2, \tag{4.38}$$

$$\psi_{16} = \left(a_0 - \frac{1}{2}d_1\sqrt{\beta_1}\coth{(F)} + \frac{1}{2}d_1\sqrt{\beta_1}\right)e^{i\theta}, \tag{4.39}$$

$$\phi_{16} = \frac{r_4}{r_3}\left(a_0 - \frac{1}{2}d_1\sqrt{\beta_1}\coth{(F)} + \frac{1}{2}d_1\sqrt{\beta_1}\right)^2. \tag{4.40}$$

For $\beta_1 > 0$,

$$\psi_{17} = \left(a_0 - d_1\sqrt{\beta_1}\right)e^{i\theta} \tag{4.41}$$

$$\phi_{17} = \frac{r_4}{r_3}\left(a_0 - d_1\sqrt{\beta_1}\right)^2. \tag{4.42}$$

For $\beta_1 > 0, \beta_2 = 0$,

$$\psi_{18} = \left(a_0 + d_1\sqrt{\beta_1}\right)e^{i\theta}, \tag{4.43}$$

$$\phi_{18} = \frac{r_4}{r_3}\left(a_0 + d_1\sqrt{\beta_1}\right)^2, \tag{4.44}$$

Set 4 :

$a_0 = 0, a_1 = a_1, d_1 = 0, b_1 = 0, \beta_1 = \beta_1, \beta_2 = 0,$

$\beta_3 = -\dfrac{{a_1}^2 r_4 r_2}{2r_1 r_3}, \lambda_3 = r_1\left(-{\lambda_1}^2 + \beta_1\right).$

Substituting these coefficients along with the auxiliary solutions of Eq. (1.48), we get solutions of Eq. ((4.1) as follows.

$u(\zeta) = S = a_1\mathbb{Q}(\zeta).$

For $\beta_1 > 0, \Delta > 0$,

$$\psi_{19} = \left(\frac{a_1\beta_1}{\cosh(\sqrt{\beta_1}(\zeta))}\frac{1}{\sqrt{-\beta_1\beta_3}}\right)e^{i\theta}, \tag{4.45}$$

$$\phi_{19} = \frac{r_4}{r_3}\left(\frac{a_1\beta_1}{\cosh(\sqrt{\beta_1}(\zeta))}\frac{1}{\sqrt{-\beta_1\beta_3}}\right)^2, \tag{4.46}$$

$$\psi_{20} = \left(\frac{a_1\beta_1}{\sinh(\sqrt{\beta_1}(\zeta))}\frac{1}{\sqrt{\beta_1\beta_3}}\right)e^{i\theta}, \tag{4.47}$$

$$\phi_{20} = \frac{r_4}{r_3}\left(\frac{a_1\beta_1}{\sinh(\sqrt{\beta_1}(\zeta))}\frac{1}{\sqrt{\beta_1\beta_3}}\right)^2. \tag{4.48}$$

For $\beta_1 < 0, \beta_3 > 0$,

$$\psi_{21} = \left(\frac{a_1 \beta_1 \cosh(G)}{2\cosh(F)^2 \sinh(G)\sqrt{\beta_1\beta_3}}\right) e^{i\theta}, \tag{4.49}$$

$$\phi_{21} = \frac{r_4}{r_3}\left(\frac{a_1 \beta_1 \cosh(G)}{2\cosh(F)^2 \sinh(G)\sqrt{\beta_1\beta_3}}\right)^2, \tag{4.50}$$

$$\psi_{22} = \left(\frac{a_1 \beta_1 \sinh(G)}{2\sinh(F)^2 \cosh(G)\sqrt{\beta_1\beta_3}}\right) e^{i\theta}, \tag{4.51}$$

$$\phi_{22} = \frac{r_4}{r_3}\left(\frac{a_1 \beta_1 \sinh(G)}{2\sinh(F)^2 \cosh(G)\sqrt{\beta_1\beta_3}}\right)^2. \tag{4.52}$$

For $\beta_1 < 0, \Delta > 0$,

$$\psi_{23} = \left(\frac{a_1 \beta_1}{\cos(\sqrt{\beta_1}(\zeta))\sqrt{-\beta_1\beta_3}}\right) e^{i\theta}, \tag{4.53}$$

$$\phi_{23} = \frac{r_4}{r_3}\left(\frac{a_1 \beta_1}{\cos(\sqrt{\beta_1}(\zeta))\sqrt{-\beta_1\beta_3}}\right)^2, \tag{4.54}$$

$$\psi_{24} = \left(\frac{a_1 \beta_1}{\sin(\sqrt{-\beta_1}(\zeta))\sqrt{-\beta_1\beta_3}}\right) e^{i\theta}, \tag{4.55}$$

$$\phi_{24} = \frac{r_4}{r_3}\left(\frac{a_1 \beta_1}{\sin(\sqrt{-\beta_1}(\zeta))\sqrt{-\beta_1\beta_3}}\right)^2. \tag{4.56}$$

For, $\beta_1 < 0, \beta_3 > 0$,

$$\psi_{25} = \left(\frac{-a_1 \beta_1}{\sin\left(\sqrt{-\beta_1}(\zeta)\right)\sqrt{-\beta_1\beta_3}}\right) e^{i\theta}, \tag{4.57}$$

$$\phi_{25} = \frac{r_4}{r_3}\left(\frac{-a_1 \beta_1}{\sin\left(\sqrt{-\beta_1}(\zeta)\right)\sqrt{-\beta_1\beta_3}}\right)^2. \tag{4.58}$$

For, $\beta_1 > 0$,

$$\psi_{26} = \left(\frac{4a_1\beta_1 \mathrm{e}^{\sqrt{\beta_1}(\zeta)} r_3 r_1}{2\beta_1 {a_1}^2 r_4 r_2 + \mathrm{e}^{2\sqrt{\beta_1}(\zeta)} r_3 r_1}\right) e^{i\theta}, \tag{4.59}$$

$$\phi_{26} = \frac{r_4}{r_3}\left(\frac{4a_1\beta_1 \mathrm{e}^{\sqrt{\beta_1}(\zeta)} r_3 r_1}{2\beta_1 {a_1}^2 r_4 r_2 + \mathrm{e}^{2\sqrt{\beta_1}(\zeta)} r_3 r_1}\right)^2. \tag{4.60}$$

For, $\beta_1 > 0, \beta_2 = 0$,

$$\psi_{27} = \left(\frac{4a_1\beta_1 \mathrm{e}^{\sqrt{\beta_1}(\zeta)} r_3 r_1}{2\beta_1 {a_1}^2 r_4 r_2 \mathrm{e}^{2\sqrt{\beta_1}(\zeta)} + r_3 r_1}\right) e^{i\theta}, \tag{4.61}$$

$$\phi_{27} = \frac{r_4}{r_3}\left(\frac{4a_1\beta_1 \mathrm{e}^{\sqrt{\beta_1}(\zeta)} r_3 r_1}{2\beta_1 {a_1}^2 r_4 r_2 \mathrm{e}^{2\sqrt{\beta_1}(\zeta)} + r_3 r_1}\right)^2, \tag{4.62}$$

Set 5 :

$$a_0 = a_0, a_1 = a_1, d_1 = 0, b_1 = 0, \beta_1 = -2\frac{{a_0}^2 r_4 r_2}{r_1 r_3},$$

$$\beta_2 = -2\frac{a_0 a_1 r_4 r_2}{r_1 r_3}, \beta_3 = -\frac{2{a_1}^2 r_4 r_2}{r_1 r_3}, \lambda_3 = \frac{{a_0}^2 r_4 r_2 - {\lambda_1}^2 r_1 r_3}{r_3}.$$

Substituting these coefficients along with the auxiliary solutions of Eq. (1.48), we get solutions of Eq. ((4.1) as follows.

$$u(\zeta) = S = a_0 + a_1\mathbb{Q}(\zeta).$$

For $\beta_1 > 0$,

$$\psi_{28} = \frac{a_0\left(\sinh\left(\sqrt{\beta_1}(\zeta)\right) - 2\cosh(F)^2 + 3\right)}{-2\cosh(\mathrm{F})^2 + \sinh\left(\sqrt{\beta_1}(\zeta)\right) - 1} e^{i\theta}, \tag{4.63}$$

$$\phi_{28} = \frac{r_4}{r_3}\left(\frac{a_0\left(\sinh\left(\sqrt{\beta_1}(\zeta)\right) - 2\cosh(F)^2 + 3\right)}{-2\cosh(\mathrm{F})^2 + \sinh\left(\sqrt{\beta_1}(\zeta)\right) - 1}\right)^2, \tag{4.64}$$

$$\psi_{29} = \frac{a_0\left(\sinh\left(\sqrt{\beta_1}(\zeta)\right) - 2\cosh(F)^2 - 1\right)}{-2\cosh(\mathrm{F})^2 + \sinh\left(\sqrt{\beta_1}(\zeta)\right) + 3} e^{i\theta}, \tag{4.65}$$

$$\phi_{29} = \frac{r_4}{r_3}\left(\frac{a_0\left(\sinh\left(\sqrt{\beta_1}(\zeta)\right) - 2\cosh(F)^2 - 1\right)}{-2\cosh(\mathrm{F})^2 + \sinh\left(\sqrt{\beta_1}(\zeta)\right) + 3}\right)^2. \tag{4.66}$$

For $\beta_1 > 0, \Delta > 0$,

$$\psi_{30} = -a_0 e^{i\theta}\,, \tag{4.67}$$

$$\phi_{30} = \frac{r_4}{r_3}(-a_0)^2. \tag{4.68}$$

For $\beta_1 > 0, \beta_3 > 0$,

$$\psi_{31} = \frac{a_0\begin{pmatrix}\sqrt{\frac{\beta_2}{2}}\sinh(G) r_3 r_1 \cosh(F)^2 \\ -a_0 a_1 r_4 r_2 \cosh(\mathrm{G})(\cosh(\mathrm{F})^2 - 1)\end{pmatrix}}{\cosh(\mathrm{F})^2\begin{pmatrix}-a_0 a_1 r_4 r_2 \cosh(\mathrm{G}) \\ +\sqrt{\frac{\beta_2}{2}}\sinh(\mathrm{G}) r_3 r_1\end{pmatrix}} e^{i\theta}\,, \tag{4.69}$$

$$\phi_{31} = \frac{r_4}{r_3}\left(\frac{a_0\begin{pmatrix}\sqrt{\frac{\beta_2}{2}}\sinh(G) r_3 r_1 \cosh(F)^2 \\ -a_0 a_1 r_4 r_2 \cosh(\mathrm{G})(\cosh(\mathrm{F})^2 - 1)\end{pmatrix}}{\cosh(\mathrm{F})^2\begin{pmatrix}-a_0 a_1 r_4 r_2 \cosh(\mathrm{G}) \\ +\sqrt{\frac{\beta_2}{2}}\sinh(\mathrm{G}) r_3 r_1\end{pmatrix}}\right)^2, \tag{4.70}$$

$$\psi_{32} = \frac{a_0\begin{pmatrix}\sqrt{\frac{\beta_2}{2}}\cosh(G)(\cosh(F)^2 - 1) r_3 r_1 \\ -a_0 a_1 r_4 r_2 \sinh(\mathrm{G})\cosh(\mathrm{F})^2\end{pmatrix}}{\sinh(\mathrm{F})^2\begin{pmatrix}-a_0 a_1 r_4 r_2 \sinh(\mathrm{G}) \\ +\sqrt{\frac{\beta_2}{2}}\cosh(\mathrm{G}) r_3 r_1\end{pmatrix}} e^{i\theta}\,, \tag{4.71}$$

$$\phi_{32} = \frac{r_4}{r_3}\left(\frac{a_0\begin{pmatrix}\sqrt{\frac{\beta_2}{2}}\cosh(G)(\cosh(F)^2 - 1) r_3 r_1 \\ -a_0 a_1 r_4 r_2 \sinh(\mathrm{G})\cosh(\mathrm{F})^2\end{pmatrix}}{\sinh(\mathrm{F})^2\begin{pmatrix}-a_0 a_1 r_4 r_2 \sinh(\mathrm{G}) \\ +\sqrt{\frac{\beta_2}{2}}\cosh(\mathrm{G}) r_3 r_1\end{pmatrix}}\right)^2. \tag{4.72}$$

For $\beta_1 > 0, \Delta = 0$,

$$\psi_{33} = \left(-a_0 \tanh(F)\right)e^{i\theta}, \tag{4.73}$$

$$\phi_{33} = \frac{r_4}{r_3}\left(-a_0 \tanh(F)\right)^2, \tag{4.74}$$

$$\psi_{34} = \left(-a_0 \coth(F)\right)e^{i\theta}, \tag{4.75}$$

$$\phi_{34} = \frac{r_4}{r_3}\left(-a_0 \coth(F)\right)^2. \tag{4.76}$$

For $\beta_1 < 0, \Delta > 0$,

$$\psi_{35} = \frac{a_0\begin{pmatrix}\frac{\beta_2}{2}\sin\left(\sqrt{-\beta_1}(\zeta)\right)\frac{r_3 r_1}{2}\\ +a_0 a_1 r_4 r_2 \sin(F')^2\end{pmatrix}}{\begin{pmatrix}-a_0 a_1 r_4 r_2 \cos(\mathrm{F}')^2\\ \frac{\beta_2}{2} + \sin\left(\sqrt{-\beta_1}(\zeta)\right)\frac{r_3 r_1}{2}\end{pmatrix}} e^{i\theta}, \tag{4.77}$$

$$\phi_{35} = \frac{r_4}{r_3}\left(\frac{a_0\begin{pmatrix}\frac{\beta_2}{2}\sin\left(\sqrt{-\beta_1}(\zeta)\right)\frac{r_3 r_1}{2}\\ +a_0 a_1 r_4 r_2 \sin(F')^2\end{pmatrix}}{\begin{pmatrix}-a_0 a_1 r_4 r_2 \cos(\mathrm{F}')^2\\ \frac{\beta_2}{2} + \sin\left(\sqrt{-\beta_1}(\zeta)\right)\frac{r_3 r_1}{2}\end{pmatrix}}\right)^2, \tag{4.78}$$

$$\psi_{36} = \frac{a_0\begin{pmatrix}\frac{\beta_2}{2}\sin\left(\sqrt{-\beta_1}(\zeta)\right)\frac{r_3 r_1}{2}\\ +a_0 a_1 r_4 r_2 \cos(F')^2\end{pmatrix}}{\begin{pmatrix}-a_0 a_1 r_4 r_2 \sin(\mathrm{F}')^2\\ +\frac{\beta_2}{2}\sin\left(\sqrt{-\beta_1}(\zeta)\right)\frac{r_3 r_1}{2}\end{pmatrix}} e^{i\theta} \tag{4.79}$$

$$\phi_{36} = \frac{r_4}{r_3}\left(\frac{a_0\begin{pmatrix}\frac{\beta_2}{2}\sin\left(\sqrt{-\beta_1}(\zeta)\right)\frac{r_3 r_1}{2}\\ +a_0 a_1 r_4 r_2 \cos(F')^2\end{pmatrix}}{\begin{pmatrix}-a_0 a_1 r_4 r_2 \sin(\mathrm{F}')^2\\ +\frac{\beta_2}{2}\sin\left(\sqrt{-\beta_1}(\zeta)\right)\frac{r_3 r_1}{2}\end{pmatrix}}\right)^2. \tag{4.80}$$

For $\beta_1 > 0$,

$$\psi_{37} = \frac{a_0\left(-4a_0 a_1 r_4 r_2 + \mathrm{e}^{\sqrt{\beta_1}(\zeta)} r_1 r_3\right)}{4a_0 a_1 r_4 r_2 + \mathrm{e}^{\sqrt{\beta_1}(\zeta)} r_1 r_3} e^{i\theta}, \tag{4.81}$$

$$\phi_{37} = \frac{r_4}{r_3}\left(\frac{a_0\left(-4a_0 a_1 r_4 r_2 + \mathrm{e}^{\sqrt{\beta_1}(\zeta)} r_1 r_3\right)}{4a_0 a_1 r_4 r_2 + \mathrm{e}^{\sqrt{\beta_1}(\zeta)} r_1 r_3}\right)^2. \tag{4.82}$$

For $\beta_1 > 0, \beta_2 = 0$,

$$\psi_{38}=\frac{a_0\left(4(a_0a_1r_4r_2)^2\mathrm{e}^{\sqrt{\beta_1}(\zeta)}+8\mathrm{e}^{\sqrt{\beta_1}(\zeta)}a_0a_1r_4r_2r_1r_3-(r_1r_3)^2\right)}{4(a_0a_1r_4r_2)^2\mathrm{e}^{\sqrt{\beta_1}(\zeta)}-(r_1r_3)^2}e^{i\theta}, \tag{4.83}$$

$$\phi_{38}=\frac{r_4}{r_3}\left(\frac{a_0\left(4(a_0a_1r_4r_2)^2\mathrm{e}^{\sqrt{\beta_1}(\zeta)}+8\mathrm{e}^{\sqrt{\beta_1}(\zeta)}a_0a_1r_4r_2r_1r_3-(r_1r_3)^2\right)}{4(a_0a_1r_4r_2)^2\mathrm{e}^{\sqrt{\beta_1}(\zeta)}-(r_1r_3)^2}\right)^2, \tag{4.84}$$

Set 6 :

$a_0=0, a_1=a_1, d_1=\dfrac{\sqrt{2}\sqrt{-\frac{r_1r_3}{r_2r_4}}}{2}, b_1=0, \beta_1=\beta_1, \beta_2=\beta_2,$
$\beta_3=-2\frac{{a_1}^2r_4r_2}{r_1r_3}, \lambda_3=-1/2r_1(2{\lambda_1}^2+\beta_1).$

Substituting these coefficients along with the auxiliary solutions Eq. (1.48), we get solutions of Eq. ((4.1) as follows.

$$u(\zeta)=S=a_1\mathbb{Q}(\zeta)+\frac{d_1\left(\frac{d}{d\zeta}\mathbb{Q}(\zeta)\right)}{\mathbb{Q}(\zeta)}.$$

For $\beta_1>0$,

$$\psi_{39}=\left(\frac{-\beta_1^{\frac{3}{2}}(\tanh(F))^2\beta_3d_1+\tanh(F)\,d_1\left({\beta_2}^2\sqrt{\beta_1}-2\beta_3\beta_1^{\frac{3}{2}}\right)+a_1\beta_1\beta_2\operatorname{sech}(F)^2-{\beta_1}^{3/2}\beta_3d_1}{\tanh(F)^2\,\beta_1\beta_3+2\tanh(F)\beta_1\beta_3+\beta_1\beta_3-{\beta_2}^2}\right)e^{i\theta}, \tag{4.85}$$

$$\phi_{39}=\frac{r_4}{r_3}\left(\frac{-\beta_1^{\frac{3}{2}}(\tanh(F))^2\beta_3d_1+\tanh(F)\,d_1\left({\beta_2}^2\sqrt{\beta_1}-2\beta_3\beta_1^{\frac{3}{2}}\right)+a_1\beta_1\beta_2\operatorname{sech}(F)^2-{\beta_1}^{3/2}\beta_3d_1}{\tanh(F)^2\,\beta_1\beta_3+2\tanh(F)\beta_1\beta_3+\beta_1\beta_3-{\beta_2}^2}\right)^2, \tag{4.86}$$

$$\psi_{40}=\left(\frac{\begin{array}{c}\sinh(F)\cosh(F)\,d_1\left(\beta_2{}^2\sqrt{\beta_1}-2\beta_3\beta_1{}^{\frac{3}{2}}\right)\\-a_1\beta_1\beta_2-\beta_1{}^{\frac{3}{2}}\beta_3d_1-2\beta_1{}^{\frac{3}{2}}(\cosh(F))^2\beta_3d_1\end{array}}{\begin{array}{c}\left(2\beta_1\beta_3-\beta_2{}^2\right)\cosh(F)^2\,\beta_1\beta_3+\\\sinh\left(\sqrt{\beta_1}\xi\right)\beta_1\beta_3-\beta_1\beta_3+\beta_2{}^2\end{array}}\right)e^{i\theta}, \tag{4.87}$$

$$\phi_{40}=\frac{r_4}{r_3}\left(\frac{\begin{array}{c}\sinh(F)\cosh(F)\,d_1\left(\beta_2{}^2\sqrt{\beta_1}-2\beta_3\beta_1{}^{\frac{3}{2}}\right)\\-a_1\beta_1\beta_2-\beta_1{}^{\frac{3}{2}}\beta_3d_1-2\beta_1{}^{\frac{3}{2}}(\cosh(F))^2\beta_3d_1\end{array}}{\begin{array}{c}\left(2\beta_1\beta_3-\beta_2{}^2\right)\cosh(F)^2\,\beta_1\beta_3+\\\sinh\left(\sqrt{\beta_1}\xi\right)\beta_1\beta_3-\beta_1\beta_3+\beta_2{}^2\end{array}}\right)^2. \tag{4.88}$$

For $\beta_1>0, \Delta>0,$

$$\psi_{41}=\left(\frac{-d_1\beta_1{}^{\frac{3}{2}}\tanh\left(\sqrt{\beta_1}\xi\right)\Delta+2a_1\beta_1{}^2\operatorname{sech}(\sqrt{\beta_1}\xi)}{\left(\Delta-\beta_2\operatorname{sech}(\sqrt{\beta_1}\xi)\right)\beta_1}\right)e^{i\theta}, \tag{4.89}$$

$$\phi_{41}=\frac{r_4}{r_3}\left(\frac{-d_1\beta_1{}^{\frac{3}{2}}\tanh\left(\sqrt{\beta_1}\xi\right)\Delta+2a_1\beta_1{}^2\operatorname{sech}(\sqrt{\beta_1}\xi)}{\left(\Delta-\beta_2\operatorname{sech}(\sqrt{\beta_1}\xi)\right)\beta_1}\right)^2. \tag{4.90}$$

For $\beta_1>0, \beta_3>0,$

$$\psi_{42}=\left(\frac{\begin{array}{c}-a_1\beta_1{}^2\operatorname{sech}(F)^2+d_1\left(\left(\sqrt{\beta_1\beta_3}\tanh(G)^2-1\right)\sqrt{2}\right.\\-2\tanh(F)\left(\sqrt{\beta_1\beta_3}\tanh(G)+1/2\beta_2\right)\left.\right)\beta_1{}^{\frac{3}{2}}\end{array}}{(\beta_2+2\sqrt{\beta_1\beta_3}\tanh(G)\,\beta_1}\right)e^{i\theta}, \tag{4.91}$$

$$\phi_{42}=\frac{r_4}{r_3}\left(\frac{\begin{array}{c}-a_1\beta_1{}^2\operatorname{sech}(F)^2+d_1\left(\left(\sqrt{\beta_1\beta_3}\tanh(G)^2-1\right)\sqrt{2}\right.\\-2\tanh(F)\left(\sqrt{\beta_1\beta_3}\tanh(G)+1/2\beta_2\right)\left.\right)\beta_1{}^{\frac{3}{2}}\end{array}}{(\beta_2+2\sqrt{\beta_1\beta_3}\tanh(G)\,\beta_1}\right)^2, \tag{4.92}$$

$$\psi_{43}=\left(\frac{\begin{array}{c}a_1{\beta_1}^2\operatorname{csch}(F)^2+d_1\left(\left(\sqrt{\beta_1\beta_3}\coth(G)^2-1\right)\sqrt{2}\right.\\ \left.-2\tanh(F)\left(\sqrt{\beta_1\beta_3}\coth(G)+1/2\beta_2\right)\right){\beta_1}^{\frac{3}{2}}\end{array}}{(\beta_2+2\sqrt{\beta_1\beta_3}\coth(G)\,\beta_1}\right)e^{i\theta}, \tag{4.93}$$

$$\phi_{43}=\frac{r_4}{r_3}\left(\frac{\begin{array}{c}a_1{\beta_1}^2\operatorname{csch}(F)^2+d_1\left(\left(\sqrt{\beta_1\beta_3}\coth(G)^2-1\right)\sqrt{2}\right.\\ \left.-2\tanh(F)\left(\sqrt{\beta_1\beta_3}\coth(G)+1/2\beta_2\right)\right){\beta_1}^{\frac{3}{2}}\end{array}}{(\beta_2+2\sqrt{\beta_1\beta_3}\coth(G)\,\beta_1}\right)^2. \tag{4.94}$$

For $\beta_1>0, \Delta=0$,

$$\psi_{44}=\left(\frac{\left(-d_1\sqrt{\beta_1}\beta_2-2a_1\beta_1\right)\tanh(F)+d_1\sqrt{\beta_1}\beta_2-2a_1\beta_1}{2\beta_2}\right)e^{i\theta}, \tag{4.95}$$

$$\phi_{44}=\frac{r_4}{r_3}\left(\frac{\left(-d_1\sqrt{\beta_1}\beta_2-2a_1\beta_1\right)\tanh(F)+d_1\sqrt{\beta_1}\beta_2-2a_1\beta_1}{2\beta_2}\right)^2, \tag{4.96}$$

$$\psi_{45}=\left(\frac{\left(-d_1\sqrt{\beta_1}\beta_2-2a_1\beta_1\right)\coth(F)+d_1\sqrt{\beta_1}\beta_2-2a_1\beta_1}{2\beta_2}\right)e^{i\theta}, \tag{4.97}$$

$$\phi_{45}=\frac{r_4}{r_3}\left(\frac{\left(-d_1\sqrt{\beta_1}\beta_2-2a_1\beta_1\right)\coth(F)+d_1\sqrt{\beta_1}\beta_2-2a_1\beta_1}{2\beta_2}\right)^2. \tag{4.98}$$

For $\beta_1<0, \Delta>0$,

$$\psi_{46}=\left(\frac{d_1\sqrt{-\beta_1}\tan\left(\sqrt{-\beta_1}\zeta\right)\Delta+2a_1\beta_1\sec\left(\sqrt{-\beta_1}\zeta\right)}{\sqrt{-4\beta_1\beta_3+{\beta_2}^2}-\beta_2\sec\left(\sqrt{-\beta_1}\zeta\right)}\right)e^{i\theta}, \tag{4.99}$$

$$\phi_{46}=\frac{r_4}{r_3}\left(\frac{d_1\sqrt{-\beta_1}\tan\left(\sqrt{-\beta_1}\zeta\right)\Delta+2a_1\beta_1\sec\left(\sqrt{-\beta_1}\zeta\right)}{\Delta-\beta_2\sec\left(\sqrt{-\beta_1}\zeta\right)}\right)^2, \tag{4.100}$$

$$\psi_{47}=\left(\frac{-d_1\sqrt{-\beta_1}\cot\left(\sqrt{-\beta_1}\zeta\right)\Delta+2a_1\beta_1\csc\left(\sqrt{-\beta_1}\zeta\right)}{\Delta-\beta_2\csc\left(\sqrt{-\beta_1}\zeta\right)}\right)e^{i\theta}, \tag{4.101}$$

$$\phi_{47} = \frac{r_4}{r_3}\left(\frac{-d_1\sqrt{-\beta_1}\cot\left(\sqrt{-\beta_1}\zeta\right)\Delta + 2a_1\beta_1\csc\left(\sqrt{-\beta_1}\zeta\right)}{\Delta - \beta_2\csc\left(\sqrt{-\beta_1}\zeta\right)}\right)^2. \tag{4.102}$$

For $\beta_1 < 0, \beta_3 > 0$,

$$\psi_{48} = \left(\frac{\left(\sqrt{-\beta_1\beta_3}\tan(F')^2 + \tan(F')\beta_2 - \sqrt{-\beta_1\beta_3}\right)d_1\sqrt{-\beta_1} - a_1\beta_1\sec(F')^2}{\beta_2 + 2\sqrt{-\beta_1\beta_3}\tan(F')}\right)e^{i\theta}, \tag{4.103}$$

$$\phi_{48} = \frac{r_4}{r_3}\left(\frac{\left(\sqrt{-\beta_1\beta_3}\tan(F')^2 + \tan(F')\beta_2 - \sqrt{-\beta_1\beta_3}\right)d_1\sqrt{-\beta_1} - a_1\beta_1\sec(F')^2}{\beta_2 + 2\sqrt{-\beta_1\beta_3}\tan(F')}\right)^2, \tag{4.104}$$

$$\psi_{49} = \left(\frac{-\left(\sqrt{-\beta_1\beta_3}\cot(F')^2 + \cot(F')\beta_2 - \sqrt{-\beta_1\beta_3}\right)d_1\sqrt{-\beta_1} - a_1\beta_1\csc(F')^2}{\beta_2 + 2\sqrt{-\beta_1\beta_3}\cot(F')}\right)e^{i\theta}, \tag{4.105}$$

$$\phi_{49} = \frac{r_4}{r_3}\left(\frac{-\left(\sqrt{-\beta_1\beta_3}\cot(F')^2 + \cot(F')\beta_2 - \sqrt{-\beta_1\beta_3}\right)d_1\sqrt{-\beta_1} - a_1\beta_1\csc(F')^2}{\beta_2 + 2\sqrt{-\beta_1\beta_3}\cot(F')}\right)^2. \tag{4.106}$$

For $\beta_1 > 0$,

$$\psi_{50} = \left(\frac{-4{\beta_1}^{3/2}\beta_3 d_1 - \sqrt{\beta_1}(\mathrm{e}^{\sqrt{\beta_1}\xi})^2 d_1 + \sqrt{\beta_1}{\beta_2}^2 d_1 + 4a_1\beta_1\mathrm{e}^{\sqrt{\beta_1}\xi}}{(\mathrm{e}^{\sqrt{\beta_1}\xi})^2 - 2\mathrm{e}^{\sqrt{\beta_1}\xi}\beta_2 - 4\beta_1\beta_3 + {\beta_2}^2}\right)e^{i\theta}, \tag{4.107}$$

$$\phi_{50} = \frac{r_4}{r_3}\left(\frac{-4{\beta_1}^{3/2}\beta_3 d_1 - \sqrt{\beta_1}(\mathrm{e}^{\sqrt{\beta_1}\xi})^2 d_1 + \sqrt{\beta_1}{\beta_2}^2 d_1 + 4a_1\beta_1\mathrm{e}^{\sqrt{\beta_1}\xi}}{(\mathrm{e}^{\sqrt{\beta_1}\xi})^2 - 2\mathrm{e}^{\sqrt{\beta_1}\xi}\beta_2 - 4\beta_1\beta_3 + {\beta_2}^2}\right)^2. \tag{4.108}$$

For $\beta_1 > 0, \beta_3 = 0$,

$$\psi_{51} = \left(\frac{-4{\beta_1}^{3/2}\mathrm{e}^{2\sqrt{\beta_1}\xi}\beta_3 d_1 - 4a_1\beta_1\mathrm{e}^{\sqrt{\beta_1}\xi} - \sqrt{\beta_1}d_1}{4\beta_1\beta_3\mathrm{e}^{2\sqrt{\beta_1}\xi} - 1}\right)e^{i\theta}, \tag{4.109}$$

$$\phi_{51} = \frac{r_4}{r_3}\left(\frac{-4{\beta_1}^{3/2}\mathrm{e}^{2\sqrt{\beta_1}\xi}\beta_3 d_1 - 4a_1\beta_1\mathrm{e}^{\sqrt{\beta_1}\xi} - \sqrt{\beta_1}d_1}{4\beta_1\beta_3\mathrm{e}^{2\sqrt{\beta_1}\xi} - 1}\right)^2, \tag{4.110}$$

4.4 Results and discussion:

This is the important section of a study as it helps us to understand physical importance and dynamical features of solitons for this model by demonstrating real and imaginary parts of many useful solutions in the form of 3-D, 2-D and contour plots. The novel generalized auxiliary equation mapping method successfully generates bright, dark, periodic, and singular soliton solutions. Bright solitons exhibit high intensity whereas dark solitons have lower intensity than its background. Kink solitons have permanent profile in medium, while periodic wave have dynamical profile and can depend on time. Singular solitons are waves with discontinuous derivatives. Each type of solution has its significance in real life. It is significant to mention that the obtained soliton solutions are more generalized and newer and might be a good addition in literature.

In Figure 4.1, graphical profile of Real value of Eq (4.45) expressed as ψ_{19} has been exhibit, in the form of 3-dimensional , and 2-dimensional and contour plot which demonstrates singular bright soliton by choosing parameters, $-10 \le x \le 10, t = 0..10,$ $r_1 = 0.5, r_2 = 1.5, r_3 = 1.5, r_4 = 1, \beta_1 = 3, \beta_2 = 0, a_1 = 4, \lambda_1 = 0.6, k_1 = 1.5, k_2 = 0.9, k_3 = 0.6, y = 2$.

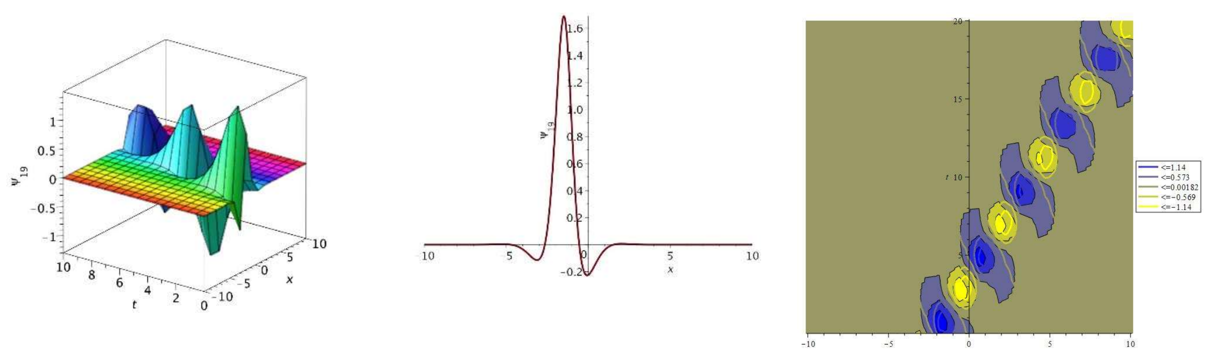


Figure 4.1:-graphs of singular bright soliton ψ_{19}

In Figure 4.2:graphs of singualr bright soliton $\boldsymbol{\phi_{19}}$., graphical depiction of Real value of Eq (4.46) expressed as ϕ_{19} has been exhibit in the form of 3-dimensional , and 2-dimensional and contour plot which demonstrates singular bright soliton for $-10 \le x \le 10, t = 0..10,$ $r_1 = 0.5, r_2 = 1.5, r_3 = 1.5, r_4 = 1, \beta_1 = 3, \beta_2 = 0, a_1 = 4, \lambda_1 = 0.6, k_1 = 1.5, k_2 = 0.9, k_3 = 0.6, y = 2$.

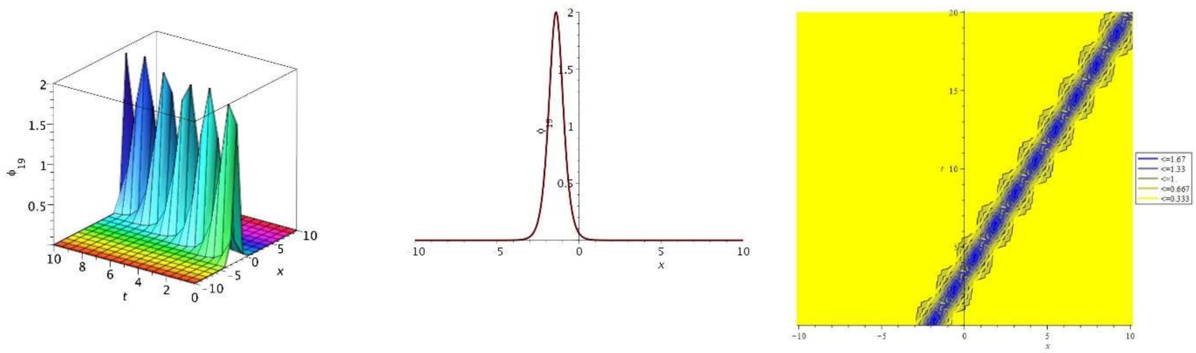


Figure 4.2:graphs of singualr bright soliton $\boldsymbol{\phi_{19}}$.

Graphical depiction of Real value of Eq (4.64) expressed as ψ_{28} has been exhibit in Figure 4.3, in the form of 3-dimensional, and 2-dimensional and contour plot which demonstrates as periodic wave solution by choosing parameters, $-10 \leq x \leq 10, t = 0..10,\ r_1 = 1, r_2 = -1.5, r_3 = 1, r_4 = 1, \beta_1 = 1, \beta_2 = 3, a_0 = 1.9, \lambda_1 = 0.6, k_1 = 1, k_2 = 0.9, k_3 = 0.2, y = 1$.

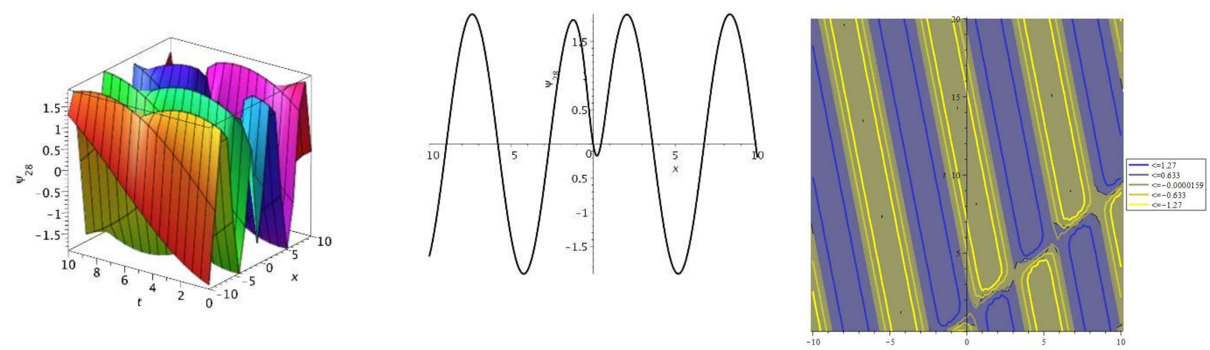


Figure 4.3: graphs of periodic solitary wave solution $\boldsymbol{\psi_{28}}$.

Graphical depiction of Real value of Eq (4.64) expressed as ϕ_{28} has been exhibit in Figure 4.4, in the form of 3-dimensional, 2-dimensional and their contour plot which demonstrates singular dark soliton solution for $-10 \leq x \leq 10, t = 0..10,\ r_1 = 1, r_2 = -1.5, r_3 = 1, r_4 = 1, \beta_1 = 1, \beta_2 = 3, a_0 = 1.9, \lambda_1 = 0.6, k_1 = 1, k_2 = 0.9, k_3 = 0.2, y = 1$.

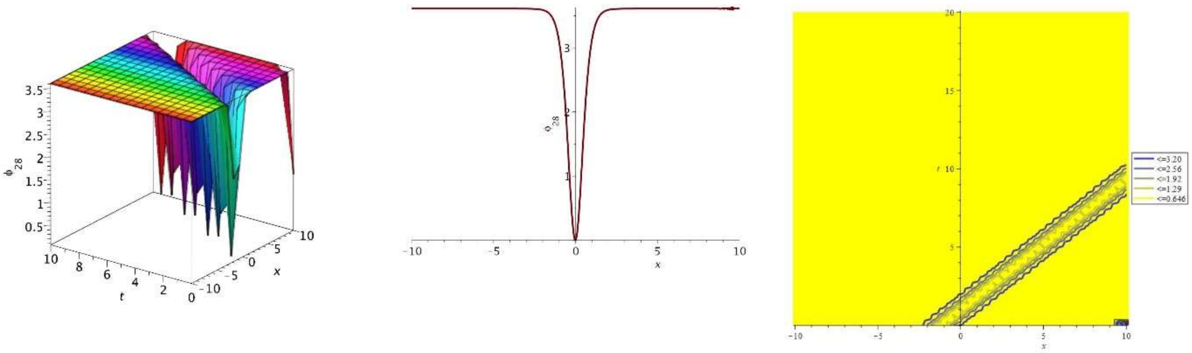


Figure 4.4: -graphs of singular dark soliton ϕ_{28}.

Graphical illustration of Real value of Eq (4.102) expressed as ψ_{47} has been exhibit in Figure 4.5, in the form of 3-dimensional, and 2-dimensional and contour plot which demonstrates localized excitation wave pattern as singular periodic soliton by selecting parameters, $-10 \leq x \leq 10, t = 0..10$, $r_1 = -1, r_2 = 1.5, r_3 = 1.5, r_4 = 1, \beta_1 = -3, \beta_2 = 4, a_1 = 4, \lambda_1 = 0.6, k_1 = 1, k_2 = 0.9, k_3 = 0.2, y = 2$.

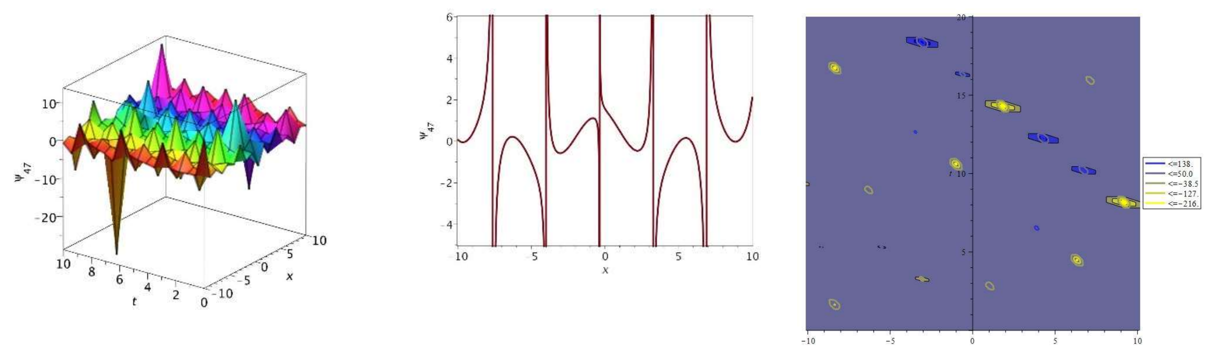


Figure 4.5: -graphical simulation of singular periodic solitary wave solution ψ_{47}.

Graphical illustration of Real value of Eq (4.102) expressed as ϕ_{47} has been exhibit in Figure 4.6, in the form of 3-dimensional , and 2-dimensional and contour plot which demonstrates localized excitation wave pattern as periodic soliton for $-10 \leq x \leq 10, t = 0..10$, $r_1 = -1, r_2 = 1.5, r_3 = 1.5, r_4 = 1, \beta_1 = -3, \beta_2 = 4, a_1 = 4, \lambda_1 = 0.6, k_1 = 1, k_2 = 0.9, k_3 = 0.2, y = 2$.

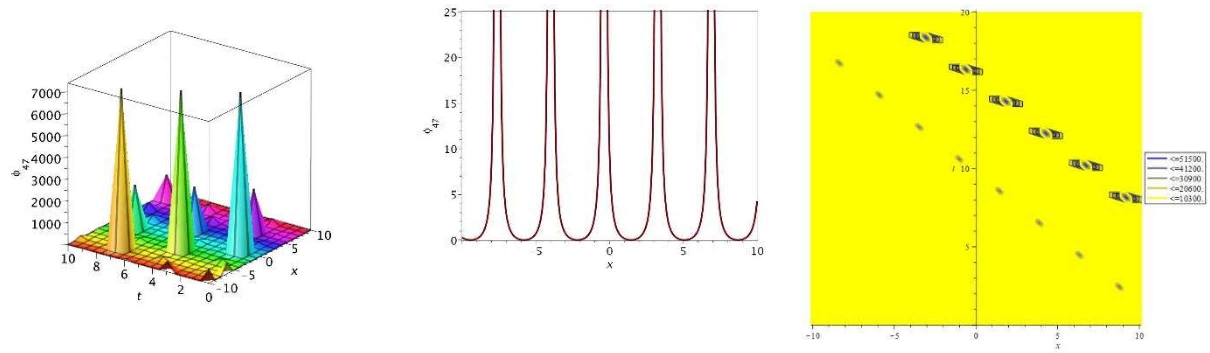


Figure 4.6: -graphical simulation of periodic solitary wave solution ϕ_{47}.

4.5 $(2+1)$ Darvey-Stewartson (DS) system:

Here, we will investigate the $(2+1)$ Davey-Stewartson (DS) system for complex valued function ψ and real valued function ϕ of x, y and t:

$$\frac{\partial}{\partial t\,\partial x}\tau + \frac{1}{2}\sigma^2\left(\frac{\partial^2}{\partial x^2}\tau + \sigma^2\frac{\partial^2}{\partial y^2}\tau\right) + \lambda|\tau|^2\tau - \frac{\partial}{\partial x}\varphi\tau = 0\,,$$
$$\frac{\partial^2}{\partial x^2}\varphi - \sigma^2\frac{\partial^2}{\partial y^2}\varphi - 2\lambda\frac{\partial}{\partial x}|\varphi|^2 = 0. \tag{4.111}$$

Where the parameters, $\lambda, \sigma = \pm 1$ establish four possible types of the system. Especially, if $\sigma = 1$ and $\sigma = -1$, describes well known Davey-Stewartson I (DSI) and Davey-Stewartson II (DSII) equations respectively. Similarly, the focusing and de-focusing cases are characterized by $\lambda = 1, \lambda = -1$. Here, $\tau(x, y, t)$ exhibit the amplitude of a surface wave packet whereas, $\varphi(x, y, t)$ exhibits velocity potential of the mean flow depending on wave surface [162]

The Davey-Stewartson (DSS) equation is a very important model that describes the short wave and long wave resonance in water exhibiting limited depth. This is an important model in two-dimensional space that explains higher order generalization of nonlinear Schrodinger equation. To acquire a better understanding of its applications in real world problems, analytical solutions are required. Many researchers have solved this model analytically and numerically to generate a variety of solutions. Such as, HA Zedan [163] established periodic and solitary wave solutions of DS model by using compound Riccati equation rational expansion method. RF Zinati [164] investigated DS equation by various techniques. Gaballah.et.al.[165] studied this model by generalized Jacobi elliptic expansion method to obtain periodic and optical solitons. Frauendiener.et.al. [166] studied this model via hybrid numerical technique. Saima.et.al [167] finds soliton solutions using three integrating techniques. After careful literature review, we realized still a lot of work can be done on this model. Motivated by above mentioned work we are using modified auxiliary equation method on $(2+1)$-dimensional Davey-Stewartson (DS) equation. It is evident from studies that higher-dimensional nonlinear models exhibit rich phenomena as compared to one-dimensional models.

Let us use the following complex transformations to solve Eq (4.111)

$$\tau(x,y,t) = u(\zeta)e^{i\theta}, \qquad \varphi(x,y,t) = V(\zeta).$$

Where,

$$\zeta = k(x + \ell y - \eta t), \qquad \theta = k_1 x + k_2 y + k_3 t,$$

using the above-mentioned wave transformation in Eq (4.111) , converts the system into the following nonlinear system of ODE,

$$\sigma^2 k^2 (l^2\sigma^2 + 1)\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}u(\zeta) + 2ik(k_2 l\sigma^4 + k_1\sigma^2 - \eta)\frac{\mathrm{d}}{\mathrm{d}\xi}u(\zeta) - 2u(\zeta)\left(\frac{\sigma^4 {k_2}^2}{2} + \frac{\sigma^2 {k_1}^2}{2} + \left(\frac{\mathrm{d}}{\mathrm{d}\xi}V(\zeta)\right)k + k_3\right) + 2\lambda\big(u(\zeta)\big)^3 = 0. \tag{4.112}$$

Separating Eq. (4.112) into real and imaginary parts we have,

Real part:

$$\sigma^2 k^2 (l^2\sigma^2 + 1)\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}u(\zeta) + 2\lambda\big(u(\zeta)\big)^3 - 2\left(\frac{\mathrm{d}}{\mathrm{d}\xi}V(\zeta)\right)ku(\zeta) - \left(\sigma^4 {k_2}^2 + \sigma^2 {k_1}^2 + 2k_3\right)u(\zeta) = 0, \tag{4.113}$$

Imaginary part:

$$\eta = k_2 l\sigma^4 + k_1\sigma^2. \tag{4.114}$$

Also, we have, from second equation of (4.111),

$$k(l^2\sigma^2 - 1)\left(\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}V(\zeta)\right) + 4\lambda u(\zeta)\frac{du}{d\zeta} = 0, \tag{4.115}$$

integrating Eq (4.115), we get

$$\frac{\mathrm{d}}{\mathrm{d}\xi}V(\zeta) = -\frac{2\lambda u(\zeta)^2}{k(l^2\sigma^2-1)}, \tag{4.116}$$

$$\Rightarrow V(\zeta) = -\frac{2\lambda\int u(\zeta)^2 d\,\zeta}{k(l^2\sigma^2-1)}. \tag{4.117}$$

Substituting Eq (4.116) along with the value of η into Eq (4.112) we get

$$\begin{aligned}&\sigma^2k^2(l^2\sigma^2+1)\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}u(\xi)+2\lambda\left(u(\xi)\right)^3\\&-2u(\xi)\left(\frac{\sigma^4{k_2}^2}{2}+\frac{\sigma^2{k_1}^2}{2}+\left(-\frac{2\lambda u(\zeta)^2}{(l^2\sigma^2-1)}\right)+k_3\right)=0,\end{aligned} \tag{4.118}$$

balancing the highest order of linear term with the nonlinear term in Eq. (4.118) we usually determine the value of N. Here $3N = N + 2 \Rightarrow N = 1$. This gives solution of the form.

$$u(\zeta) = S = a_0 + a_1\mathbb{Q}(\zeta) + \frac{b_1}{\mathbb{Q}(\zeta)} + \frac{d_1\left(\frac{d}{d\zeta}\mathbb{Q}(\zeta)\right)}{\mathbb{Q}(\zeta)^2}, \tag{4.119}$$

replacing Eq. (4.119) into Eq. (4.118) along with Eq. (1.48), we get algebraic system and by equating this system to 0 we get values of coefficients $a_0, a_1, a_2, b_1, d_1, \beta_1, \beta_2, \beta_3, k$ as follows.

Set 1 :

$$a_0 = 0, a_1 = a_1, b_1 = 0, k = k, \beta_2 = \beta_2, \beta_1 = \frac{-2\sigma^4{k_2}^2 - 2\sigma^2{k_1}^2 - 4k_3}{k^2\sigma^2(l^2\sigma^2+1)},$$

$$\beta_3 = \frac{-4\lambda{a_1}^2}{k^2\sigma^2(l^2\sigma^2-1)}, d_1 = \frac{\sqrt{\frac{-l^2\sigma^2+1}{\lambda}}\sigma k}{2}.$$

For these set of coefficients, we have following solutions,

$$u(\zeta) = S = a_1\mathbb{Q}(\zeta) + \frac{d_1\left(\frac{d}{d\zeta}\mathbb{Q}(\zeta)\right)}{\mathbb{Q}(\zeta)},$$

where,

$$\tau_j(x,y,t) = u_j(\zeta)e^{i\theta}, \qquad \varphi_j(x,y,t) = V_j(\zeta) = -\frac{2\lambda\int u(\zeta)^2 d\,\zeta}{k(l^2\sigma^2-1)},$$

substituting these coefficients along with the auxiliary solutions of Eq. (1.48), we get solutions of Eq. (4.111) as follows.

F or $\beta_1 > 0, \Delta > 0$,

$$\tau_1 = \left(\frac{-\beta_1^{\frac{1}{2}} d_1 (\Delta) \sinh(\sqrt{\beta_1}\xi) + 2a_1 {\beta_1}^2}{\left(\Delta \cosh(\sqrt{\beta_1}\xi) - \beta_2\right)} \right) e^{i\theta}, \tag{4.120}$$

$$\begin{aligned}\phi_1 &= \frac{\lambda}{k(l^2\sigma^2 - 1)\beta_1^{\frac{3}{2}} {\beta_3}^2 ET} \\ &\times \left(E\beta_3\sqrt{2}{\beta_1}^2\beta_2(\Delta - \beta_2)\left(\beta_3 {d_1}^2 - {a_1}^2\right) \arctan\left(\frac{\beta_1\beta_3 \tanh(F)\sqrt{2}}{T}\right) \right. \\ &\left. -2T \begin{pmatrix} -2\beta_3 a_1 d_1 (\Delta - 2\beta_2)\beta_1^{\frac{3}{2}} + \\ E{\beta_3}^2 \ln(1 + \tanh(F))\, {d_1}^2 {\beta_1}^2 \\ -E{\beta_3}^2 \ln(\tanh(F) - 1)\, {d_1}^2 {\beta_1}^2 \\ -\frac{\beta_2\Delta}{2}\begin{pmatrix} \left(\beta_3 {d_1}^2 + {a_1}^2\right)\tanh(F)\,\beta_1 \\ -2\sqrt{\beta_1}a_1 d_1 \beta_2 \end{pmatrix} \\ -\sqrt{\beta_1}a_1 d_1 {\beta_2}^3 - 2\left(\beta_1\beta_3 - \frac{1}{4{\beta_2}^2}\right) \\ \times \beta_1\left(\beta_3 {d_1}^2 + {a_1}^2\right)\tanh(F) \end{pmatrix} \right), \end{aligned} \tag{4.121}$$

$$\tau_2 = \left(\frac{-\sqrt{\beta_1} d_1(\Delta)\cosh\left(\sqrt{\beta_1}\xi\right) + 2a_1\beta_1}{\left(\Delta \sinh\left(\sqrt{\beta_1}\xi\right) - \beta_2\right)}\right) e^{i\theta}, \tag{4.122}$$

$$\begin{aligned}
\varphi_2 = {} & \frac{-2\lambda}{k(l^2\sigma^2 - 1)} \times \\
& \times \sqrt{\beta_1} {d_1}^2 \ln(1 + \tanh(F)) - \sqrt{\beta_1} {d_1}^2 \ln(\tanh(F) - 1) \\
& + 8 \frac{\beta_1 \tanh(F)\, \Delta a_1 d_1}{E{\beta_2}^2} + 8 \frac{{\beta_1}^{\frac{3}{2}} \beta_3 \tanh(F)\, {d_1}^2}{E{\beta_2}^2} \\
& - 2 \frac{\sqrt{\beta_1} \tanh(F)\, {d_1}^2}{E} + 8 \frac{{\beta_1}^{\frac{3}{2}} \tanh(F)\, {a_1}^2}{E{\beta_2}^2} \\
& - 2 \frac{\sqrt{\beta_1} \tanh(F)\, {a_1}^2}{E\beta_3} - 2 \frac{\sqrt{\beta_1} \Delta {d_1}^2}{E\beta_2} - 2 \frac{\sqrt{\beta_1} \Delta {a_1}^2}{E\beta_3\beta_2} \\
& - \frac{\sqrt{\beta_1} \beta_2 {d_1}^2}{\sqrt{\beta_1\beta_3}} \operatorname{arctanh}\left(\frac{\tanh(F)\, \beta_2 + \Delta}{2\sqrt{\beta_1\beta_3}}\right) \\
& + \frac{\sqrt{\beta_1} \beta_2 {a_1}^2}{\beta_3\sqrt{\beta_1\beta_3}} \operatorname{arctanh}\left(\frac{\tanh(F)\, \beta_2 + \Delta}{2\sqrt{\beta_1\beta_3}}\right).
\end{aligned} \tag{4.123}$$

For $\beta_1 > 0, \Delta = 0$,

$$\tau_3 = \frac{-\sqrt{\beta_1} d_1 \tanh(F)\, \beta_2 + d_1\sqrt{\beta_1}\beta_2 - 2a_1\beta_1 \tanh(F) - 2a_1\beta_1}{2\beta_2} e^{i\theta}, \tag{4.124}$$

$$\varphi_3 = \frac{-2\lambda}{k(l^2\sigma^2 - 1)} \times \begin{pmatrix} -2 \dfrac{\beta_1 \tanh(F)\, a_1 d_1}{\beta_2} + \sqrt{\beta_1} {d_1}^2 \ln(1 + \tanh(F)) \\ - \dfrac{\sqrt{\beta_1} \tanh(F)\, {d_1}^2}{2} - 4 \dfrac{{\beta_1}^{\frac{3}{2}} {a_1}^2 \ln(\tanh(F) - 1)}{{\beta_2}^2} \\ -2 \dfrac{{\beta_1}^{3/2} \tanh(F) {a_1}^2}{{\beta_2}^2} \end{pmatrix}, \tag{4.125}$$

$$\tau_4 = \frac{-\sqrt{\beta_1} d_1 \coth(F)\, \beta_2 + d_1 \sqrt{\beta_1} \beta_2 - 2a_1\beta_1 \coth(F) - 2a_1\beta_1}{2\beta_2} e^{i\theta}, \tag{4.126}$$

$$\varphi_4 = \frac{-2\lambda}{k(l^2\sigma^2 - 1)} \times \left(\begin{array}{c} -2\frac{\beta_1 \coth(F)\, a_1 d_1}{\beta_2} + \sqrt{\beta_1} {d_1}^2 \ln(1 + \coth(F)) \\ -\frac{\sqrt{\beta_1} \coth(F)\, {d_1}^2}{2} - 4\frac{{\beta_1}^{\frac{3}{2}} {a_1}^2 \ln(\coth(F) - 1)}{{\beta_2}^2} \\ -2\frac{{\beta_1}^{3/2} \coth\,(F) {a_1}^2}{{\beta_2}^2} \end{array} \right). \tag{4.127}$$

For $\beta_1 < 0, \Delta > 0$,

$$\tau_5 = \frac{\left(\begin{array}{c} 2a_1\beta_1 \sec\left(\sqrt{-\beta_1}\xi\right) \\ + \tan\left(\sqrt{-\beta_1}\xi\right) d_1 \sqrt{-\beta_1}\Delta \end{array} \right)}{\Delta - \beta_2 \sec\,(\sqrt{-\beta_1}\xi)} e^{i\theta}, \tag{4.128}$$

$$\varphi_5 = \frac{-2\lambda}{k(l^2\sigma^2 - 1)\sqrt{-\beta_1}\beta_3} \times \left(\begin{array}{c} \left(\frac{\left(\beta_3 {d_1}^2 + {a_1}^2\right)(\mathrm{J}) \tan(F')}{2\beta_3} \right. \\ \left. -\frac{\sqrt{-\beta_1} a_1 d_1 (-2\Delta\beta_1\beta_3 + (\mathrm{J})\beta_2)}{\beta_1\beta_3} \right) \\ \times \left(\tan(F')^2 - \frac{\Delta\beta_2}{2\beta_1\beta_3} - 1 + \frac{{\beta_2}^2}{2\beta_1\beta_3} \right)^{-1} \\ + \frac{\beta_1\left(-\beta_3 {d_1}^2 + {a_1}^2\right)(\Delta - \beta_2)\beta_2\sqrt{2}}{2\sqrt{T}} \\ \times \operatorname{arctanh}\left(\frac{\tan(F')\, \beta_1\beta_3\sqrt{2}}{\sqrt{T}} \right) \\ + \frac{{d_1}^2 \arctan\,(\tan(F'))}{4} \end{array} \right), \tag{4.129}$$

$$\tau_6 = \frac{\begin{pmatrix} 2a_1\beta_1 \csc\left(\sqrt{-\beta_1}\xi\right) \\ -2co\,\mathrm{t}\left(\sqrt{-\beta_1}\xi\right)\sqrt{-\beta_1}\Delta\, d_1 \end{pmatrix}}{\Delta - \beta_2 \csc\left(\sqrt{-\beta_1}\xi\right)} e^{i\theta}, \tag{4.130}$$

$$\varphi_6 = \frac{-2\lambda}{k(l^2\sigma^2 - 1)} \times \left(\begin{array}{c} -\frac{8\beta_1 \tan(F')\,\Delta a_1 d_1}{H{\beta_2}^2} - \frac{8{\beta_1}^2\beta_3 \tan(F')\,{d_1}^2}{\sqrt{-\beta_1}H{\beta_2}^2} - \frac{2\beta_1\Delta{d_1}^2}{\sqrt{-\beta_1}H\beta_2} \\ +\frac{2\beta_1 \tan(F')\,{d_1}^2}{\sqrt{-\beta_1}H} - \frac{8{\beta_1}^2 \tan(F')\,{a_1}^2}{\sqrt{-\beta_1}H{\beta_2}^2} + \frac{2\beta_1 \tan(F')\,{a_1}^2}{\sqrt{-\beta_1}H\beta_3} \\ -\frac{2\beta_1\Delta{a_1}^2}{\sqrt{-\beta_1}H\beta_2\beta_3} + \frac{\beta_1\beta_2{d_1}^2}{\sqrt{-\beta_1}\sqrt{\beta_1\beta_3}}\arctan\left(\frac{-2\tan(F')\,\beta_2 + 2\Delta}{4\sqrt{\beta_1\beta_3}}\right) \\ -\frac{\beta_1\beta_2{a_1}^2}{\sqrt{-\beta_1}\beta_3\sqrt{\beta_1\beta_3}}\arctan\left(\frac{-2\tan(F')\,\beta_2 + 2\Delta}{4\sqrt{\beta_1\beta_3}}\right) \\ +2\frac{\beta_1{d_1}^2\arctan\left(\tan(F')\right)}{\sqrt{-\beta_1}} \end{array} \right). \tag{4.131}$$

For $\beta_1 < 0, \beta_3 > 0$,

$$\tau_6 = \frac{\begin{pmatrix} \left(\sqrt{-\beta_1\beta_3}\tan(F')^2 + \tan(F')\,\beta_2 - \sqrt{-\beta_1\beta_3}\right) \\ \times d_1\sqrt{-\beta_1} - a_1\beta_1 \sec(F')^2 \end{pmatrix}}{\beta_2 + 2\sqrt{-\beta_1\beta_3}\tan(F')} e^{i\theta}, \tag{4.132}$$

$$\begin{aligned} \varphi_6 = {} & \frac{-2\lambda}{k(l^2\sigma^2 - 1)} \\ & \times \frac{2\beta_1}{\sqrt{-\beta_1}}\left(\frac{\left(2\sqrt{-\beta_1\beta_3}\sqrt{-\beta_1}a_1 d_1 + \beta_1\beta_3{d_1}^2 + {a_1}^2\beta_1\right)\tan(F')}{4\beta_1\beta_3}\right. \\ & + \frac{1}{4\beta_1\beta_3}\left(\frac{\begin{pmatrix} -8\sqrt{-\beta_1\beta_3}\sqrt{-\beta_1}a_1 d_1\beta_1\beta_3 + 2\sqrt{-\beta_1\beta_3}\sqrt{-\beta_1}a_1{\beta_2}^2 d_1 \\ -4{\beta_1}^2{\beta_3}^2{d_1}^2 + \beta_1\beta_3{d_1}^2{\beta_2}^2 - 4{a_1}^2{\beta_1}^2\beta_3 + {a_1}^2\beta_1{\beta_2}^2 \end{pmatrix}}{4\beta_1\beta_3\tan(F') - 2\sqrt{-\beta_1\beta_3}\beta_2}\right. \\ & \left. + \frac{i\sqrt{\beta_1\beta_3}\beta_2\left(-\beta_3{d_1}^2 + {a_1}^2\right)\ln\left(-2\beta_1\beta_3\tan(F') + \sqrt{-\beta_1\beta_3}\beta_2\right)}{\beta_3}\right) \\ & \left. -{d_1}^2\left(\pi/2 - \arctan(\tan(F'))\right)\right), \end{aligned} \tag{4.133}$$

$$\tau_7 = \frac{\begin{pmatrix} -\left(\sqrt{-\beta_1\beta_3}\cot(F')^2 + \cot(F')\,\beta_2 - \sqrt{-\beta_1\beta_3}\right) \\ \times\, d_1\sqrt{-\beta_1} - a_1\beta_1\csc(F')^2 \end{pmatrix}}{\beta_2 + 2\sqrt{-\beta_1\beta_3}\cot(F')} e^{i\theta}, \tag{4.134}$$

$$\begin{aligned} \varphi_7 &= \frac{-2\lambda}{k(l^2\sigma^2 - 1)} \\ &\times \frac{2\beta_1}{\sqrt{-\beta_1}}\left(\frac{\left(2\sqrt{-\beta_1\beta_3}\sqrt{-\beta_1}a_1d_1 + \beta_1\beta_3{d_1}^2 + {a_1}^2\beta_1\right)\cot(F')}{4\beta_1\beta_3}\right. \\ &+ \frac{1}{4\beta_1\beta_3}\left(\frac{\begin{pmatrix} 8\sqrt{-\beta_1\beta_3}\sqrt{-\beta_1}a_1d_1\beta_1\beta_3 - 2\sqrt{-\beta_1\beta_3}\sqrt{-\beta_1}a_1{\beta_2}^2d_1 \\ -4{\beta_1}^2{\beta_3}^2{d_1}^2 + \beta_1\beta_3{d_1}^2{\beta_2}^2 - 4{a_1}^2{\beta_1}^2\beta_3 + {a_1}^2\beta_1{\beta_2}^2 \end{pmatrix}}{4\beta_1\beta_3\cot(F') - 2\sqrt{-\beta_1\beta_3}\beta_2}\right. \\ &+ \left.\frac{i\sqrt{\beta_1\beta_3}\beta_2\left(-\beta_3{d_1}^2 + {a_1}^2\right)\ln\left(-2\beta_1\beta_3\cot(F') + \sqrt{-\beta_1\beta_3}\beta_2\right)}{\beta_3}\right) \\ &\left. - {d_1}^2\left(\pi/2 - \operatorname{arccot}(\cot(F'))\right)\right). \end{aligned} \tag{4.135}$$

For $\beta_1 > 0$,

$$\tau_8 = \frac{-4{\beta_1}^{3/2}\beta_3d_1 - \sqrt{\beta_1}\left(e^{\sqrt{\beta_1}\xi}\right)^2 d_1 + \sqrt{\beta_1}{\beta_2}^2d_1 + 4a_1\beta_1e^{\sqrt{\beta_1}\xi}}{\left(e^{\sqrt{\beta_1}\xi}\right)^2 - 2e^{\sqrt{\beta_1}\xi}\beta_2 - 4\beta_1\beta_3 + {\beta_2}^2} e^{i\theta}, \tag{4.136}$$

$$\begin{aligned} \varphi_8 &= \frac{-2\lambda}{k(l^2\sigma^2 - 1){\beta_1}^{1/2}{\beta_3}^{3/2}\left(e^{2\sqrt{\beta_1}\xi} - 2e^{\sqrt{\beta_1}\xi}\beta_2 - 4\beta_1\beta_3 + {\beta_2}^2\right)} \\ &\times \begin{pmatrix} \sqrt{\beta_1}\left(\beta_2(-\beta_3{d_1}^2 + {a_1}^2)\left(e^{2\sqrt{\beta_1}\xi} - 2e^{\sqrt{\beta_1}\xi}\beta_2 - 4\beta_1\beta_3 + {\beta_2}^2\right)\right. \\ \operatorname{arctanh}\left(\frac{e^{\sqrt{\beta_1}\xi} - \beta_2}{2\sqrt{\beta_1\beta_3}}\right) + \sqrt{\beta_1\beta_3}\left(\ln\left(e^{\sqrt{\beta_1}\xi}\right)e^{2\sqrt{\beta_1}\xi}\beta_3{d_1}^2\right. \\ -\left(4\beta_1\beta_3 - {\beta_2}^2 + 2e^{\sqrt{\beta_1}\xi}\beta_2\right)\beta_3{d_1}^2\ln\left(e^{\sqrt{\beta_1}\xi}\right) \\ +\left(8\sqrt{\beta_1}a_1\beta_3d_1 - 2\beta_2\left(\beta_3{d_1}^2 + {a_1}^2\right)\right)e^{\sqrt{\beta_1}\xi} \\ \left.\left. -\left(8\beta_3{d_1}^2 + 8{a_1}^2\right)\left(\beta_1\beta_3 - \frac{{\beta_2}^2}{4}\right)\right)\right) \end{pmatrix}. \end{aligned} \tag{4.137}$$

For $\beta_1 > 0, \beta_2 = 0$,

$$\tau_9 = \frac{-4e^{2\sqrt{\beta_1}\xi}{\beta_1}^{3/2}\beta_3 d_1 - 4a_1\beta_1 e^{\sqrt{\beta_1}\xi} - \sqrt{\beta_1}d_1}{4\beta_1\beta_3 e^{2\sqrt{\beta_1}\xi} - 1} e^{i\theta}, \tag{4.138}$$

$$\varphi_9 = -2\lambda\left(\frac{\begin{array}{c}4{\beta_1}^{\frac{3}{2}}e^{2\sqrt{\beta_1}\xi}\ln\left(e^{\sqrt{\beta_1}\xi}\right){\beta_3}^2{d_1}^2 - \sqrt{\beta_1}{d_1}^2\ln\left(e^{\sqrt{\beta_1}\xi}\right)\beta_3 \\ -8a_1\beta_1 d_1 e^{\sqrt{\beta_1}\xi}\beta_3 - 2\sqrt{\beta_1}(\beta_3{d_1}^2 + {a_1}^2)\end{array}}{k(l^2\sigma^2 - 1)\left(4{\beta_3}^2\beta_1 e^{2\sqrt{\beta_1}\xi} - \beta_3\right)}\right), \tag{4.139}$$

Set 2 :

$$a_0 = \frac{-(l^2\sigma^2 - 1)\sqrt{2}}{2\lambda\sqrt{\frac{l^4\sigma^4 - 1}{(\sigma^4{k_2}^2 + \sigma^2{k_1}^2 + 2k_3)\lambda}}}, a_1 = \frac{1}{2\sqrt{2}}\sqrt{\frac{l^4\sigma^4 - 1}{(\sigma^4{k_2}^2 + \sigma^2{k_1}^2 + 2k_3)\lambda}}\sigma^2\beta_2 k^2,$$

$$b_1 = 0, k = k, \beta_2 = \beta_2, \beta_1 = \frac{-2\sigma^4{k_2}^2 - 2\sigma^2{k_1}^2 - 4k_3}{k^2\sigma^2(l^2\sigma^2 + 1)},$$

$$\beta_3 = -\frac{{\beta_2}^2(l^2\sigma^2 + 1)k^2\sigma^2}{8\sigma^4{k_2}^2 + 8\sigma^2{k_1}^2 + 16k_3}, d_1 = 0.$$

Substituting these coefficients along with the auxiliary solutions of Eq. (1.48), we get solutions of Eq. (4.111) as follows.

$$u(\zeta) = S = a_0 + a_1\mathbb{Q}(\zeta). \tag{4.140}$$

For $\beta_1 > 0$, we have

$$\tau_{10} = \frac{\begin{pmatrix}-a_1\beta_1\beta_2\operatorname{csch}(F)^2 - a_0{\beta_2}^2 \\ +(\coth(F) + 1)^2 a_0\beta_1\beta_3\end{pmatrix}}{(\coth(F) + 1)^2\beta_1\beta_3 - {\beta_2}^2} e^{i\theta}, \tag{4.141}$$

$$\varphi_{10}=\frac{-2\lambda}{k(l^2\sigma^2-1)\begin{pmatrix}\sqrt{\beta_1}\beta_2(\tanh f^4\beta_1\beta_3+4\tanh f^3\beta_1\beta_3\\+(6\beta_1\beta_3-4{\beta_2}^2)\tanh f^2+4\beta_1\beta_3\tanh f+\beta_1\beta_3)\beta_3{}^3\end{pmatrix}}$$

$$\times\begin{pmatrix}a_1\sqrt{\beta_1{\beta_2}^2{\beta_3}^3}(\tanh f^4\beta_1\beta_3+4\tanh f^3\beta_1\beta_3\\+(6\beta_1\beta_3-4{\beta_2}^2)\tanh f^2+4\beta_1\beta_3\tanh f+\beta_1\beta_3)\\\left(a_0\beta_3-\frac{a_1\beta_2}{4}\right)\ln\left(2\sqrt{\beta_1{\beta_2}^2{\beta_3}^3}\tanh f-\beta_1{\beta_3}^2(\tanh f+1)^2\right)\\-a_1\sqrt{\beta_1{\beta_2}^2{\beta_3}^3}(\tanh f^4\beta_1\beta_3+4(\tanh(f))^3\beta_1\beta_3\\+(6\beta_1\beta_3-4{\beta_2}^2)\tanh f^2+4\beta_1\beta_3\tanh f+\beta_1\beta_3)\\(a_0\beta_3-\frac{a_1\beta_2}{4})\ln(2\sqrt{\beta_1{\beta_2}^2{\beta_3}^3}\tanh f\\+\beta_1{\beta_3}^2(\tanh f+1)^2)-\beta_2({a_0}^2(\tanh f^4\beta_1\beta_3\\+4\tanh f^3\beta_1\beta_3+\left(6\beta_1\beta_3-4{\beta_2}^2\right)\tanh f^2\\+4\beta_1\beta_3\tanh f+\beta_1\beta_3)\beta_3\ln(\tanh f-1)\\-{a_0}^2(\tanh f^4\beta_1\beta_3+4\tanh f^3\beta_1\beta_3\\+(6\beta_1\beta_3-4{\beta_2}^2)\tanh f^2+4\beta_1\beta_3\tanh f+\beta_1\beta_3)\beta_3\\\ln(\tanh f+1)-{a_1}^2\beta_1{\beta_2}^2\tanh f(\tanh f-1)^2){\beta_3}^2\end{pmatrix}. \quad (4.142)$$

For $\beta_1>0, \Delta>0$,

$$\tau_{11}=\frac{(l^2\sigma^2-1)\sqrt{2}}{2\lambda\sqrt{\frac{l^4\sigma^4-1}{\left(\sigma^4{k_2}^2+\sigma^2{k_1}^2+2k_3\right)\lambda}}}e^{i\theta}, \quad (4.143)$$

$$\varphi_{11}=\frac{-2\left(\sigma^4{k_2}^2+\sigma^2{k_1}^2+2k_3\right)\lambda}{k(l^2\sigma^2+1)}\xi. \quad (4.144)$$

For $\beta_1>0, \Delta=0$,

$$\tau_{12}=\frac{-a_1\beta_1\tanh(F)+a_0\beta_2-a_1\beta_1}{\beta_2}e^{i\theta}, \quad (4.145)$$

$$\varphi_{12}=\frac{2\lambda}{k(l^2\sigma^2-1)}\left(\frac{\begin{matrix}(a_0\beta_2-2a_1\beta_1)^2\ln(\tanh F-1)+2{\beta_1}^2{a_1}^2\tanh F\\-{a_0}^2\ln(\tanh F+1){\beta_2}^2\end{matrix}}{{\beta_2}^2\sqrt{\beta_1}}\right), \quad (4.146)$$

$$\tau_{13} = \frac{-a_1\beta_1 \coth(F) + a_0\beta_2 - a_1\beta_1}{\beta_2} e^{i\theta}, \tag{4.147}$$

$$\varphi_{13} = \frac{2\lambda}{k(l^2\sigma^2 - 1)} \left(\frac{\begin{array}{c}(a_0\beta_2 - 2a_1\beta_1)^2 \ln(\coth\mathrm{F} - 1) - a_0{}^2 \ln(1 + \coth\mathrm{F})\,\beta_2{}^2 \\ +2\beta_1{}^2 a_1{}^2 \coth\mathrm{F})\end{array}}{\sqrt{\beta_1}\beta_2{}^2} \right). \tag{4.148}$$

For $\beta_1 < 0, \beta_3 > 0$,

$$\tau_{14} = \frac{-a_1\beta_1 \sec(F')^2 + 2\tan(F')\sqrt{-\beta_1\beta_3}a_0 + a_0\beta_2}{\beta_2 + 2\tan(F')\sqrt{-\beta_1\beta_3}} e^{i\theta}, \tag{4.149}$$

$$\varphi_{14} = 2\lambda \frac{\begin{array}{c} 16\left(\tan(F')\sqrt{-\beta_1\beta_3} + \frac{\beta_2}{2}\right)\beta_1 a_1 \left(a_0\beta_3 - \frac{a_1\beta_2}{4}\right)\ln\left(\beta_2 + 2\tan(F')\sqrt{-\beta_1\beta_3}\right) \\ -8\beta_3 a_0{}^2\left(-2\tan(F')\,\beta_1\beta_3 + \sqrt{-\beta_1\beta_3}\beta_2\right)\arctan(\tan(F')) \\ +2\beta_1 a_1{}^2\left(-2(\tan(F'))^2\beta_1\beta_3 + \tan(F')\sqrt{-\beta_1\beta_3}\beta_2 + 2\beta_1\beta_3 - \frac{\beta_2{}^2}{2}\right) \end{array}}{k(l^2\sigma^2 - 1)\sqrt{-\beta_1\beta_3}\sqrt{-\beta_1}(8\beta_3\sqrt{-\beta_1\beta_3}\tan(F') + 4\beta_3\beta_2)}, \tag{4.150}$$

$$\tau_{15} = \frac{-a_1\beta_1 \csc(F')^2 + 2\cot(F')\sqrt{-\beta_1\beta_3}a_0 + a_0\beta_2}{\beta_2 + 2\cot(F')\sqrt{-\beta_1\beta_3}} e^{i\theta}, \tag{4.151}$$

$$
\begin{aligned}
\varphi_{15} = &\frac{-2{\beta_1}^2{a_1}^2}{\sqrt{-\beta_1}\beta_2\left(\beta_2\tan(F') + 2\sqrt{-\beta_1\beta_3}\right)} \\
&+ \frac{\beta_1{a_1}^2\beta_2}{2\sqrt{-\beta_1}\beta_3\left(\beta_2\tan(F') + 2\sqrt{-\beta_1\beta_3}\right)} \\
&+ \frac{2a_1\beta_1\ln\left(\beta_2\tan(F) + 2\sqrt{-\beta_1\beta_3}\right)a_0}{\sqrt{-\beta_1}\sqrt{-\beta_1\beta_3}} \\
&- \frac{\beta_1{a_1}^2\ln\left(\beta_2\tan(F) + 2\sqrt{-\beta_1\beta_3}\right)\beta_2}{2\sqrt{-\beta_1}\sqrt{-\beta_1\beta_3}\beta_3} \\
&+ \frac{\beta_1{a_1}^2}{2\sqrt{-\beta_1}\beta_3\tan(F)} - 2\frac{a_1\beta_1\ln(\tan(F))\,a_0}{\sqrt{-\beta_1}\sqrt{-\beta_1\beta_3}} + \frac{\beta_1{a_1}^2\ln(\tan(F))\,\beta_2}{2\sqrt{-\beta_1}\sqrt{-\beta_1\beta_3}\beta_3} \\
&+ \frac{4{a_0}^2(-\beta_1\beta_3)^{\frac{3}{2}}\ln(1 + (\tan(F))^2)\,\beta_2}{\sqrt{-\beta_1}\left(\beta_2 + 2\sqrt{-\beta_1\beta_3}\right)^2\left(\beta_2 + \sqrt{-\beta_1\beta_3}\right)^2} \\
&+ \frac{4{a_0}^2\sqrt{-\beta_1\beta_3}\ln(1 + (\tan(F))^2)\,\beta_1\beta_2\beta_3}{\sqrt{-\beta_1}\left(\beta_2 + 2\sqrt{-\beta_1\beta_3}\right)^2\left(\beta_2 + \sqrt{-\beta_1\beta_3}\right)^2} \\
&- \frac{24{a_0}^2\sqrt{-\beta_1\beta_3}\arctan(\tan(F))\,\beta_1\beta_2\beta_3}{\sqrt{-\beta_1}\left(\beta_2 + 2\sqrt{-\beta_1\beta_3}\right)^2\left(\beta_2 + \sqrt{-\beta_1\beta_3}\right)^2} \\
&+ \frac{12{a_0}^2\sqrt{-\beta_1\beta_3}\arctan(\tan(F))\,{\beta_2}^3}{\sqrt{-\beta_1}\left(\beta_2 + 2\sqrt{-\beta_1\beta_3}\right)^2\left(\beta_2 + \sqrt{-\beta_1\beta_3}\right)^2} \\
&+ \frac{8{a_0}^2\arctan(\tan(F))\,{\beta_1}^2{\beta_3}^2}{\sqrt{-\beta_1}\left(\beta_2 + 2\sqrt{-\beta_1\beta_3}\right)^2\left(\beta_2 + \sqrt{-\beta_1\beta_3}\right)^2} \\
&- \frac{26{a_0}^2\arctan(\tan(F))\,\beta_1{\beta_2}^2\beta_3}{\sqrt{-\beta_1}\left(\beta_2 + 2\sqrt{-\beta_1\beta_3}\right)^2\left(\beta_2 + \sqrt{-\beta_1\beta_3}\right)^2} \\
&+ \frac{2{a_0}^2\arctan\,(\tan(F))\beta_2{}^4}{\sqrt{-\beta_1}(\beta_2 + 2\sqrt{-\beta_1\beta_3})^2(\beta_2 + \sqrt{-\beta_1\beta_3})^2}.
\end{aligned}
\tag{4.152}
$$

For $\beta_1 > 0$,

$$\tau_{16} = \frac{\left(e^{\sqrt{\beta_1}\xi}\right)^2 a_0 - 2e^{\sqrt{\beta_1}\xi}a_0\beta_2 + 4a_1\beta_1 e^{\sqrt{\beta_1}\xi} - 4a_0\beta_1\beta_3 + a_0{\beta_2}^2}{\left(e^{\sqrt{\beta_1}\xi}\right)^2 - 2e^{\sqrt{\beta_1}\xi}\beta_2 - 4\beta_1\beta_3 + {\beta_2}^2} e^{i\theta}, \tag{4.153}$$

$$\begin{aligned} \varphi_{16} = {} & \frac{2\lambda}{k(l^2\sigma^2 - 1)\sqrt{\beta_1}\beta_3\left(\left(e^{\sqrt{\beta_1}\xi} - \beta_2\right)^2 - 4\beta_1\beta_3\right)} \\ & \times 4a_1\left(\left(\left(\left(e^{\sqrt{\beta_1}\xi} - \beta_2\right)^2 - 4\beta_1\beta_3\right)\beta_1\right.\right. \\ & \times \left(a_0\beta_3 - \frac{a_1\beta_2}{4}\right)\text{arctanh}\left(\frac{e^{\sqrt{\beta_1}\xi} - \beta_2}{2\sqrt{\beta_1\beta_3}}\right) + (\ln(e^{\sqrt{\beta_1}\xi})e^{2\sqrt{\beta_1}\xi}{a_0}^2\beta_3 \\ & -4\beta_3\left(\beta_1\beta_3 - \frac{{\beta_2}^2}{4} + \frac{e^{\sqrt{\beta_1}\xi}\beta_2}{2}\right){a_0}^2\ln\left(e^{\sqrt{\beta_1}\xi}\right) \\ & \left.\left. -8{a_1}^2\left(\beta_1\beta_3 - \frac{{\beta_2}^2}{4} + \frac{e^{\sqrt{\beta_1}\xi}\beta_2}{4}\right)\beta_1\right)\right), \end{aligned} \tag{4.154}$$

Set 3 :

$$a_0 = \sqrt{\frac{(\sigma^4{k_2}^2 + \sigma^2{k_1}^2 + 2k_3)(l^2\sigma^2 - 1)}{2\lambda(l^2\sigma^2 + 1)}}, a_1 = 0,$$

$$b_1 = 0, k = k, \beta_2 = \beta_2, \beta_1 = \beta_1, \beta_3 = \beta_3, d_1 = 0\,.$$

For these set of coefficients, we have following solutions,

$$u(\zeta) = S = a_0, \tag{4.155}$$

$$\tau_{17} = a_0 e^{i\theta}, \qquad \varphi_{17} = -\frac{2\lambda(a_0)^2\zeta}{k(l^2\sigma^2 - 1)}, \tag{4.156}$$

Set 4 :

$$a_0 = \frac{\sqrt{\frac{(\sigma^4 {k_2}^2 + \sigma^2 {k_1}^2 + 2k_3)(l^2\sigma^2 - 1)}{\lambda(l^2\sigma^2 + 1)}}}{\sqrt{2}}, a_1 = 0,$$

$$b_1 = 0, k = k, \beta_2 = 0, \beta_3 = 0, d_1 = d_1\,,$$

$$\beta_1 = \frac{(\sigma^4 {k_2}^2 + \sigma^2 {k_1}^2 + 2k_3)(l\sigma - 1)(l\sigma + 1)}{8\lambda {d_1}^2(l^2\sigma^2 + 1)}.$$

Substituting these coefficients along with the auxiliary solutions of Eq. (1.48), we get solutions of Eq. (4.111) as follows.

$$u(\zeta) = S = a_0 + \frac{d_1\left(\frac{d}{d\zeta}\mathbb{Q}(\zeta)\right)}{\mathbb{Q}(\zeta)},$$

$$\tau_{18} = \left(a_0 - \sqrt{\beta_1}d_1\right)e^{i\theta}\,, \tag{4.157}$$

$$\varphi_{18} = \frac{-2\lambda\left(a_0 - \sqrt{\beta_1}d_1\right)^2\zeta}{k(l^2\sigma^2 - 1)}, \tag{4.158}$$

$$\tau_{19} = \left(a_0 + \sqrt{\beta_1}d_1\right)e^{i\theta}\,, \tag{4.159}$$

$$\varphi_{19} = \frac{-2\lambda\left(a_0 + \sqrt{\beta_1}d_1\right)^2\zeta}{k(l^2\sigma^2 - 1)}, \tag{4.160}$$

Set 5 :

$$a_0 = 0, a_1 = 0, b_1 = 0, k = k, \beta_2 = 0, \beta_1 = \frac{-\sigma^4 {k_2}^2 - \sigma^2 {k_1}^2 - 2k_3}{2k^2\sigma^2(l^2\sigma^2 + 1)},$$

$$\beta_3 = \beta_3, d_1 = \sqrt{\frac{-l^2\sigma^2 + 1}{\lambda}}\sigma k\,.$$

Substituting these coefficients along with the auxiliary solutions of Eq. (1.48), we get solutions of Eq. (4.111) as follows.

$$u(\zeta) = S = \frac{d_1\left(\frac{d}{d\zeta}\mathbb{Q}(\zeta)\right)}{\mathbb{Q}(\zeta)},$$

$$\tau_{20} = -\sqrt{\beta_1} d_1 e^{i\theta}, \tag{4.161}$$

$$\varphi_{20} = \frac{2\beta_1 \zeta}{k}, \tag{4.162}$$

$$\tau_{21} = -\sqrt{\beta_1} \tanh\,(\sqrt{\beta_1}\xi) d_1 e^{i\theta}\,, \tag{4.163}$$

$$\varphi_{21} = \frac{1}{k}\begin{pmatrix} 2\sqrt{\beta_1}\tanh\left(\sqrt{\beta_1}\xi\right) + \sqrt{\beta_1}\ln\left(\tanh\left(\sqrt{\beta_1}\xi\right) - 1\right) \\ -\sqrt{\beta_1}\ln\,(\tanh\,(\sqrt{\beta_1}\xi) + 1) \end{pmatrix}, \tag{4.164}$$

$$\tau_{22} = -\sqrt{\beta_1} \coth\,(\sqrt{\beta_1}\xi) d_1 e^{i\theta}\,, \tag{4.165}$$

$$\varphi_{22} = \frac{1}{k}\begin{pmatrix} 2\sqrt{\beta_1}\coth\left(\sqrt{\beta_1}\xi\right) + \sqrt{\beta_1}\ln\left(\coth\left(\sqrt{\beta_1}\xi\right) - 1\right) \\ -\sqrt{\beta_1}\ln\,(coth\,(\sqrt{\beta_1}\xi) + 1) \end{pmatrix}, \tag{4.166}$$

$$\tau_{23} = \frac{\sqrt{\beta_1} d_1 \begin{pmatrix} \left(\tanh\left(\frac{\sqrt{\beta_1}\xi}{\sqrt{2}}\right)^2 - 1\right)\sqrt{2} - \\ \left(2\tanh\left(\frac{\sqrt{\beta_1}\xi}{\sqrt{2}}\right)\right)\tanh(F) \end{pmatrix}}{2\tanh(F)} e^{i\theta}\,, \tag{4.167}$$

$$\varphi_{23} = \frac{2\lambda}{k(l^2\sigma^2 - 1)}\begin{pmatrix} 2d_1\ln\left(\mathrm{e}^{\sqrt{\beta_1}\xi} + 1\right) - d_1\ln\left(\mathrm{e}^{\sqrt{2}\sqrt{\beta_1}\xi} + 1\right) \\ +d_1\ln\,(\mathrm{e}^{\sqrt{2}\sqrt{\beta_1}\xi} - 1) - \sqrt{\beta_1} d_1 \xi \end{pmatrix}, \tag{4.168}$$

$$\tau_{24} = \frac{\sqrt{\beta_1} d_1 \begin{pmatrix} \left(\coth\left(\frac{\sqrt{\beta_1}\xi}{\sqrt{2}}\right)^2 - 1\right)\sqrt{2} - \\ \left(2\coth\left(\frac{\sqrt{\beta_1}\xi}{\sqrt{2}}\right)\right)\coth(F) \end{pmatrix}}{2\coth(F)} e^{i\theta}\,, \tag{4.169}$$

$$\varphi_{24} = \frac{2\lambda}{k(l^2\sigma^2 - 1)}\begin{pmatrix} 2d_1\ln\left(\mathrm{e}^{\sqrt{\beta_1}\xi} - 1\right) - d_1\ln\left(\mathrm{e}^{\sqrt{2}\sqrt{\beta_1}\xi} - 1\right) \\ +d_1\ln\,(\mathrm{e}^{\sqrt{2}\sqrt{\beta_1}\xi} + 1) - \sqrt{\beta_1} d_1 \xi \end{pmatrix}, \tag{4.170}$$

$$\tau_{25} = \frac{1}{2}(1 - \tanh(F))\sqrt{\beta_1} d_1\, e^{i\theta}, \tag{4.171}$$

$$\varphi_{25} = \frac{2}{k}\left(\sqrt{\beta_1}\ln\,(\tanh\,(F) + 1) - \frac{1}{2}\sqrt{\beta_1}\tanh\,(\mathrm{F})\right), \tag{4.172}$$

$$\tau_{26} = \frac{1}{2}(1 - \coth(F))\sqrt{\beta_1}d_1\, e^{i\theta}, \tag{4.173}$$

$$\varphi_{26} = \frac{2}{k}\left(\sqrt{\beta_1}\ln\,(\coth\,(F) + 1) - \frac{1}{2}\sqrt{\beta_1}\coth\,(\mathrm{F})\right), \tag{4.174}$$

$$\tau_{27} = d_1\sqrt{-\beta_1}\tan\,(\sqrt{-\beta_1}\xi)e^{i\theta}\,, \tag{4.175}$$

$$\varphi_{27} = \frac{-2}{\sqrt{-\beta_1}k}\left(\beta_1\tan\,(\sqrt{-\beta_1}\xi) - \beta_1\arctan\,\left(\tan(\sqrt{-\beta_1}\xi)\right)\right), \tag{4.176}$$

$$\tau_{28} = d_1\sqrt{-\beta_1}\cot\,(\sqrt{-\beta_1}\xi)e^{i\theta}\,, \tag{4.177}$$

$$\varphi_{28} = \frac{2}{\sqrt{-\beta_1}k}\left(\beta_1\cot\left(\sqrt{-\beta_1}\xi\right) + \frac{\pi}{2} + \beta_1\operatorname{arccot}\,\left(\cot(\sqrt{-\beta_1}\xi)\right)\right), \tag{4.178}$$

$$\tau_{29} = \frac{d_1\sqrt{-\beta_1}(\tan(F')^2 - 1)}{2\tan(F')}e^{i\theta}\,, \tag{4.179}$$

$$\varphi_{29} = \frac{-\sqrt{-\beta_1}}{k\tan\,(\mathrm{F}')}\left(\begin{matrix}\tan(F')^2 - 1\\ -4\arctan(\tan(\mathrm{F}'))\tan(\mathrm{F}')\end{matrix}\right), \tag{4.180}$$

$$\tau_{30} = \frac{-d_1\sqrt{-\beta_1}(\cot(F')^2 - 1)}{2\cot(F')}e^{i\theta}\,, \tag{4.181}$$

$$\varphi_{30} = \frac{-\sqrt{-\beta_1}}{k\,cot\,(\mathrm{F}')}\left(\begin{matrix}\cot(F')^2 - 1\\ -4\arctan(\cot(\mathrm{F}'))\cot(\mathrm{F}')\end{matrix}\right), \tag{4.182}$$

$$\tau_{31} = \frac{-(\mathrm{e}^{\sqrt{\beta_1}\xi})^2\sqrt{\beta_1}d_1 - 4{\beta_1}^{3/2}\beta_3 d_1}{(\mathrm{e}^{\sqrt{\beta_1}\xi})^2 - 4\beta_1\beta_3}e^{i\theta}\,, \tag{4.183}$$

$$\varphi_{31} = \frac{2}{k}\left(\sqrt{\beta_1}\ln\,(\mathrm{e}^{\sqrt{\beta_1}\xi}) - \frac{8{\beta_1}^{3/2}\beta_3}{(\mathrm{e}^{\sqrt{\beta_1}\xi})^2 - 4\beta_1\beta_3}\right), \tag{4.184}$$

$$\tau_{32} = -\frac{d_1\sqrt{\beta_1}\left(4\beta_1\beta_3\mathrm{e}^{2\sqrt{\beta_1}\xi} + 1\right)}{4\beta_1\beta_3\mathrm{e}^{2\sqrt{\beta_1}\xi} - 1}e^{i\theta}\,, \tag{4.185}$$

$$\varphi_{32} = \frac{-2}{k}\left(\frac{2\sqrt{\beta_1}}{4\beta_1\beta_3\mathrm{e}^{2\sqrt{\beta_1}\xi} - 1} - \frac{1}{2}\sqrt{\beta_1}\ln\,(\mathrm{e}^{2\sqrt{\beta_1}\xi})\right), \tag{4.186}$$

Set 6 :

$a_0 = 0, a_1 = a_1, b_1 = 0, d_1 = 0,$

$$k = k, \beta_2 = 0, \beta_1 = \frac{\sigma^4 {k_2}^2 + \sigma^2 {k_1}^2 + 2k_3}{k^2\sigma^2(l^2\sigma^2 + 1)}, \beta_3 = \frac{-\lambda {a_1}^2}{k^2\sigma^2(l^2\sigma^2 - 1)}.$$

Substituting these coefficients along with the auxiliary solutions of Eq. (1.48), we get solutions of Eq. (4.111) as follows.

$u(\zeta) = S = a_1\mathbb{Q}(\zeta),$

$$\tau_{33} = \left(\frac{2a_1\beta_1 \operatorname{sech}(\sqrt{\beta_1}\xi)}{\sqrt{-4\beta_1\beta_3}}\right)e^{i\theta}, \tag{4.187}$$

$$\varphi_{33} = \frac{2\lambda}{k(l^2\sigma^2 - 1)}\left(\frac{{a_1}^2\sqrt{\beta_1}\tanh(\sqrt{\beta_1}\xi)}{\beta_3}\right), \tag{4.188}$$

$$\tau_{34} = \left(\frac{2a_1\beta_1 \operatorname{csch}(\sqrt{\beta_1}\xi)}{\sqrt{4\beta_1\beta_3}}\right)e^{i\theta}, \tag{4.189}$$

$$\varphi_{34} = \frac{2\lambda}{k(l^2\sigma^2 - 1)}\left(\frac{{a_1}^2\sqrt{\beta_1}\coth(\sqrt{\beta_1}\xi)}{\beta_3}\right), \tag{4.190}$$

$$\tau_{35} = \left(\frac{2a_1\beta_1 \sec(\sqrt{-\beta_1}\xi)}{\sqrt{-4\beta_1\beta_3}}\right)e^{i\theta}, \tag{4.191}$$

$$\varphi_{35} = \frac{2\lambda {a_1}^2\beta_1 \tan(\sqrt{-\beta_1}\xi)}{k(l^2\sigma^2 - 1)\beta_3\sqrt{-\beta_1}}, \tag{4.192}$$

$$\tau_{36} = \left(\frac{2a_1\beta_1 \csc(\sqrt{-\beta_1}\xi)}{\sqrt{-4\beta_1\beta_3}}\right)e^{i\theta}, \tag{4.193}$$

$$\varphi_{36} = \frac{-2\lambda {a_1}^2\beta_1 \cot(\sqrt{-\beta_1}\xi)}{k(l^2\sigma^2 - 1)\beta_3\sqrt{-\beta_1}}, \tag{4.194}$$

$$\tau_{37} = \left(\frac{-a_1\beta_1 \sec(F')^2}{2\sqrt{-\beta_1\beta_3}\tan(F')}\right)e^{i\theta}, \tag{4.195}$$

$$\varphi_{37} = \frac{\lambda}{2k(l^2\sigma^2 - 1)}\left(\frac{{a_1}^2\beta_1}{\beta_3 \cos(F')^2 \sin(F')^2}\right), \tag{4.196}$$

$$\tau_{38} = \left(\frac{-a_1\beta_1 \csc(F')^2}{2\sqrt{-\beta_1\beta_3}\cot(F')}\right)e^{i\theta}, \tag{4.197}$$

$$\varphi_{38} = \frac{-\lambda}{k(l^2\sigma^2-1)}\left(\frac{{a_1}^2\beta_1(2\cos(F')^2-1)}{\beta_3\sqrt{-\beta_1}\sin(F')\cos(F')}\right), \tag{4.198}$$

$$\tau_{39} = \left(4\frac{a_1\beta_1 \mathrm{e}^{\sqrt{\beta_1}\xi}}{(\mathrm{e}^{\sqrt{\beta_1}\xi})^2-4\beta_1\beta_3}\right)e^{i\theta}, \tag{4.199}$$

$$\varphi_{39} = \frac{16\lambda}{k(l^2\sigma^2-1)}\left(\frac{{a_1}^2{\beta_1}^{3/2}}{(\mathrm{e}^{\sqrt{\beta_1}\xi})^2-4\beta_1\beta_3}\right), \tag{4.200}$$

$$\tau_{40} = \left(\frac{-4a_1\beta_1 \mathrm{e}^{\sqrt{\beta_1}\xi}}{4\beta_1\beta_3\mathrm{e}^{2\sqrt{\beta_1}\xi}-1}\right)e^{i\theta}, \tag{4.201}$$

$$\varphi_{40} = \frac{4\lambda}{k(l^2\sigma^2-1)}\left(\frac{{a_1}^2\sqrt{\beta_1}}{\beta_3(4\beta_1\beta_3(\mathrm{e}^{\sqrt{\beta_1}\xi})^2-1)}\right), \tag{4.202}$$

Set 7 :

$a_0 = 0, a_1 = 0,$

$b_1 = 0, k = k, \beta_2 = 0, \beta_3 = 0, d_1 = d_1,$

$$\beta_1 = \frac{(\sigma^4{k_2}^2 + \sigma^2{k_1}^2 + 2k_3)(l\sigma-1)(l\sigma+1)}{2\lambda{d_1}^2(l^2\sigma^2+1)}.$$

Substituting these coefficients along with the auxiliary solutions of Eq. (1.48), we get solutions of Eq. (4.111) as follows.

$$u(\zeta) = S = \frac{d_1\left(\frac{d}{d\zeta}\mathbb{Q}(\zeta)\right)}{\mathbb{Q}(\zeta)},$$

$$\tau_{41} = \left(\pm\sqrt{\beta_1}d_1\right)e^{i\theta}, \tag{4.203}$$

$$\varphi_{41} = -\frac{2\lambda\beta_1(d_1)^2\zeta}{k(l^2\sigma^2-1)}, \tag{4.204}$$

4.6 Results and discussion

In this section, graphical simulation of $(2+1)$ Davey-Stewartson (DS) system has been given. With the assistance of modified auxiliary equation mapping method, we succeed in obtaining various possible physical wave patterns by choosing appropriate parameters. The obtained soliton solutions are more generalized and newer and might be good addition in literature. To analyze this model. 3-D, 2-D and contour plots have been plotted to explain divergence and physics of these waves by choosing suitable values of parameters included in solutions.

Graphical depiction of Real value of Eq (4.205) expressed as τ_{10} has been exhibit in Figure 4.7, in the form of 3-dimensional, and 2-dimensional and contour plot which demonstrates as singular periodic wave solution by choosing parameters, $-10 \leq x \leq 10, t = 0..10, k = 4, c_2 = 1, k_1 = 3, k_2 = 1, k_3 = -6.1, l = 1.3, \sigma = 1, \beta_2 = 3, \lambda = 1, y = 2$.

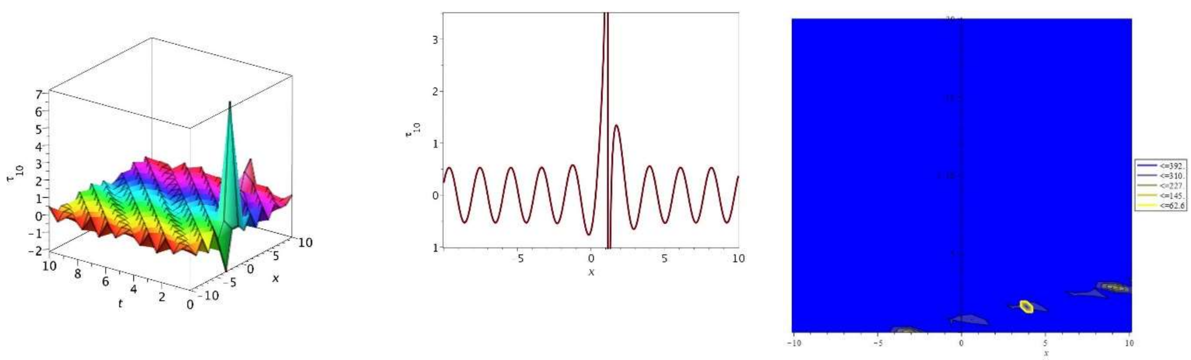


Figure 4.7: graphs of solitary wave solution τ_{10}

Graphical depiction of Real value of Eq (4.142) expressed as φ_{10} has been exhibit in Figure 4.8, in the form of 3-dimensional, 2-dimensional and their contour plot which demonstrates singular periodic wave solution by choosing parameters $-10 \leq x \leq 10, t = 0..10, k = 4, c_2 = 1, k_1 = 3, k_2 = 1, k_3 = -6.1, l = 1.3, \sigma = 1, \beta_2 = 3, \lambda = 1, y = 2$.

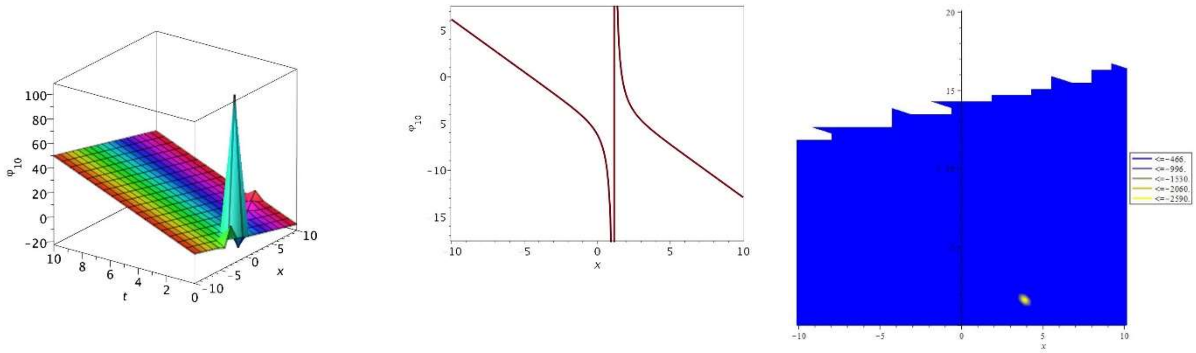


Figure 4.8: -graphs of singular periodic wave solution φ_{10}

Graphical profile of Real value of Eq (4.206) expressed as τ_{22} has been exhibit in Figure 4.9, in the form of 3-dimensional , and 2-dimensional and contour plot which demonstrates singular periodic soliton by choosing parameters, $-10 \leq x \leq 10, t = 0..10, k = 0.4, c_2 = 1, k_1 = 3, k_2 = 1, k_3 = -10, l = 6, \sigma = 1, \beta_2 = -2, a_1 = 4, \lambda = 5, y = 2$.

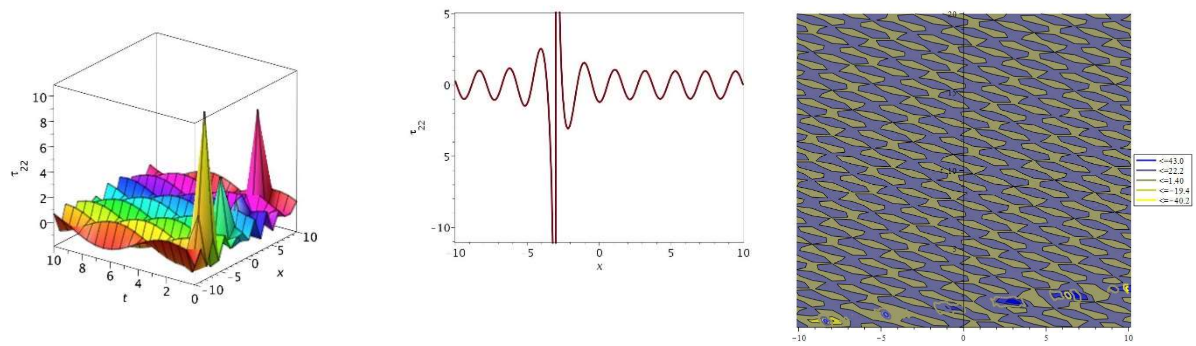


Figure 4.9:-graphs of periodic solitary wave solution τ_{22}

Graphical depiction of Real value of Eq (4.166) expressed as φ_{22} has been exhibit in Figure 4.10, in the form of 3-dimensional , and 2-dimensional and contour plot which demonstrates singular kink soliton by choosing parameters, $-10 \leq x \leq 10, t = 0..10, k = 0.4, c_2 = 1, k_1 = 3, k_2 = 1, k_3 = -10, l = 6, \sigma = 1, \beta_2 = -2, a_1 = 4, \lambda = 5, y = 2$.

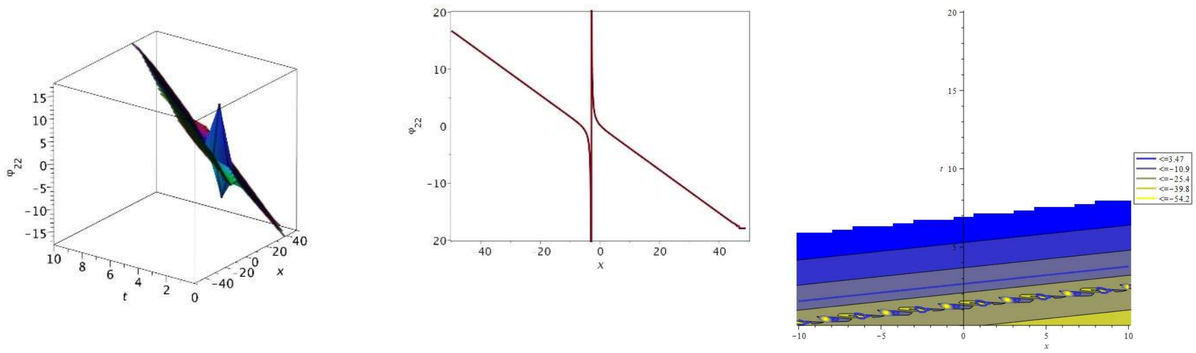


Figure 4.10:graphs of singular kink soliton φ_{22}

Graphical illustration of imaginary value of Eq (4.188) expressed as τ_{33} has been exhibit in Figure 4.11, in the form of 3-dimensional , and 2-dimensional and contour plot which demonstrates localized excitation wave pattern as soliton with parameters $-10 \leq x \leq 10, t = 0..10, k = 0.1, c_2 = 5, k_1 = 1, k_2 = 1, k_3 = 0.1, l = 2, \sigma = 1, a_1 = 4, \lambda = 1, y = 2.$

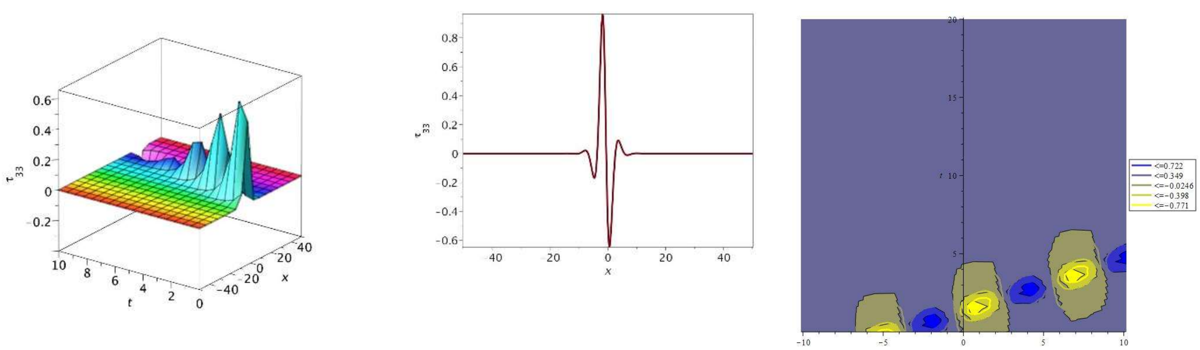


Figure 4.11: -graphical simulation of solitary wave solution τ_{33}

Graphical illustration of imaginary value of Eq (4.188) expressed as φ_{33} has been exhibit in Figure 4.12, in the form of 3-dimensional , and 2-dimensional and contour plot which demonstrates localized excitation wave pattern as kink soliton with parameters $-10 \leq x \leq 10, t = 0..10, k = 0.1, c_2 = 5, k_1 = 1, k_2 = 1, k_3 = 0.1, l = 2, \sigma = 1, a_1 = 4, \lambda = 1, y = 2.$

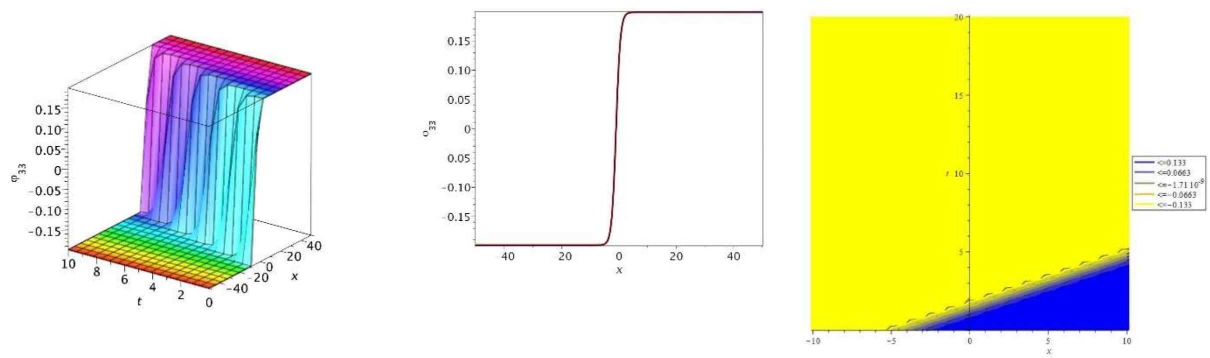


Figure 4.12: -graphical simulation of solitary wave solution φ_{33}.

4.7 Conclusion:

Optical solitons of Fokas system and (2+1)-Dimensional Davey-Stewartson equations have been investigated and analyzed by generalized auxiliary equation mapping method and thus, numerous types of exact solutions are obtained which includes hyperbolic, trigonometric, exponential, and rational solutions that exhibit bright and dark solitons, kink solitons, periodic wave, and singular solitons profiles. Furthermore, by choosing appropriate parameters in solutions, 3-D, 2-D and contour plots have been examined graphically to study dynamics and physical behavior of obtained solitons. Wave velocity and parameters involved in wave number are responsible for the types and profile of solitons. The applied technique has been recognized as efficient, robust, and useful in constructing optical solitons as it provides more generalized solutions. This technique has some advantages over previously studies techniques in literature as it depends on second degree differential equation and generates fourteen solutions that covers many types of soliton solutions and still this method is evolving and modifying continuously, also it can be applied on many nonlinear models to check their physical significance.

4.8 Summary:

In this chapter we have studied Fokas system and $(2+1)$ Davey-Stewartson (DS) system via generalized auxiliary equation mapping method. Obtained solutions are in the form of solitons. Solutions of both equations provide valuable insights of wave propagation, signal processing in optical fibers, imaging techniques and have applications in many areas such as mathematical physics, biology, and oceanography. The accuracy of the obtained results provides the efficiency of the method. Graphical simulation of these results has been discussed in the form of 3-D, 2-D and contour plots. This chapter consists of an introduction of governing equations along with main steps of methods used and derivation of solutions by proposed method. Finally graphical representation of some results followed by conclusion.

Chapter 5 includes interesting results of some fractional PDEs. FNLPDEs are used to model such phenomena where the dependent variable is reliant on more than one independent variable.

Chapter 5. Exact solutions of Fractional nonlinear PDEs by Improved generalized Riccati Equation mapping method.

5.1 Introduction:

The use of fractional calculus to model certain real-life phenomena is getting a great attention nowadays. Nonlinear fractional differential equations (NLFDEs) appear as a direct result of this attention. Nonlinear fractional partial differential equations (NLFPDEs) cover a major share of those NLFDEs, and they are used to model such phenomena where the dependent variable is reliant on more than one independent variable. NLFPDEs are generalizations of nonlinear partial differential equations (NPDEs) in which the orders of derivatives involved are fractional. These equations have numerous applications in different fields of engineering and physical sciences such as in fluid mechanic, fractional dynamics, and wave propagation etc. [168]. It is very important not only to formulate the governing FPDE of a certain phenomenon but also to find out its exact solutions. Solutions of an equation, governing a certain real-life phenomenon, give us very useful details of the phenomenon itself and can be used to understand and predict the variations in the depended variable (and the quantities driven by it).

In this study we are interested in a special type of exact solutions of NLFPDEs known as solitary wave solutions. Since solitons have been proved to be the exact solutions of a large class of NLPDEs, their complete understanding would lead us to a broad understanding of the real-life phenomena themselves. Some of the methods that are already being used to find solution of fractional order nonlinear partial differential equations are Homotopy perturbation method (HPM) [123], Variational iteration method (VIM) [126, 127], F-expansion method [128], Exp-function method [129, 130], Fan sub-equation method [131], $\left(\frac{G'}{G}\right)$-expansion method [132], Improved tan $(\frac{\phi}{2})$-expansion method [112], Exp $\left(-\phi(\xi)\right)$ method [133] and Kudryashov method [134] etc. Some of these methods provide exact solutions to NLFPDEs (like Exp-function method, Fan sub-equation method, $\left(\frac{G'}{G}\right)$-expansion method etc.) while the others provide series solution (like VIM and HPM). Nowadays mathematicians are trying to extend conventional methods to make them capable of solving fractional order partial differential equations. These extended methods would enable scientists working on fractional models to deal with them more effectively. Finding exact solutions of NLFPDEs used to be a herculean task, however, modern symbolic computation tools have made the task relatively easier. In a result of these computational tools, the efforts to extend the methods used to solve integer order NLPDEs to their fractional counterparts, and apply them to solve real life fractional models, have gain a tremendous popularity.

5.2 Illustrative Examples:

5.3 Space-time fractional nonlinear DDE for Murnaghan's rod:

In this section we apply improved generalized Riccati equation mapping method on space-time fractional nonlinear elastic inhomogeneous double dispersive equation for Murnaghan's rod which is given as:

$$D_t^{2\alpha}u(x,t)-\frac{E}{\rho}D_x^{2\alpha}u(x,t)$$
$$=\frac{\epsilon}{2}\left(\frac{1}{\theta}\left(l\beta D_x^{2\alpha}u^2(x,t)+\theta\nu^2 D_{tx}^{4\alpha}u(x,t)-b\delta\nu^2 D_x^{4\alpha}u(x,t)\right)\right), \tag{5.1}$$

where $u(x,t)$ is strain wave function, $b=\frac{M}{E}<1, l=\frac{B}{E}$ are combinations of the constant scale factors [169]. Parameter $0<\alpha\leq 1$, is the order of fractional time and space derivatives. Where $D_t^{\alpha}u$ and $D_x^{\alpha}u$ are the Caputo fractional derivative [36] of u with respect to t and x respectively. The doubly dispersive equation (DDE), which is an important nonlinear physical model describing the nonlinear wave propagation in the elastic inhomogeneous circular cylinder Murnaghan's rod. The global existence and blow-up of solutions for doubly dispersive equation was discussed by Harby et al. [170]. Cattani et al. [169] had used extended Sinh-Gordon equation expansion method (ShGEEM) and the modified $\exp(-\phi(\zeta))$-expansion function method, to find the topological, non-topological, singular, compound topological-non-topological bell-type and compound singular, soliton-like, singular periodic wave and exponential function solutions to the doubly dispersive equation for inhomogeneous Murnaghan's rod. Moreover, Baskonus et al [171] solved inhomogeneous Murnaghan's rod by F-expansion method and obtained Jacobi elliptic function solutions including bright and dark solitons, topological, non-topological, singular, periodic, their combinations and compound solitons.

Now, by using the following nonlinear fractional order wave transformation:

$$u(x,t)=U(\xi),$$

where,

$$\xi = \frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)},$$

the above mentioned NLFPDE can be transform into nonlinear ODE as follows:

$$\frac{\epsilon v^2(-\lambda^2\theta + b\delta)}{\theta}\left(-\frac{l\beta\big(U(\xi)\big)^2}{2v^2(-\lambda^2\theta + b\delta)} + \frac{\lambda^2\theta U(\xi)}{\epsilon v^2(-\lambda^2\theta + b\delta)} - \frac{EU(\xi)}{\epsilon v^2(-\lambda^2\theta + b\delta)}\right)$$
$$+\frac{1}{2}\frac{\epsilon v^2(-\lambda^2\theta + b\delta)U''(\xi)}{\theta}\left(-\frac{\lambda^2\theta}{-\lambda^2\theta + b\delta} + \frac{b\delta}{-\lambda^2\theta + b\delta}\right) = 0, \tag{5.2}$$

Eq. (5.2) obtained by applying integration process twice to the resulting equation and both of time consider constant of integration equal to zero. By using homogeneous balance principle between the highest order derivative and nonlinearity yields $N = 2$. Therefore, Eq. (5.2) has a solution,

$$U(\xi\,) = \frac{a_{-2}}{\big(\varphi(\xi)\big)^2} + \frac{a_{-1}}{\varphi(\xi)} + a_0 + a_1\varphi(\xi) + a_2\big(\varphi(\xi)\big)^2. \tag{5.3}$$

Now, substituting Eq.(5.3) along with Eq. (1.68) into Eq. (5.2) after collecting all terms with the same order in ϕ^i and ϕ^{-i}, where, $(i = 0,1,2, \ldots\ldots)$. and equating each coefficient to 0, we get a system of NL algebraic equations. Solving these equations yields the following non-trivial solutions.

Set 1 :

$$a_1 = 0, \qquad a_2 = 0, \qquad a_0 = \left(-2\frac{v^2(p^2 + 2rq)}{\beta l(2 + \epsilon(p^2 - 4rq)v^2)}\right),$$
$$a_{-1} = \left(-\frac{(12E - 12b\delta)prv^2}{\beta l(2 + \epsilon(p^2 - 4rq)v^2)}\right), \qquad a_{-2} = \left(-\frac{(12E - 12b\delta)r^2v^2}{\beta l(2 + \epsilon(p^2 - 4rq)v^2)}\right),$$
$$\lambda = \left(\sqrt{\frac{b\delta\epsilon p^2v^2 - 4b\delta\epsilon qrv^2 + 2E}{\theta(\epsilon p^2v^2 - 4\epsilon qrv^2 + 2)}}\right), \tag{5.4}$$

$$U_1(\xi) = a_0 + \frac{a_{-2}}{(\varphi(\xi))^2} + \frac{a_{-1}}{\varphi(\xi)}. \tag{5.5}$$

Set 2 :

$$a_1 = 0, \qquad a_2 = 0, \qquad a_0 = \left(12\frac{qrv^2(E - b\delta)}{\beta(\epsilon(p^2 - 4rq)v^2 - 2)l}\right),$$

$$a_{-1} = \left(\frac{(12E - 12b\delta)prv^2}{\beta(\epsilon(p^2 - 4rq)v^2 - 2)l}\right), \qquad a_{-2} = \left(12\frac{(E - b\delta)r^2v^2}{\beta(\epsilon(p^2 - 4rq)v^2 - 2)l}\right), \tag{5.6}$$

$$\lambda = \left(\sqrt{-\frac{2E - b\delta\epsilon p^2v^2 + 4b\delta\epsilon qrv^2}{\theta(\epsilon p^2v^2 - 4\epsilon qrv^2 - 2)}}\right),$$

$$U_2(\xi) = a_0 + \frac{a_{-2}}{(\varphi(\xi))^2} + \frac{a_{-1}}{\varphi(\xi)}. \tag{5.7}$$

Set 3 :

$$a_{-1} = 0, \qquad a_{-2} = 0, \qquad a_0 = \left(-2\frac{v^2(p^2 + 2rq)(E - b\delta)}{\beta l(2 + \epsilon(p^2 - 4rq)v^2)}\right),$$

$$a_1 = \left(-\frac{(12E - 12b\delta)pqv^2}{\beta l(2 + \epsilon(p^2 - 4rq)v^2)}\right), \qquad a_2 = \left(-\frac{(12E - 12b\delta)q^2v^2}{\beta l(2 + \epsilon(p^2 - 4rq)v^2)}\right), \tag{5.8}$$

$$\lambda = \left(\sqrt{\frac{b\delta\epsilon p^2v^2 - 4b\delta\epsilon qrv^2 + 2E}{\theta(\epsilon p^2v^2 - 4\epsilon qrv^2 + 2)}}\right),$$

$$U_3(\xi) = a_0 + a_1\varphi(\xi) + a_2(\varphi(\xi))^2. \tag{5.9}$$

Set 4 :

$$a_{-1} = 0, \qquad a_{-2} = 0, \qquad a_0 = \left(12\frac{qrv^2(-b\delta + E)}{\beta(-2 + \epsilon(p^2 - 4rq)v^2)l}\right),$$

$$a_1 = \left(\frac{(12E - 12b\delta)pqv^2}{\beta(-2 + \epsilon(p^2 - 4rq)v^2)l}\right), \qquad a_2 = \left(\frac{(12E - 12b\delta)q^2v^2}{\beta(-2 + \epsilon(p^2 - 4rq)v^2)l}\right), \tag{5.10}$$

$$\lambda = \left(\sqrt{-\frac{2E - b\delta\epsilon p^2v^2 + 4b\delta\epsilon qrv^2}{\theta(\epsilon p^2v^2 - 4\epsilon qrv^2 - 2)}}\right),$$

$$U_4(\xi) = a_0 + a_1\varphi(\xi) + a_2(\varphi(\xi))^2. \tag{5.11}$$

For the case 1, substituting the values from Eq. (5.4) into Eq. (5.5) along with the Riccati equations solutions, we can get many different types of solutions including solitary wave solutions, periodic wave solutions and rational solutions. Where,

$$\xi = \frac{x^{\alpha}}{\Gamma(1+\alpha)} - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}.$$

Family 1:

When $\Delta > 0$ and $pq \neq 0$ or $qr \neq 0$, the hyperbolic function solutions of Eq. (5.1) are as follows,

$$U_{1,1} = -\frac{48(E-b\delta)r^2 v^2 q^2}{(2+\epsilon(\Delta)v^2)l\beta\left(p+\sqrt{\Delta}\tanh\left(\frac{1}{2\sqrt{\Delta}\xi}\right)\right)^2} + \frac{24(E-b\delta)prv^2 q}{(2+\epsilon(\Delta)v^2)l\beta\left(p+\sqrt{\Delta}\tanh\left(\frac{1}{2\sqrt{\Delta}\xi}\right)\right)} - A_0, \tag{5.12}$$

$$U_{1,2} = -\frac{48(E-b\delta)r^2 v^2 q^2}{(2+\epsilon(\Delta)v^2)l\beta\left(p+\sqrt{\Delta}\coth\left(\frac{1}{2\sqrt{\Delta}\xi}\right)\right)^2} + \frac{24(E-b\delta)prv^2 q}{(2+\epsilon(\Delta)v^2)l\beta\left(p+\sqrt{\Delta}\coth\left(\frac{1}{2\sqrt{\Delta}\xi}\right)\right)} - A_0, \tag{5.13}$$

$$U_{1,3} = -\frac{48(E-b\delta)r^2 v^2 q^2}{(2+\epsilon(\Delta)v^2)l\beta\left(\begin{matrix}(p+\sqrt{\Delta})\times \\ (\tanh(\sqrt{\Delta}\xi) \pm i\,\mathrm{sech}(\sqrt{\Delta}\xi))\end{matrix}\right)^2} + \frac{24(E-b\delta)prv^2 q}{(2+\epsilon(\Delta)v^2)l\beta\left(\begin{matrix}(p+\sqrt{\Delta})\times \\ (\tanh(\sqrt{\Delta}\xi) \pm i\,\mathrm{sech}(\sqrt{\Delta}\xi))\end{matrix}\right)} - A_0, \tag{5.14}$$

$$U_{1,4} = -\frac{48(E-b\delta)r^2 v^2 q^2}{(2+\epsilon(\Delta)v^2)l\beta\left(\begin{matrix}(p+\sqrt{\Delta})\times \\ (\coth(\sqrt{\Delta}\xi) \pm \mathrm{csch}(\sqrt{\Delta}\xi))\end{matrix}\right)^2} + \frac{24(E-b\delta)prv^2 q}{(2+\epsilon(\Delta)v^2)l\beta\left(\begin{matrix}(p+\sqrt{\Delta})\times \\ (\coth(\sqrt{\Delta}\xi) \pm \mathrm{csch}(\sqrt{\Delta}\xi))\end{matrix}\right)} - A_0, \tag{5.15}$$

$$U_{1,5} = -\frac{48(E-b\delta)r^2 v^2 q^2}{\beta l(2+\epsilon(\Delta)v^2)\left(\begin{array}{c}\left(2p+\sqrt{\Delta}\right)\times\\ \tanh\left(\frac{\sqrt{\Delta}\xi}{4}\right)+\coth\left(\frac{\sqrt{\Delta}\xi}{4}\right)\end{array}\right)^2} + \frac{24(E-b\delta)prv^2 q}{\beta l(2+\epsilon(\Delta)v^2)\left(\begin{array}{c}\left(2p+\sqrt{\Delta}\right)\times\\ \tanh\left(\frac{\sqrt{\Delta}\xi}{4}\right)+\coth\left(\frac{\sqrt{\Delta}\xi}{4}\right)\end{array}\right)} - A_0, \tag{5.16}$$

$$U_{1,6} = -\frac{48(E-b\delta)r^2 v^2 q^2}{(2+\epsilon(\Delta)v^2)l\beta}\left(-p+\frac{\begin{array}{c}\sqrt{(A^2+B^2)(\Delta)}\\ -A\sqrt{\Delta}\cosh\left(\sqrt{\Delta}\xi\right)\end{array}}{\text{A}\sinh\left(\sqrt{\Delta}\xi\right)+B}\right)^{-2} - \frac{24(E-b\delta)prv^2 q}{(2+\epsilon(\Delta)v^2)l\beta}\left(-p+\frac{\begin{array}{c}\sqrt{(A^2+B^2)(\Delta)}\\ -A\sqrt{\Delta}\cosh\left(\sqrt{\Delta}\xi\right)\end{array}}{\text{A}\sinh\left(\sqrt{\Delta}\xi\right)+B}\right)^{-1} - A_0, \tag{5.17}$$

$$U_{1,7} = -\frac{48(E-b\delta)r^2 v^2 q^2}{\beta l(2+\epsilon(\Delta)v^2)}\left(-p-\frac{\begin{array}{c}\sqrt{(-A^2+B^2)(\Delta)}\\ +A\sqrt{\Delta}\sinh\left(\sqrt{\Delta}\xi\right)\end{array}}{\text{A}\cosh\left(\sqrt{\Delta}\xi\right)+B}\right)^{-2} - \frac{24(E-b\delta)prv^2 q}{\beta l(2+\epsilon(\Delta)v^2)}\left(-p-\frac{\begin{array}{c}\sqrt{(-A^2+B^2)(\Delta)}\\ +A\sqrt{\Delta}\sinh\left(\sqrt{\Delta}\xi\right)\end{array}}{\text{A}\cosh\left(\sqrt{\Delta}\xi\right)+B}\right)^{-1} - A_0. \tag{5.18}$$

Where two non-zero real constants A and B satisfies $A^2 - B^2 > 0$.

$$U_{1,8} = -\frac{3(E-b\delta)v^2\begin{pmatrix}\sqrt{\Delta}\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\\ -\mathrm{p}\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\end{pmatrix}^2}{\beta l(2+\epsilon(\Delta)v^2)\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)^2} - \frac{6(E-b\delta)pv^2\begin{pmatrix}\sqrt{\Delta}\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\\ -\mathrm{p}\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\end{pmatrix}}{\beta l(2+\epsilon(\Delta)v^2)\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)} - A_0, \tag{5.19}$$

$$U_{1,9} = -\frac{3(E-b\delta)v^2\begin{pmatrix}-\sqrt{\Delta}\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\\ +\mathrm{p}\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\end{pmatrix}^2}{(2+\epsilon(\Delta)v^2)\beta l\left(\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)^2} + \frac{6(E-b\delta)pv^2\begin{pmatrix}-\sqrt{\Delta}\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\\ +\mathrm{p}\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\end{pmatrix}}{(2+\epsilon(\Delta)v^2)\beta\ \mathrm{l}\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)} - A_0, \tag{5.20}$$

$$U_{1,10} = -\frac{3(E-b\delta)v^2\begin{pmatrix}(\sqrt{\Delta}\sinh(\sqrt{\Delta}\xi)-\\ p(\cosh(\sqrt{\Delta}\xi)\pm i\sqrt{\Delta})\end{pmatrix}^2}{\beta l(2+\epsilon(\Delta)v^2)\left(\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)^2} - \frac{6(E-b\delta)pv^2\begin{pmatrix}(\sqrt{\Delta}\sinh(\sqrt{\Delta}\xi)-\\ p(\cosh(\sqrt{\Delta}\xi)\pm i\sqrt{\Delta})\end{pmatrix}}{\beta l(2+\epsilon(\Delta)v^2)\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)} - A_0, \tag{5.21}$$

$$U_{1,11} = -\frac{3(E-b\delta)v^2\begin{pmatrix}(\sqrt{\Delta}\cosh(\sqrt{\Delta}\xi))\\ -\mathrm{psinh}(\sqrt{\Delta}\xi)\pm\sqrt{\Delta}))\end{pmatrix}^2}{(2+\epsilon(\Delta)v^2)\beta l\left(\sinh\frac{\sqrt{\Delta}\xi}{2}\right)^2} - \frac{6(E-b\delta)pv^2\begin{pmatrix}(\sqrt{\Delta}\cosh(\sqrt{\Delta}\xi))\\ -\mathrm{psinh}(\sqrt{\Delta}\xi)\pm\sqrt{\Delta}))\end{pmatrix}}{(2+\epsilon(\Delta)v^2)\beta l\left(\sinh\frac{\sqrt{\Delta}\xi}{2}\right)} - A_0, \tag{5.22}$$

$$U_{1,12} = -\frac{3(E-b\delta)v^2\begin{pmatrix}-2p\sinh\left(\frac{\sqrt{\Delta}\xi}{4}\right)\cosh\left(\frac{\sqrt{\Delta}\xi}{4}\right)\\ +2\sqrt{\Delta}\cosh\left(\frac{\sqrt{\Delta}\xi}{4}\right)^2-\sqrt{\Delta}\end{pmatrix}^2}{(8+4\epsilon(\Delta)v^2)\beta l\sinh\left(\frac{\sqrt{\Delta}\xi}{4}\right)^2\cosh\left(\frac{\sqrt{\Delta}\xi}{4}\right)^2} - \frac{6(E-b\delta)pv^2\begin{pmatrix}-2p\sinh\left(\frac{\sqrt{\Delta}\xi}{4}\right)\cosh\left(\frac{\sqrt{\Delta}\xi}{4}\right)\\ +2\sqrt{\Delta}\cosh\left(\frac{\sqrt{\Delta}\xi}{4}\right)^2-\sqrt{\Delta}\end{pmatrix}}{2(2+\epsilon(\Delta)v^2)\beta\,\mathrm{lsinh}\left(\frac{\sqrt{\Delta}\xi}{4}\right)\cosh\left(\frac{\sqrt{\Delta}\xi}{4}\right)} - A_0. \tag{5.23}$$

Family 2:

If $\Delta < 0$ and $pq \neq 0$ $(or\ qr \neq 0)$, we have the following trigonometric solutions.

$$U_{1,13}=-\frac{48(E-b\delta)r^2v^2q^2}{\beta(2+\epsilon(\Delta)v^2)l\left(-p+\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)^2}-\frac{24(E-b\delta)prv^2q}{\beta(2+\epsilon(\Delta)v^2)l\left(-p+\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}-\mathrm{A}_0, \tag{5.24}$$

$$U_{1,14}=-\frac{48(E-b\delta)r^2v^2q^2}{\beta(2+\epsilon(\Delta)v^2)l\left(p+\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)^2}+\frac{24(E-b\delta)prv^2q}{\beta(2+\epsilon(\Delta)v^2)l\left(p+\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}-\mathrm{A}_0, \tag{5.25}$$

$$U_{1,15}=-\frac{48(E-b\delta)r^2v^2q^2}{\beta(2+\epsilon(\Delta)v^2)l\left(\begin{matrix}(-p+\sqrt{-\Delta})\times\\(\tan(\sqrt{-\Delta}\xi)\pm\sec(\sqrt{-\Delta}\xi))\end{matrix}\right)^2}-\frac{24(E-b\delta)prv^2q}{\beta(2+\epsilon(\Delta)v^2)l\left(\begin{matrix}(-p+\sqrt{-\Delta})\times\\(\tan(\sqrt{-\Delta}\xi)\pm\sec(\sqrt{-\Delta}\xi))\end{matrix}\right)}-\mathrm{A}_0, \tag{5.26}$$

$$U_{1,16}=-\frac{48(E-b\delta)r^2v^2q^2}{\beta(2+\epsilon(\Delta)v^2)l\left(\begin{matrix}\left(p+\sqrt{-\Delta}\right)\times\\\cot(\sqrt{-\Delta}\xi)\pm\csc(\sqrt{-\Delta}\xi)\end{matrix}\right)^2}+\frac{24(E-b\delta)prv^2q}{\beta(2+\epsilon(\Delta)v^2)l\left(\begin{matrix}\left(p+\sqrt{-\Delta}\right)\times\\(\cot(\sqrt{-\Delta}\xi)\pm\csc(\sqrt{-\Delta}\xi)))\end{matrix}\right)}-\mathrm{A}_0, \tag{5.27}$$

$$U_{1,17} = -\frac{192(E-b\delta)r^2v^2q^2}{\beta(2+\epsilon(\Delta)v^2)l\left(\begin{array}{c}\left(-2p+\sqrt{-\Delta}\right)\times\\ \left(\tan\left(\frac{\sqrt{-\Delta}\xi}{4}\right)-\cot\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\right)\end{array}\right)^2}$$

$$-\frac{48(E-b\delta)r^2v^2q^2}{\beta(2+\epsilon(\Delta)v^2)l\left(\begin{array}{c}\left(-2p+\sqrt{-\Delta}\right)\times\\ \left(\tan\left(\frac{\sqrt{-\Delta}\xi}{4}\right)-\cot\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\right)\end{array}\right)} \tag{5.28}$$

$$-\mathrm{A}_0,$$

$$U_{1,18} = -\frac{48(E-b\delta)r^2v^2q^2}{\beta(2+\epsilon(\Delta)v^2)l}\left(-p+\frac{\begin{array}{c}\sqrt{(A^2-B^2)(-\Delta)}-\\ A\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)\end{array}}{\mathrm{A}\sin\left(\sqrt{-\Delta}\xi\right)+B}\right)^{-2}$$

$$-\frac{24(E-b\delta)prv^2q}{\beta(2+\epsilon(\Delta)v^2)l}\left(-p+\frac{\begin{array}{c}\sqrt{(A^2-B^2)(-\Delta)}-\\ A\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)\end{array}}{\mathrm{A}\sin\left(\sqrt{-\Delta}\xi\right)+B}\right)^{-1} \tag{5.29}$$

$$-\mathrm{A}_0,$$

$$U_{1,19} = -\frac{48(E-b\delta)r^2v^2q^2}{\beta(2+\epsilon(\Delta)v^2)l}\left(-p+\frac{\begin{array}{c}\sqrt{(A^2-B^2)(-\Delta)}+\\ A\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)\end{array}}{\mathrm{A}\sin\left(\sqrt{-\Delta}\xi\right)+B}\right)^{-2}$$

$$-\frac{24(E-b\delta)prv^2q}{\beta(2+\epsilon(\Delta)v^2)l}\left(-p+\frac{\begin{array}{c}\sqrt{(A^2-B^2)(-\Delta)}+\\ A\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)\end{array}}{\mathrm{A}\sin\left(\sqrt{-\Delta}\xi\right)+B}\right)^{-1} \tag{5.30}$$

$$-\mathrm{A}_0.$$

Where two non-zero real constants A and B satisfies $A^2 - B^2 > 0$.

$$U_{1,20} = -\frac{3(E - b\delta)v^2\left(\begin{array}{c}\left(\sqrt{-\Delta}\right)\times\\ \sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right) + \mathrm{p}\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\end{array}\right)^2}{\beta(2 + \epsilon(\Delta)v^2)l\left(\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)^2} + \frac{6(E - b\delta)pv^2\left(\begin{array}{c}\left(\sqrt{-\Delta}\right)\times\\ \sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right) + \mathrm{p}\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\end{array}\right)}{\beta(2 + \epsilon(\Delta)v^2)\,\mathrm{l}\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)} - \mathrm{A}_0, \tag{5.31}$$

$$U_{1,21} = -\frac{3(E - b\delta)v^2\left(\begin{array}{c}\left(\sqrt{-\Delta}\right)\times\\ \cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right) - \mathrm{p}\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\end{array}\right)^2}{\beta(2 + \epsilon(\Delta)v^2)l\left(\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)^2} - \frac{6(E - b\delta)pv^2\left(\begin{array}{c}\left(\sqrt{-\Delta}\right)\times\\ \cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right) - \mathrm{p}\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\end{array}\right)}{\beta(2 + \epsilon(\Delta)v^2)\,\mathrm{l}\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)} - \mathrm{A}_0, \tag{5.32}$$

$$U_{1,22}=-\frac{3(E-b\delta)\upsilon^2\begin{pmatrix}\sqrt{-\Delta}\sin\left(\sqrt{-\Delta}\xi\right)\\+p\cos\left(\sqrt{-\Delta}\xi\right)\pm\sqrt{-\Delta}\end{pmatrix}^2}{\beta(2+\epsilon(\Delta)\upsilon^2)l\left(\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)^2}+\frac{6(E-b\delta)p\upsilon^2\begin{pmatrix}\sqrt{-\Delta}\sin\left(\sqrt{-\Delta}\xi\right)\\+p\cos\left(\sqrt{p^2-4qr}\xi\right)\pm\sqrt{-\Delta}\end{pmatrix}}{\beta(2+\epsilon(\Delta)\upsilon^2)\,\mathrm{l}\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)}-\mathrm{A}_0, \tag{5.33}$$

$$U_{1,23}=-\frac{3(E-b\delta)\upsilon^2\begin{pmatrix}\left(\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)\right)\\-p\sin\left(\sqrt{-\Delta}\xi\right)\pm\sqrt{-\Delta}\end{pmatrix}^2}{\beta(2+\epsilon(\Delta)\upsilon^2)l\left(\sin\left(\sqrt{-\Delta}\xi\right)\right)^2}-\frac{6(E-b\delta)p\upsilon^2\begin{pmatrix}\left(\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)\right)\\-p\sin\left(\sqrt{-\Delta}\xi\right)\pm\sqrt{-\Delta}\end{pmatrix}}{\beta(2+\epsilon(\Delta)\upsilon^2)\,\mathrm{l}\sin\left(\sqrt{-\Delta}\xi\right)}-\mathrm{A}_0, \tag{5.34}$$

$$U_{1,24}=-\frac{3(E-b\delta)\upsilon^2\begin{pmatrix}\left(-2p\sin\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\cos\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\right)\\+2\sqrt{-\Delta}\left(\cos\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\right)^2\\-\sqrt{-\Delta}\end{pmatrix}^2}{4\beta(2+\epsilon(\Delta)\upsilon^2)l\left(\sin\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\right)^2\left(\cos\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\right)^2}-\frac{6(E-b\delta)p\upsilon^2\begin{pmatrix}\left(-2p\sin\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\cos\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\right)\\+2\sqrt{-\Delta}\left(\cos\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\right)^2\\-\sqrt{4qr-p^2}\end{pmatrix}}{2\beta(2+\epsilon(\Delta)\upsilon^2)\,\mathrm{l}\sin\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\cos\left(\frac{\sqrt{-\Delta}\xi}{4}\right)}-\mathrm{A}_0. \tag{5.35}$$

Where $A_0 = 2\frac{v^2(p^2+2qr)(-b\delta+E)}{(2+\epsilon(p^2-4qr)v^2)l\beta}$.

In case 2, substituting values from Eq. (5.6) and Riccati equation solutions in Eq. (5.7) with

$$\xi = \frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)},$$

Family1:

When $\Delta > 0$ and $pq \neq 0$ or $qr \neq 0$, the hyperbolic function solutions of Eq. (5.1) are as follows,

$$U_{2,1} = \frac{(-48b\delta + 48E)r^2v^2q^2}{\beta l\left(-2 + \epsilon(\Delta)v^2\right)\left(p + \left(\sqrt{\Delta} \times \tanh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\right)^2} - \frac{(-24b\delta + 24E)prv^2q}{\beta l\left(-2 + \epsilon(\Delta)v^2\right)\left(p + \left(\sqrt{\Delta} \times \tanh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\right)} + 12\frac{qrv^2(-b\delta + E)}{\beta l(-2 + \epsilon(\Delta)v^2)}, \tag{5.36}$$

$$U_{2,2} = \frac{(-48b\delta + 48E)r^2v^2q^2}{\beta l(-2 + \epsilon(\Delta)v^2)\left(p + \left(\sqrt{\Delta} \times \coth\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\right)^2} - \frac{(-24b\delta + 24E)prv^2q}{\beta l(-2 + \epsilon(\Delta)v^2)\left(p + \left(\sqrt{\Delta} \times \coth\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\right)} + 12\frac{qrv^2(-b\delta + E)}{\beta l(-2 + \epsilon(\Delta)v^2)}, \tag{5.37}$$

$$U_{2,3} = \frac{(-48b\delta + 48E)r^2 v^2 q^2}{\beta l(-2+\epsilon(\Delta)v^2)\left(\begin{array}{c}(p+\sqrt{\Delta})\times \\ \left(\tanh\left(\sqrt{\Delta}\xi\right) \pm \mathrm{isech}\left(\sqrt{\Delta}\xi\right)\right)\end{array}\right)^2}$$
$$- \frac{(-24b\delta + 24E)prv^2 q}{\beta l(-2+\epsilon(\Delta)v^2)\left(\begin{array}{c}\left(p+\sqrt{\Delta}\right)\times \\ \left(\tanh\left(\sqrt{\Delta}\xi\right) \pm \mathrm{isech}\left(\sqrt{\Delta}\xi\right)\right)\end{array}\right)} + 12\frac{qrv^2(-b\delta+E)}{\beta l(-2+\epsilon(\Delta)v^2)}, \tag{5.38}$$

$$U_{2,4} = \frac{(-48b\delta + 48E)r^2 v^2 q^2}{\beta l(-2+\epsilon(\Delta)v^2)\left(\begin{array}{c}\left(p+\sqrt{\Delta}\right)\times \\ \left(\coth\left(\sqrt{\Delta}\xi\right) \pm \mathrm{csch}\left(\sqrt{\Delta}\xi\right)\right)\end{array}\right)^2}$$
$$- \frac{(-24b\delta + 24E)prv^2 q}{\beta l(-2+\epsilon(\Delta)v^2)\left(\begin{array}{c}\left(p+\sqrt{\Delta}\right)\times \\ \left(\coth\left(\sqrt{\Delta}\xi\right) \pm \mathrm{csch}\left(\sqrt{\Delta}\xi\right)\right)\end{array}\right)} + 12\frac{qrv^2(-b\delta+E)}{\beta l(-2+\epsilon(\Delta)v^2)}, \tag{5.39}$$

$$U_{2,5} = \frac{(-48b\delta + 48E)r^2 v^2 q^2}{\beta l(-2+\epsilon(\Delta)v^2)\times \left(\begin{array}{c}\left(2p+\sqrt{\Delta}\right)\times \\ \left(\tanh\left(\frac{\sqrt{\Delta}\xi}{4}\right) + \coth\left(\frac{\sqrt{\Delta}\xi}{4}\right)\right)\end{array}\right)^2}$$
$$- \frac{(-24b\delta + 24E)prv^2 q}{\beta l(-2+\epsilon(\Delta)v^2) \left(\begin{array}{c}\left(2p+\sqrt{\Delta}\right)\times \\ \left(\tanh\left(\frac{\sqrt{\Delta}\xi}{4}\right) + \coth\left(\frac{\sqrt{\Delta}\xi}{4}\right)\right)\end{array}\right)} + 12\frac{qrv^2(-b\delta+E)}{\beta l(-2+\epsilon(\Delta)v^2)}, \tag{5.40}$$

$$U_{2,6}=\frac{(-48b\delta+48E)r^2v^2q^2}{\beta l(-2+\epsilon(\Delta)v^2)}\left(-p+\frac{\begin{pmatrix}\sqrt{(A^2+B^2)(\Delta)}-\\ A\sqrt{\Delta}\cosh\left(\sqrt{\Delta}\xi\right)\end{pmatrix}}{\text{Asinh}\left(\sqrt{\Delta}\xi\right)+B}\right)^{-2}$$

$$+\frac{(-24b\delta+24E)prv^2q}{\beta l(-2+\epsilon(\Delta)v^2)}\left(-p+\frac{\begin{pmatrix}\sqrt{(A^2+B^2)(\Delta)}\\ -A\sqrt{\Delta}\cosh\left(\sqrt{\Delta}\xi\right)\end{pmatrix}}{\text{Asinh}\left(\sqrt{\Delta}\xi\right)+B}\right)^{-1} \tag{5.41}$$

$$+12\frac{qrv^2(-b\delta+E)}{\beta l(-2+\epsilon(\Delta)v^2)},$$

$$U_{2,7}=\frac{(-48b\delta+48E)r^2v^2q^2}{\beta l(-2+\epsilon(p^2-4qr)v^2)}\times$$

$$\left(-p-\frac{\begin{pmatrix}\sqrt{(-A^2+B^2)(\Delta)}+\\ A\sqrt{\Delta}\sinh\left(\sqrt{\Delta}\xi\right)\end{pmatrix}}{\text{Acosh}\left(\sqrt{\Delta}\xi\right)+B}\right)^{-2}+\frac{(-24b\delta+24E)prv^2q}{\beta l(-2+\epsilon(\Delta)v^2)}\times \tag{5.42}$$

$$\left(-p-\frac{\begin{pmatrix}\sqrt{(-A^2+B^2)(\Delta)}+\\ A\sqrt{\Delta}\sinh\left(\sqrt{\Delta}\xi\right)\end{pmatrix}}{\text{Acosh}\left(\sqrt{\Delta}\xi\right)+B}\right)^{-1}+12\frac{qrv^2(-b\delta+E)}{\beta l(-2+\epsilon(\Delta)v^2)}.$$

Where two non-zero real constants A and B satisfies $A^2-B^2>0$.

$$U_{2,8} = \frac{(-3b\delta + 3E)v^2 \left(\sqrt{\Delta}\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right) - \mathrm{p}\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)^2}{\beta l(-2 + \epsilon(\Delta)v^2)\left(\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)^2} + \frac{(-6b\delta + 6E)pv^2 \left(\sqrt{\Delta}\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right) - \mathrm{p}\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}{\beta l(-2 + \epsilon(\Delta)v^2)\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)} + 12\frac{qrv^2(-b\delta + E)}{\beta l(-2 + \epsilon(\Delta)v^2)}, \tag{5.43}$$

$$U_{2,9} = \frac{(-3b\delta + 3E)v^2 \left(-\sqrt{\Delta}\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right) + \mathrm{p}\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)^2}{\beta l(-2 + \epsilon(\Delta)v^2)\left(\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)^2} - \frac{(-6b\delta + 6E)pv^2 \left(-\sqrt{\Delta}\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right) + \mathrm{p}\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}{\beta l(-2 + \epsilon(\Delta)v^2)\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)} + 12\frac{qrv^2(-b\delta + E)}{\beta l(-2 + \epsilon(\Delta)v^2)}, \tag{5.44}$$

$$U_{2,10} = \frac{(-3b\delta + 3E)v^2 \begin{pmatrix} (\sqrt{\Delta}\sinh(\sqrt{\Delta}\xi) \\ -p(\cosh(\sqrt{\Delta}\xi) \pm i\sqrt{\Delta}) \end{pmatrix}^2}{\beta l(-2 + \epsilon(\Delta)v^2)\left(\cosh\left(\frac{1}{2\sqrt{\Delta}\xi}\right)\right)^2} + \frac{(-6b\delta + 6E)pv^2 \begin{pmatrix} (\sqrt{\Delta}\sinh(\sqrt{\Delta}\xi) \\ -p(\cosh(\sqrt{\Delta}\xi) \pm i\sqrt{\Delta}) \end{pmatrix}}{\beta l(-2 + \epsilon(\Delta)v^2)\cosh\left(\frac{1}{2\sqrt{\Delta}\xi}\right)} + 12\frac{qrv^2(-b\delta + E)}{\beta l(-2 + \epsilon(\Delta)v^2)}, \tag{5.45}$$

$$U_{2,11} = \frac{(-3b\delta + 3E)v^2 \begin{pmatrix} \sqrt{\Delta}\cosh(\sqrt{\Delta}\xi) \\ -p(\sinh(\sqrt{\Delta}\xi) \pm \sqrt{\Delta}) \end{pmatrix}^2}{\beta l(-2 + \epsilon(\Delta)v^2)\left(\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)^2} + \frac{(-6b\delta + 6E)pv^2 \begin{pmatrix} \sqrt{\Delta}\cosh(\sqrt{\Delta}\xi) \\ -p(\sinh(\sqrt{\Delta}\xi) \pm \sqrt{\Delta}) \end{pmatrix}}{\beta l(-2 + \epsilon(\Delta)v^2)\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)} + 12\frac{qrv^2(-b\delta + E)}{\beta l(-2 + \epsilon(\Delta)v^2)}, \tag{5.46}$$

$$U_{2,12} = \frac{(-3b\delta + 3E)v^2 \begin{pmatrix} \left(-p\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right) \\ +2\sqrt{\Delta}\left(\cosh\left(\frac{1}{4\sqrt{\Delta}\xi}\right)\right)^2 - \sqrt{\Delta} \end{pmatrix}^2}{4\,\beta l(-2 + \epsilon(\Delta)v^2)\left(\frac{1}{2}\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)^2} + \frac{(-3b\delta + 3E)pv^2 \begin{pmatrix} \left(-p\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right) \\ +2\sqrt{\Delta}\left(\cosh\left(\frac{\sqrt{\Delta}\xi}{4}\right)\right)^2 - \sqrt{\Delta} \end{pmatrix}}{\beta l(-2 + \epsilon(p^2 - 4qr)v^2)\frac{1}{2}\sinh\left(\frac{\sqrt{\Delta}\xi}{4}\right)} + 12\frac{qrv^2(-b\delta + E)}{\beta l(-2 + \epsilon(\Delta)v^2)}. \tag{5.47}$$

Family2:

If $\Delta < 0$ and $pq \neq 0$ $(or\ qr \neq 0)$, we have the following trigonometric solutions:

$$U_{2,13} = \frac{(-48b\delta + 48E)r^2 v^2 q^2}{l\beta(-2 + \epsilon(\Delta)v^2)\left(-p + \sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)^2} + \frac{(-24b\delta + 24E)prv^2 q}{l\beta(-2 + \epsilon(\Delta)v^2)\left(-p + \sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)} + 12\frac{qrv^2(-b\delta + E)}{l\beta(-2 + \epsilon(\Delta)v^2)}, \tag{5.48}$$

$$U_{2,14} = \frac{(-48b\delta + 48E)r^2 v^2 q^2}{l\beta(-2 + \epsilon(\Delta)v^2)\left(p + \sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)^2} - \frac{(-24b\delta + 24E)prv^2 q}{l\beta(-2 + \epsilon(\Delta)v^2)\left(p + \sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)} + 12\frac{qrv^2(-b\delta + E)}{l\beta(-2 + \epsilon(\Delta)v^2)}, \tag{5.49}$$

$$U_{2,15} = \frac{(-48b\delta + 48E)r^2 v^2 q^2}{l\beta(-2 + \epsilon(\Delta)v^2)\left(\begin{matrix}\left(-p + \sqrt{-\Delta}\right)\times \\ \left(\tan\left(\sqrt{-\Delta}\xi\right) \pm \sec\left(\sqrt{-\Delta}\xi\right)\right)\end{matrix}\right)^2} + \frac{(-24b\delta + 24E)prv^2 q}{l\beta(-2 + \epsilon(\Delta)v^2)\left(\begin{matrix}\left(-p + \sqrt{-\Delta}\right)\times \\ \left(\tan\left(\sqrt{-\Delta}\xi\right) \pm \sec\left(\sqrt{-\Delta}\xi\right)\right)\end{matrix}\right)} + 12\frac{qrv^2(-b\delta + E)}{l\beta(-2 + \epsilon(\Delta)v^2)}, \tag{5.50}$$

$$U_{2,16} = \frac{(-48b\delta + 48E)r^2 v^2 q^2}{l\beta(-2 + \epsilon(\Delta)v^2)\left(\begin{matrix}\left(p + \sqrt{-\Delta}\right)\times \\ \left(\cot\left(\sqrt{-\Delta}\xi\right) \pm \csc\left(\sqrt{-\Delta}\xi\right)\right)\end{matrix}\right)^2} - \frac{(-24b\delta + 24E)prv^2 q}{l\beta(-2 + \epsilon(\Delta)v^2)\left(\begin{matrix}\left(p + \sqrt{-\Delta}\right)\times \\ \left(\cot\left(\sqrt{-\Delta}\xi\right) \pm \csc\left(\sqrt{-\Delta}\xi\right)\right)\end{matrix}\right)} + 12\frac{qrv^2(-b\delta + E)}{l\beta(-2 + \epsilon(\Delta)v^2)}, \tag{5.51}$$

$$U_{2,17}=\frac{(-192b\delta+192E)r^2v^2q^2}{l\beta(-2+\epsilon(\Delta)v^2)\left(\begin{array}{c}\left(-2p+\sqrt{-\Delta}\right)\times\\ \left(\tan\left(\frac{\sqrt{-\Delta}\xi}{4}\right)-\cot\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\right)\end{array}\right)^2}$$
$$+\frac{(-48b\delta+48E)prv^2q}{l\beta(-2+\epsilon(\Delta)v^2)\left(\begin{array}{c}\left(-2p+\sqrt{-\Delta}\right)\times\\ \left(\tan\left(\frac{\sqrt{-\Delta}\xi}{4}\right)-\cot\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\right)\end{array}\right)}+12\frac{qrv^2(-b\delta+E)}{l\beta(-2+\epsilon(\Delta)v^2)}, \quad (5.52)$$

$$U_{2,18}=\frac{(-48b\delta+48E)r^2v^2q^2}{l\beta(-2+\epsilon(\Delta)v^2)}\times\left(-p+\frac{\left(\begin{array}{c}\pm i\sqrt{(-A^2+B^2)(-\Delta)}\\ -A\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)\end{array}\right)}{A\sin\left(\sqrt{-\Delta}\xi\right)+B}\right)^{-2}$$
$$+\frac{(-24b\delta+24E)prv^2q}{l\beta(-2+\epsilon(\Delta)v^2)}\times\left(-p+\frac{\left(\begin{array}{c}\pm i\sqrt{(-A^2+B^2)(-\Delta)}\\ -A\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)\end{array}\right)}{A\sin\left(\sqrt{-\Delta}\xi\right)+B}\right)^{-1} \quad (5.53)$$
$$+12\frac{qrv^2(-b\delta+E)}{l\beta(-2+\epsilon(\Delta)v^2)},$$

$$U_{2,19}=\frac{(-48b\delta+48E)r^2v^2q^2}{l\beta(-2+\epsilon(\Delta)v^2)}\left(-p+\frac{\left(\begin{array}{c}\pm i\sqrt{(-A^2+B^2)(-\Delta)}\\ -A\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)\end{array}\right)}{A\sin\left(\sqrt{-\Delta}\xi\right)+B}\right)^{-2}$$
$$+\frac{(-24b\delta+24E)prv^2q}{l\beta(-2+\epsilon(\Delta)v^2)}\left(-p+\frac{\left(\begin{array}{c}\pm i\sqrt{(-A^2+B^2)(-\Delta)}\\ -A\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)\end{array}\right)}{A\sin\left(\sqrt{-\Delta}\xi\right)+B}\right)^{-1} \quad (5.54)$$
$$+12\frac{qrv^2(-b\delta+E)}{l\beta(-2+\epsilon(\Delta)v^2)}.$$

Where two non-zero real constants A and B satisfies $A^2-B^2>0$.

$$U_{2,20}=\frac{(-3b\delta+3E)v^2\left(\sqrt{-\Delta}\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)+\mathrm{p}\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)^2}{l\beta(-2+\epsilon(\Delta)v^2)\left(\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)^2}-\frac{(-6b\delta+6E)pv^2\left(\sqrt{-\Delta}\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)+\mathrm{p}\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}{l\beta(-2+\epsilon(\Delta)v^2)\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)}+12\frac{qrv^2(-b\delta+E)}{l\beta(-2+\epsilon(\Delta)v^2)}, \tag{5.55}$$

$$U_{2,21}=\frac{(-3b\delta+3E)v^2\left(\sqrt{-\Delta}\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)-\mathrm{p}\sin\left(\frac{\sqrt{-p^2+4qr}\xi}{2}\right)\right)^2}{l\beta(-2+\epsilon(\Delta)v^2)\left(\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)^2}+\frac{(-6b\delta+6E)pv^2\left(\sqrt{-\Delta}\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)-\mathrm{p}\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}{l\beta(-2+\epsilon(\Delta)v^2)\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)}+12\frac{qrv^2(-b\delta+E)}{l\beta(-2+\epsilon(\Delta)v^2)}, \tag{5.56}$$

$$U_{2,22}=\frac{(-3b\delta+3E)\upsilon^2\begin{pmatrix}\sqrt{-\Delta}\sin\left(\sqrt{-\Delta}\xi\right)\\+p\cos\left(\sqrt{\Delta}\xi\right)\pm\sqrt{-\Delta}\end{pmatrix}^2}{l\beta(-2+\epsilon(\Delta)\upsilon^2)\left(\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)^2}-\frac{(-6b\delta+6E)p\upsilon^2\begin{pmatrix}\sqrt{-\Delta}\sin\left(\sqrt{-\Delta}\xi\right)\\+p\cos\left(\sqrt{\Delta}\xi\right)\pm\sqrt{-\Delta}\end{pmatrix}}{l\beta(-2+\epsilon(\Delta)\upsilon^2)\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)}+12\frac{qr\upsilon^2(-b\delta+E)}{l\beta(-2+\epsilon(\Delta)\upsilon^2)},\tag{5.57}$$

$$U_{2,23}=\frac{(-3b\delta+3E)\upsilon^2\begin{pmatrix}\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)\\-p\sin\left(\sqrt{-\Delta}\xi\right)\pm\sqrt{-\Delta}\end{pmatrix}^2}{l\beta(-2+\epsilon(\Delta)\upsilon^2)\left(\sin\left(\sqrt{-\Delta}\xi\right)\right)^2}+\frac{(-6b\delta+6E)p\upsilon^2\begin{pmatrix}\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)\\-p\sin\left(\sqrt{-\Delta}\xi\right)\pm\sqrt{-\Delta}\end{pmatrix}}{l\beta(-2+\epsilon(\Delta)\upsilon^2)\sin\left(\sqrt{-\Delta}\xi\right)}+12\frac{qr\upsilon^2(-b\delta+E)}{l\beta(-2+\epsilon(\Delta)\upsilon^2)},\tag{5.58}$$

$$U_{2,24}=\frac{(-3b\delta+3E)\upsilon^2\begin{pmatrix}\left(-p\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)-\sqrt{-\Delta}\\+2\sqrt{-\Delta}\left(\cos\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\right)^2\end{pmatrix}^2}{l\beta(-2+\epsilon(\Delta)\upsilon^2)\left(\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)^2}+\frac{(-3b\delta+3E)p\upsilon^2\begin{pmatrix}\left(-p\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)-\sqrt{-\Delta}\\+2\sqrt{-\Delta}\left(\cos\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\right)^2\end{pmatrix}}{\frac{l\beta(-2+\epsilon(\Delta)\upsilon^2)}{2}\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)}+12\frac{qr\upsilon^2(-b\delta+E)}{l\beta(-2+\epsilon(\Delta)\upsilon^2)}.\tag{5.59}$$

For case 3, substituting values from. Eq. (5.8) and Riccati equation solutions in Eq. (5.9) with

$$\xi = \frac{x^{\alpha}}{\Gamma(1+\alpha)} - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)},$$

Family1:

When $p^2 - 4qr > 0$ and $pq \neq 0$ or $qr \neq 0$, the hyperbolic function solutions of Eq. (5.1) are as follows,

$$\begin{aligned} U_{3,1} = &-2\frac{v^2(p^2+2qr)(-b\delta+E)}{(2+\epsilon(\Delta)v^2)\beta l} \\ &+\frac{(-6b\delta+6E)pv^2\left(p+\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}{(2+\epsilon(\Delta)v^2)\beta l} \\ &-\frac{(-3b\delta+3E)v^2\left(p+\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)^2}{(2+\epsilon(\Delta)v^2)\beta l}, \end{aligned} \tag{5.60}$$

$$\begin{aligned} U_{3,2} = &-2\frac{v^2(p^2+2qr)(-b\delta+E)}{(2+\epsilon(\Delta)v^2)\beta l} \\ &+\frac{(-6b\delta+6E)pv^2\left(p+\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}{(2+\epsilon(\Delta)v^2)\beta l} \\ &-\frac{(-3b\delta+3E)v^2\left(p+\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)^2}{(2+\epsilon(\Delta)v^2)\beta l}, \end{aligned} \tag{5.61}$$

$$\begin{aligned} U_{3,3} = &-2\frac{v^2(p^2+2qr)(-b\delta+E)}{(2+\epsilon(\Delta)v^2)\beta l} \\ &+\frac{(-6b\delta+6E)pv^2\left(\begin{array}{c}(p+\sqrt{\Delta})\times \\ \left(\tanh(\sqrt{\Delta}\xi)\pm i\operatorname{sech}(\sqrt{\Delta}\xi)\right)\end{array}\right)}{(2+\epsilon(\Delta)v^2)\beta l} \\ &-\frac{(-3b\delta+3E)v^2\left(\begin{array}{c}(p+\sqrt{\Delta})\times \\ \tanh\left(\sqrt{\Delta}\xi\pm i\operatorname{sech}(\sqrt{\Delta}\xi)\right)\end{array}\right)^2}{(2+\epsilon(\Delta)v^2)\beta l}, \end{aligned} \tag{5.62}$$

$$
\begin{aligned}
U_{3,4} = &-2\frac{v^2(p^2+2qr)(-b\delta+E)}{(2+\epsilon(\Delta)v^2)\beta l} \\
&+\frac{(-6b\delta+6E)pv^2\begin{pmatrix}\left(p+\sqrt{\Delta}\right)\times \\ \left(\coth\left(\sqrt{\Delta}\xi\right)\pm\operatorname{csch}\left(\sqrt{\Delta}\xi\right)\right)\end{pmatrix}}{(2+\epsilon(\Delta)v^2)\beta l} \\
&-\frac{(-3b\delta+3E)v^2\begin{pmatrix}\left(p+\sqrt{\Delta}\right)\times \\ \left(\coth\left(\sqrt{\Delta}\xi\right)\pm\operatorname{csch}\left(\sqrt{\Delta}\xi\right)\right)\end{pmatrix}^2}{(2+\epsilon(\Delta)v^2)\beta l},
\end{aligned}
\tag{5.63}
$$

$$
\begin{aligned}
U_{3,5} = &-2\frac{v^2(p^2+2qr)(-b\delta+E)}{(2+\epsilon(\Delta)v^2)\beta l} \\
&+\frac{(-6b\delta+6E)pv^2\begin{pmatrix}\left(2p+\sqrt{\Delta}\right)\times \\ \left(\tanh\left(\frac{\sqrt{\Delta}\xi}{4}\right)+\coth\left(\frac{\sqrt{\Delta}\xi}{4}\right)\right)\end{pmatrix}}{(2+\epsilon(\Delta)v^2)\beta l} \\
&-\frac{(-3b\delta+3E)v^2\begin{pmatrix}\left(2p+\sqrt{\Delta}\right)\times \\ \left(\tanh\left(\frac{\sqrt{\Delta}\xi}{4}\right)+\coth\left(\frac{\sqrt{\Delta}\xi}{4}\right)\right)\end{pmatrix}^2}{(2+\epsilon(\Delta)v^2)\beta l},
\end{aligned}
\tag{5.64}
$$

$$
\begin{aligned}
U_{3,6} = &-2\frac{v^2(p^2+2qr)(-b\delta+E)}{(2+\epsilon(\Delta)v^2)\beta l} \\
&-\frac{(-6b\delta+6E)pv^2}{(2+\epsilon(p^2-4qr)v^2)\beta l}\left(-p+\frac{\begin{pmatrix}\sqrt{(A^2+B^2)(\Delta)} \\ -A\sqrt{\Delta}\cosh\left(\sqrt{\Delta}\xi\right)\end{pmatrix}}{\mathrm{Asinh}\left(\sqrt{\Delta}\xi\right)+B}\right) \\
&-\frac{(-3b\delta+3E)v^2}{(2+\epsilon(\Delta)v^2)\beta l}\left(-p+\frac{\begin{pmatrix}\sqrt{(A^2+B^2)(\Delta)} \\ -A\sqrt{\Delta}\cosh\left(\sqrt{\Delta}\xi\right)\end{pmatrix}}{\mathrm{Asinh}\left(\sqrt{\Delta}\xi\right)+B}\right)^2,
\end{aligned}
\tag{5.65}
$$

$$U_{3,7} = -2\frac{\upsilon^2(\Delta)(-b\delta+E)}{(2+\epsilon(\Delta)\upsilon^2)\beta l} - \frac{(-6b\delta+6E)p\upsilon^2}{(2+\epsilon(\Delta)\upsilon^2)\beta l}\left(-p-\frac{\left(\begin{array}{c}\sqrt{(-A^2+B^2)(\Delta)}\\ +A\sqrt{\Delta}\sinh\left(\sqrt{\Delta}\xi\right)\end{array}\right)}{\mathrm{Acosh}\left(\sqrt{\Delta}\xi\right)+B}\right) - \frac{(-3b\delta+3E)\upsilon^2}{(2+\epsilon(\Delta)\upsilon^2)\beta l}\left(-p-\frac{\left(\begin{array}{c}\sqrt{(-A^2+B^2)(\Delta)}\\ +A\sqrt{\Delta}\sinh\left(\sqrt{\Delta}\xi\right)\end{array}\right)}{\mathrm{Acosh}\left(\sqrt{\Delta}\xi\right)+B}\right)^2. \tag{5.66}$$

Where two non-zero real constants A and B satisfies $A^2 - B^2 > 0$.

$$U_{3,8} = -2\frac{\upsilon^2(\Delta)(-b\delta+E)}{(2+\epsilon(\Delta)\upsilon^2)\beta l} - \frac{(-24b\delta+24E)pq\upsilon^2\,\mathrm{rcosh}\left(\frac{\sqrt{\Delta}\xi}{2}\right)}{(2+\epsilon(\Delta)\upsilon^2)\beta l\left(\begin{array}{c}\sqrt{\Delta}\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\\ -\,\mathrm{pcosh}\left(\frac{\sqrt{\Delta}\xi}{2}\right)\end{array}\right)} - \frac{(-48b\delta+48E)q^2\upsilon^2r^2\left(\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)^2}{(2+\epsilon(\Delta)\upsilon^2)\beta l\left(\begin{array}{c}\sqrt{\Delta}\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\\ -\,\mathrm{pcosh}\left(\frac{\sqrt{\Delta}\xi}{2}\right)\end{array}\right)^2}, \tag{5.67}$$

$$U_{3,9} = -2\frac{v^2(\Delta)(-b\delta + E)}{(2+\epsilon(\Delta)v^2)\beta l} + \frac{(-24b\delta + 24E)pqv^2 \operatorname{rsinh}\left(\frac{\sqrt{\Delta}\xi}{2}\right)}{(2+\epsilon(\Delta)v^2)\beta l\left(\begin{array}{c}-\sqrt{\Delta}\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\\ +\operatorname{psinh}\left(\frac{\sqrt{\Delta}\xi}{2}\right)\end{array}\right)} - \frac{(-48b\delta + 48E)q^2v^2r^2\left(\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)^2}{(2+\epsilon(\Delta)v^2)\beta l\left(\begin{array}{c}-\sqrt{\Delta}\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\\ +\operatorname{psinh}\left(\frac{\sqrt{\Delta}\xi}{2}\right)\end{array}\right)^2}, \tag{5.68}$$

$$U_{3,10} = -2\frac{v^2(\Delta)(-b\delta + E)}{(2+\epsilon(\Delta)v^2)\beta l} - \frac{(-24b\delta + 24E)pqv^2 \operatorname{rcosh}\left(\frac{\sqrt{\Delta}\xi}{2}\right)}{(2+\epsilon(\Delta)v^2)\beta l\left(\begin{array}{c}\sqrt{\Delta}\sinh(\sqrt{\Delta}\xi)\\ -p(\cosh(\sqrt{\Delta}\xi) \pm i\sqrt{\Delta}\end{array}\right)} - \frac{(-48b\delta + 48E)q^2v^2r^2\left(\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)^2}{(2+\epsilon(\Delta)v^2)\beta l\left(\begin{array}{c}\sqrt{\Delta}\sinh(\sqrt{\Delta}\xi)\\ -p(\cosh(\sqrt{\Delta}\xi) \pm i\sqrt{\Delta}\end{array}\right)^2}, \tag{5.69}$$

$$U_{3,11} = -2\frac{v^2(\Delta)(-b\delta + E)}{(2+\epsilon(\Delta)v^2)\beta l} \tag{5.70}$$

$$-\frac{(-24b\delta+24E)pqv^2\,\mathrm{rsinh}\left(\frac{\sqrt{\Delta}\xi}{2}\right)}{(2+\epsilon(\Delta)v^2)\beta l\begin{pmatrix}\sqrt{\Delta}\cosh(\sqrt{\Delta}\xi)\\-p(\sinh(\sqrt{\Delta}\xi)\pm\sqrt{\Delta})\end{pmatrix}}$$

$$-\frac{(-48b\delta+48E)q^2v^2r^2\left(\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)^2}{(2+\epsilon(\Delta)v^2)\beta l\begin{pmatrix}\sqrt{\Delta}\cosh(\sqrt{\Delta}\xi)\\-p(\sinh(\sqrt{\Delta}\xi)\pm\sqrt{\Delta})\end{pmatrix}^2},$$

$$U_{3,12}=-2\frac{v^2(\Delta)(-b\delta+E)}{(2+\epsilon(\Delta)v^2)\beta l}$$

$$-\frac{24(-b\delta+E)pqv^2\,\mathrm{rsinh}\left(\frac{\sqrt{\Delta}\xi}{2}\right)}{(2+\epsilon(\Delta)v^2)\beta l\begin{pmatrix}\left(-p\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\\+2\sqrt{\Delta}\left(\cosh\left(\frac{\sqrt{\Delta}\xi}{4}\right)\right)^2\\-\sqrt{\Delta}\end{pmatrix}}$$

$$-\frac{96(-b\delta+E)q^2v^2r^2\left(\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)^2}{(2+\epsilon(\Delta)v^2)\beta l\begin{pmatrix}\left(-p\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\\+2\sqrt{\Delta}\left(\cosh\left(\frac{\sqrt{\Delta}\xi}{4}\right)\right)^2\\-\sqrt{\Delta}\end{pmatrix}^2}. \quad (5.71)$$

Family2:

If $\Delta < 0$ and $pq \neq 0$ $(or\ qr \neq 0)$, we have the following trigonometric solutions:

$$U_{3,13} = -2\frac{\upsilon^2(\Delta)(-b\delta+E)}{\beta(2+\epsilon(\Delta)\upsilon^2)l} - \frac{(-6b\delta+6E)p\upsilon^2\left(-p+\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}{\beta(2+\epsilon(\Delta)\upsilon^2)l} - \frac{(-3b\delta+3E)\upsilon^2\left(-p+\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)^2}{\beta(2+\epsilon(\Delta)\upsilon^2)l}, \quad (5.72)$$

$$U_{3,14} = -2\frac{\upsilon^2(\Delta)(-b\delta+E)}{\beta(2+\epsilon(\Delta)\upsilon^2)l} + \frac{(-6b\delta+6E)p\upsilon^2\left(p+\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}{\beta(2+\epsilon(\Delta)\upsilon^2)l} - \frac{(-3b\delta+3E)\upsilon^2\left(p+\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)^2}{\beta(2+\epsilon(\Delta)\upsilon^2)l}, \quad (5.73)$$

$$U_{3,15} = -2\frac{\upsilon^2(\Delta)(-b\delta+E)}{\beta(2+\epsilon(\Delta)\upsilon^2)l} - \frac{(-6b\delta+6E)p\upsilon^2\left(\begin{matrix}\left(-p+\sqrt{-\Delta}\right)\times\\ \left(\tan\left(\sqrt{-\Delta}\xi\right)\pm\sec\left(\sqrt{-\Delta}\xi\right)\right)\end{matrix}\right)}{\beta(2+\epsilon(\Delta)\upsilon^2)l} - \frac{(-3b\delta+3E)\upsilon^2\left(\begin{matrix}\left(-p+\sqrt{-\Delta}\right)\times\\ \left(\tan\left(\sqrt{-\Delta}\xi\right)\pm\sec\left(\sqrt{-\Delta}\xi\right)\right)\end{matrix}\right)^2}{\beta(2+\epsilon(\Delta)\upsilon^2)l}, \quad (5.74)$$

$$U_{3,16} = -2\frac{v^2(\Delta)(-b\delta + E)}{\beta(2+\epsilon(\Delta)v^2)l} + \frac{(-6b\delta+6E)pv^2\left(\begin{array}{c}\left(p+\sqrt{-\Delta}\right)\times \\ \left(\cot\left(\sqrt{-\Delta}\xi\right)\pm\csc\left(\sqrt{-\Delta}\xi\right)\right)\end{array}\right)}{\beta(2+\epsilon(\Delta)v^2)l} - \frac{(-3b\delta+3E)v^2\left(\begin{array}{c}\left(p+\sqrt{-\Delta}\right)\times \\ \left(\cot\left(\sqrt{-\Delta}\xi\right)\pm\csc\left(\sqrt{-\Delta}\xi\right)\right)\end{array}\right)^2}{\beta(2+\epsilon(\Delta)v^2)l}, \tag{5.75}$$

$$U_{3,17} = -2\frac{v^2(\Delta)(-b\delta + E)}{\beta(2+\epsilon(\Delta)v^2)l} - \frac{(-3b\delta+3E)pv^2\left(\begin{array}{c}(-2p+\sqrt{-\Delta})\times \\ \left(\tan\left(\frac{\sqrt{-\Delta}\xi}{4}\right)-\cot\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\right)\end{array}\right)}{\beta(2+\epsilon(\Delta)v^2)l} - \frac{(-3b\delta+3E)v^2\left(\begin{array}{c}\left(-2p+\sqrt{-\Delta}\right)\times \\ \left(\tan\left(\frac{\sqrt{-\Delta}\xi}{4}\right)-\cot\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\right)\end{array}\right)^2}{4\beta(2+\epsilon(\Delta)v^2)l}, \tag{5.76}$$

$$U_{3,18} = -2\frac{v^2(\Delta)(-b\delta + E)}{\beta(2+\epsilon(\Delta)v^2)l} - \frac{(-6b\delta+6E)pv^2}{\beta(2+\epsilon(\Delta)v^2)l}\left(-p+\frac{\begin{array}{c}\pm i\sqrt{(-A^2+B^2)(-\Delta)} \\ -A\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)\end{array}}{A\sin\left(\sqrt{-\Delta}\xi\right)+B}\right) - \frac{(-3b\delta+3E)v^2}{\beta(2+\epsilon(\Delta)v^2)l}\left(-p+\frac{\begin{array}{c}\pm i\sqrt{(-A^2+B^2)(-\Delta)} \\ -A\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)\end{array}}{A\sin\left(\sqrt{-\Delta}\xi\right)+B}\right)^2, \tag{5.77}$$

$$U_{3,19} = -2\frac{\upsilon^2(\Delta)(-b\delta + E)}{\beta(2 + \epsilon(\Delta)\upsilon^2)l} - \frac{(-6b\delta + 6E)p\upsilon^2}{\beta(2 + \epsilon(\Delta)\upsilon^2)l}\left(-p + \frac{\pm i\sqrt{(-A^2 + B^2)(-\Delta)} + A\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)}{\mathrm{Asin}\left(\sqrt{-\Delta}\xi\right) + B}\right) - \frac{(-3b\delta + 3E)\upsilon^2}{\beta(2 + \epsilon(\Delta)\upsilon^2)l}\left(-p + \frac{\pm i\sqrt{(-A^2 + B^2)(-\Delta)} + A\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)}{\mathrm{Asin}\left(\sqrt{-\Delta}\xi\right) + B}\right)^2 . \tag{5.78}$$

Where two non-zero real constants A and B satisfies $A^2 - B^2 > 0$.

$$U_{3,20} = -2\frac{\upsilon^2(\Delta)(-b\delta + E)}{\beta(2 + \epsilon(\Delta)\upsilon^2)l} + \frac{(-24b\delta + 24E)pq\upsilon^2\,\mathrm{rcos}\left(\frac{\sqrt{-\Delta}\xi}{2}\right)}{\beta(2 + \epsilon(\Delta)\upsilon^2)l\left(\sqrt{-\Delta}\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right) + \mathrm{pcos}\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)} - \frac{(-48b\delta + 48E)q^2\upsilon^2r^2\left(\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)^2}{\beta(2 + \epsilon(\Delta)\upsilon^2)l\left(\sqrt{-\Delta}\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right) + \mathrm{pcos}\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)^2}, \tag{5.79}$$

$$U_{3,21} = -2\frac{\upsilon^2(\Delta)(-b\delta+E)}{\beta(2+\epsilon(\Delta)\upsilon^2)l} - \frac{(-24b\delta+24E)pq\upsilon^2 \,\mathrm{rsin}\left(\frac{\sqrt{-\Delta}\xi}{2}\right)}{\beta(2+\epsilon(\Delta)\upsilon^2)l\begin{pmatrix}\sqrt{-\Delta}\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\\ -\,\mathrm{psin}\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\end{pmatrix}} - \frac{(-48b\delta+48E)q^2\upsilon^2r^2\left(\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)^2}{\beta(2+\epsilon(\Delta)\upsilon^2)l\begin{pmatrix}\sqrt{-\Delta}\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\\ -\,\mathrm{psin}\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\end{pmatrix}^2}, \tag{5.80}$$

$$U_{3,22} = -2\frac{\upsilon^2(\Delta)(-b\delta+E)}{\beta(2+\epsilon(p^2-4qr)\upsilon^2)l} + \frac{(-24b\delta+24E)pq\upsilon^2 \,\mathrm{rcos}\left(\frac{\sqrt{-\Delta}\xi}{2}\right)}{\beta(2+\epsilon(\Delta)\upsilon^2)l\begin{pmatrix}\sqrt{-\Delta}\sin\left(\sqrt{-\Delta}\xi\right)\\ +p\cos\left(\sqrt{\Delta}\xi\right)\pm\sqrt{-\Delta}\end{pmatrix}} - \frac{(-48b\delta+48E)q^2\upsilon^2r^2\left(\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)^2}{\beta(2+\epsilon(\Delta)\upsilon^2)l\begin{pmatrix}\sqrt{-\Delta}\sin\left(\sqrt{-\Delta}\xi\right)\\ +p\cos\left(\sqrt{\Delta}\xi\right)\pm\sqrt{-\Delta}\end{pmatrix}^2}, \tag{5.81}$$

$$U_{3,23} = -2\frac{\upsilon^2(\Delta)(-b\delta+E)}{\beta(2+\epsilon(\Delta)\upsilon^2)l} \tag{5.82}$$

$$-\frac{(-48b\delta+48E)q^2\upsilon^2r^2\left(\sin\left(\sqrt{-\Delta}\xi\right)\right)^2}{\beta(2+\epsilon(\Delta)\upsilon^2)l\begin{pmatrix}\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)\\ -p\sin\left(\sqrt{-\Delta}\xi\right)\pm\sqrt{-\Delta}\end{pmatrix}^2},$$

$$\begin{aligned}U_{3,24}=&-2\frac{\upsilon^2(\Delta)(-b\delta+E)}{\beta(2+\epsilon(\Delta)\upsilon^2)l}\\&-\frac{24(-b\delta+E)pq\upsilon^2\,\mathrm{r}\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)}{\beta(2+\epsilon(\Delta)\upsilon^2)l\begin{pmatrix}-p\sin\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\\+2\sqrt{-\Delta}\left(\cos\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\right)^2\\-\sqrt{-\Delta})\end{pmatrix}}\\&-\frac{48(-b\delta+E)q^2\upsilon^2r^2\left(\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)^2}{\beta(2+\epsilon(\Delta)\upsilon^2)l\begin{pmatrix}-p\sin\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\\+2\sqrt{-\Delta}\left(\cos\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\right)^2\\-\sqrt{-\Delta})\end{pmatrix}^2}.\end{aligned}\tag{5.83}$$

Family 3:

When $r=0$, and $pq\neq 0$, we get soliton like solutions,

$$\begin{aligned}U_{3,25}=&-2\frac{\upsilon^2p^2(-b\delta+E)}{(2+\epsilon p^2\upsilon^2)\beta l}\\&+\frac{(-12b\delta+12E)p^2q\upsilon^2\aleph}{(2+\epsilon p^2\upsilon^2)\beta lq(\aleph+\cosh(p\xi)-\sinh(p\xi))}\\&-\frac{(-12b\delta+12E)q^2\upsilon^2p^2\aleph^2}{(2+\epsilon p^2\upsilon^2)\beta l\left(q(\aleph+\cosh(p\xi)-\sinh(p\xi))\right)^2},\end{aligned}\tag{5.84}$$

$$U_{3,26} = -2\frac{v^2p^2(-b\delta+E)}{(\epsilon p^2v^2+2)\beta l} + \frac{(-12b\delta+12E)p^2qv^2(\cosh(p\xi)+\sinh(p\xi))}{(\epsilon p^2v^2+2)\beta lq(\aleph+\cosh(p\xi)+\sinh(p\xi))} - \frac{(-12b\delta+12E)q^2v^2p^2(\cosh(p\xi)+\sinh(p\xi))^2}{(\epsilon p^2v^2+2)\beta l\big(q(\aleph+\cosh(p\xi)+\sinh(p\xi))\big)^2}, \tag{5.85}$$

Where $\aleph$ is constant.

Family 4:

When $q \neq 0$, and, $r = p = 0$, we have following rational solution.

$$U_{3,27} = -\frac{(-6b\delta+6E)q^2v^2}{\beta l(q\xi+C)^2}, \tag{5.86}$$

Where C is an arbitrary constant.

In the case 4, substituting values from Eq. (5.10) and Riccati equation solutions in Eq. (5.11) with

$$\xi = \frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)},$$

Family1:

When $\Delta > 0$ and $pq \neq 0$ or $qr \neq 0$, the hyperbolic function solutions of Eq. (5.1) are as follows,

$$U_{4,1} = 12\frac{qrv^2(-b\delta+E)}{l(-2+\epsilon(\Delta)v^2)\beta} - \frac{(-6b\delta+6E)pv^2\left(p+\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}{l(-2+\epsilon(\Delta)v^2)\beta} + \frac{(-3b\delta+3E)v^2\left(p+\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)^2}{l(-2+\epsilon(\Delta)v^2)\beta}, \tag{5.87}$$

$$U_{4,2} = 12\frac{qrv^2(-b\delta+E)}{l(-2+\epsilon(\Delta)v^2)\beta} - \frac{(-6b\delta+6E)pv^2\left(p+\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}{l(-2+\epsilon(\Delta)v^2)\beta} + \frac{(-3b\delta+3E)v^2\left(p+\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)^2}{l(-2+\epsilon(\Delta)v^2)\beta}, \tag{5.88}$$

$$U_{4,3} = 12\frac{qrv^2(-b\delta+E)}{l(-2+\epsilon(\Delta)v^2)\beta} - \frac{(-6b\delta+6E)pv^2\left(\begin{matrix}\left(p+\sqrt{\Delta}\right)\times \\ \tanh\left(\sqrt{\Delta}\xi\right)\pm i\operatorname{sech}\left(\sqrt{\Delta}\xi\right)\end{matrix}\right)}{l(-2+\epsilon(\Delta)v^2)\beta} + \frac{(-3b\delta+3E)v^2\left(\begin{matrix}(p+\sqrt{\Delta})\times \\ \tanh\,(\sqrt{\Delta}\xi)\pm i\operatorname{sech}(\sqrt{\Delta}\xi)\end{matrix}\right)^2}{l(-2+\epsilon(\Delta)v^2)\beta}, \tag{5.89}$$

$$U_{4,4} = 12\frac{qrv^2(-b\delta+E)}{l(-2+\epsilon(\Delta)v^2)\beta} - \frac{(-6b\delta+6E)pv^2\left(\begin{matrix}\left(p+\sqrt{\Delta}\right)\times \\ \coth\left(\sqrt{\Delta}\xi\right)\pm \operatorname{csch}\left(\sqrt{\Delta}\xi\right)\end{matrix}\right)}{l(-2+\epsilon(\Delta)v^2)\beta} + \frac{(-3b\delta+3E)v^2\left(\begin{matrix}(p+\sqrt{\Delta}) \\ \coth(\sqrt{\Delta}\xi)\pm \operatorname{csch}(\sqrt{\Delta}\xi)\end{matrix}\right)^2}{l(-2+\epsilon(\Delta)v^2)\beta}, \tag{5.90}$$

$$U_{4,5} = 12\frac{qr\upsilon^2(-b\delta+E)}{l(-2+\epsilon(\Delta)\upsilon^2)\beta} - \frac{(-6b\delta+6E)p\upsilon^2\left(\begin{array}{c}(2p+\sqrt{\Delta})\times\\ \left(\tanh\left(\frac{\sqrt{\Delta}\xi}{4}\right)+\coth\left(\frac{\sqrt{\Delta}\xi}{4}\right)\right)\end{array}\right)}{l(-2+\epsilon(\Delta)\upsilon^2)\beta} + \frac{(-3b\delta+3E)\upsilon^2\left(\begin{array}{c}(2p+\sqrt{p^2-4qr})\times\\ \left(\tanh\left(\frac{\sqrt{\Delta}\xi}{4}\right)+\coth\left(\frac{\sqrt{\Delta}\xi}{4}\right)\right)\end{array}\right)^2}{l(-2+\epsilon(\Delta)\upsilon^2)\beta}, \tag{5.91}$$

$$U_{4,6} = 12\frac{qr\upsilon^2(-b\delta+E)}{l(-2+\epsilon(\Delta)\upsilon^2)\beta} + \frac{(-6b\delta+6E)p\upsilon^2}{l(-2+\epsilon(\Delta)\upsilon^2)\beta}\left(-p+\frac{\begin{array}{c}\sqrt{(A^2+B^2)(\Delta)}\\ -A\sqrt{\Delta}\cosh\left(\sqrt{\Delta}\xi\right)\end{array}}{\text{Asinh}\left(\sqrt{\Delta}\xi\right)+B}\right) + \frac{(-3b\delta+3E)\upsilon^2}{l(-2+\epsilon(\Delta)\upsilon^2)\beta}\left(-p+\frac{\begin{array}{c}\sqrt{(A^2+B^2)(\Delta)}\\ -A\sqrt{\Delta}\cosh\left(\sqrt{\Delta}\xi\right)\end{array}}{\text{Asinh}\left(\sqrt{\Delta}\xi\right)+B}\right)^2, \tag{5.92}$$

$$U_{4,7} = 12\frac{qr\upsilon^2(-b\delta+E)}{l(-2+\epsilon(\Delta)\upsilon^2)\beta} + \frac{(-6b\delta+6E)p\upsilon^2}{l(-2+\epsilon(\Delta)\upsilon^2)\beta}\left(-p-\frac{\begin{array}{c}\sqrt{(-A^2+B^2)(\Delta)}\\ +A\sqrt{\Delta}\sinh\left(\sqrt{\Delta}\xi\right)\end{array}}{\text{Acosh}\left(\sqrt{\Delta}\xi\right)+B}\right) + \frac{(-3b\delta+3E)\upsilon^2}{l(-2+\epsilon(\Delta)\upsilon^2)\beta}\left(-p-\frac{\begin{array}{c}\sqrt{(-A^2+B^2)(\Delta)}\\ +A\sqrt{p^2-4qr}\sinh\left(\sqrt{\Delta}\xi\right)\end{array}}{\text{Acosh}\left(\sqrt{\Delta}\xi\right)+B}\right)^2, \tag{5.93}$$

$$U_{4,7} = 12\frac{qrv^2(-b\delta+E)}{l(-2+\epsilon(\Delta)v^2)\beta} + \frac{(-6b\delta+6E)pv^2}{l(-2+\epsilon(\Delta)v^2)\beta}\left(-p-\frac{\sqrt{(-A^2+B^2)(\Delta)} + A\sqrt{\Delta}\sinh\left(\sqrt{\Delta}\xi\right)}{\mathrm{A}\cosh\left(\sqrt{\Delta}\xi\right)+B}\right) + \frac{(-3b\delta+3E)v^2}{l(-2+\epsilon(\Delta)v^2)\beta}\left(-p-\frac{\sqrt{(-A^2+B^2)(\Delta)} + A\sqrt{\Delta}\sinh\left(\sqrt{\Delta}\xi\right)}{\mathrm{A}\cosh\left(\sqrt{\Delta}\xi\right)+B}\right)^2. \quad (5.94)$$

Where two non-zero real constants A and B satisfies $A^2 - B^2 > 0$.

$$U_{4,8} = 12\frac{qrv^2(-b\delta+E)}{l(-2+\epsilon(\Delta)v^2)\beta} + \frac{(-24b\delta+24E)pqv^2\,\mathrm{r}\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)}{l(-2+\epsilon(\Delta)v^2)\beta\left(\sqrt{\Delta}\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right) - \mathrm{p}\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)} + \frac{(-48b\delta+48E)q^2v^2r^2\left(\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)^2}{l(-2+\epsilon(\Delta)v^2)\beta\left(\sqrt{\Delta}\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right) - \mathrm{p}\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)^2}, \quad (5.95)$$

$$U_{4,9} = 12\frac{qrv^2(-b\delta+E)}{l(-2+\epsilon(\Delta)v^2)\beta} - \frac{(-24b\delta+24E)pqv^2\,\mathrm{r}\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)}{l(-2+\epsilon(\Delta)v^2)\beta\left(-\sqrt{\Delta}\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right) + \mathrm{p}\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)} + \frac{(-48b\delta+48E)q^2v^2r^2\left(\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)^2}{l(-2+\epsilon(\Delta)v^2)\beta\left(-\sqrt{\Delta}\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right) + \mathrm{p}\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)^2}, \tag{5.96}$$

$$U_{4,10} = 12\frac{qrv^2(-b\delta+E)}{l(-2+\epsilon(\Delta)v^2)\beta} + \frac{(-24b\delta+24E)pqv^2\,\mathrm{r}\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)}{l(-2+\epsilon(\Delta)v^2)\beta\left(\sqrt{\Delta}\sinh(\sqrt{\Delta}\xi) - \mathrm{p}\cosh(\sqrt{\Delta}\xi) \pm i\sqrt{\Delta}\right)} + \frac{(-48b\delta+48E)q^2v^2r^2\left(\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)^2}{l(-2+\epsilon(\Delta)v^2)\beta\left(\sqrt{\Delta}\sinh(\sqrt{\Delta}\xi) - p\cosh(\sqrt{\Delta}\xi) \pm i\sqrt{\Delta}\right)^2}, \tag{5.97}$$

$$U_{4,11} = 12\frac{qrv^2(-b\delta+E)}{l(-2+\epsilon(\Delta)v^2)\beta}$$

$$+\frac{(-24b\delta+24E)pqv^2\,r\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)}{l(-2+\epsilon(\Delta)v^2)\beta\begin{pmatrix}\sqrt{\Delta}\cosh(\sqrt{\Delta}\xi)-\\p(\sinh(\sqrt{\Delta}\xi)\pm\sqrt{\Delta})\end{pmatrix}} \tag{5.98}$$

$$+\frac{(-48b\delta+48E)q^2v^2r^2\left(\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)^2}{l(-2+\epsilon(\Delta)v^2)\beta\begin{pmatrix}\sqrt{\Delta}\cosh(\sqrt{\Delta}\xi)-\\p(\sinh(\sqrt{\Delta}\xi)\pm\sqrt{\Delta})\end{pmatrix}^2},$$

$$U_{4,12}=12\frac{qrv^2(-b\delta+E)}{l(-2+\epsilon(\Delta)v^2)\beta}$$

$$+\frac{24(-b\delta+E)pqv^2\,r\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)}{l(-2+\epsilon(\Delta)v^2)\beta\begin{pmatrix}-p\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\\+2\sqrt{\Delta}\left(\cosh\left(\frac{\sqrt{\Delta}\xi}{4}\right)\right)^2\\-\sqrt{\Delta}\end{pmatrix}} \tag{5.99}$$

$$+\frac{48(-b\delta+E)q^2v^2r^2\left(\sinh\left(\frac{\sqrt{\Delta}\xi}{4}\right)\right)^2}{l(-2+\epsilon(\Delta)v^2)\beta\begin{pmatrix}-p\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\\+2\sqrt{\Delta}\left(\cosh\left(\frac{\sqrt{\Delta}\xi}{4}\right)\right)^2\\-\sqrt{\Delta}\end{pmatrix}^2}.$$

Family 2:

When $\Delta < 0$ and $pq \neq 0$ or $qr \neq 0$, the trigonometric solutions are.

$$U_{4,13}=12\frac{qrv^2(-b\delta+E)}{\beta l(-2+\epsilon(\Delta)v^2)}$$

$$+\frac{(-6b\delta+6E)pv^2\left(-p+\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}{\beta l(-2+\epsilon(\Delta)v^2)} \tag{5.100}$$
$$+\frac{(-3b\delta+3E)v^2\left(-p+\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)^2}{\beta l(-2+\epsilon(\Delta)v^2)},$$

$$U_{4,14}=12\frac{qrv^2(-b\delta+E)}{\beta l(-2+\epsilon(\Delta)v^2)}$$
$$-\frac{(-6b\delta+6E)pv^2\left(p+\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}{\beta l(-2+\epsilon(\Delta)v^2)} \tag{5.101}$$
$$+\frac{(-3b\delta+3E)v^2\left(p+\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)^2}{\beta l(-2+\epsilon(\Delta)v^2)},$$

$$U_{4,15}=12\frac{qrv^2(-b\delta+E)}{\beta l(-2+\epsilon(\Delta)v^2)}$$
$$+\frac{(-6b\delta+6E)pv^2\left(\begin{matrix}\left(-p+\sqrt{-\Delta}\right)\times\\ \left(\tan\left(\sqrt{-\Delta}\xi\right)\pm\sec\left(\sqrt{-\Delta}\xi\right)\right)\end{matrix}\right)}{\beta l(-2+\epsilon(\Delta)v^2)} \tag{5.102}$$
$$+\frac{(-3b\delta+3E)v^2\left(\begin{matrix}(-p+\sqrt{-\Delta})\times\\ \left(\tan\left(\sqrt{-\Delta}\xi\right)\pm\sec\left(\sqrt{-\Delta}\xi\right)\right)\end{matrix}\right)^2}{\beta l(-2+\epsilon(\Delta)v^2)},$$

$$U_{4,16}=12\frac{qrv^2(-b\delta+E)}{\beta l(-2+\epsilon(\Delta)v^2)}$$

$$+\frac{(-3b\delta+3E)v^2\left(\begin{array}{c}(p+\sqrt{-\Delta})\times\\ \left(\cot\left(\sqrt{-\Delta}\xi\right)\pm\csc\left(\sqrt{-\Delta}\xi\right)\right)\end{array}\right)^2}{\beta l(-2+\epsilon(\Delta)v^2)}, \tag{5.103}$$

$$\begin{aligned} U_{4,17} &= 12\frac{qrv^2(-b\delta+E)}{\beta l(-2+\epsilon(\Delta)v^2)} \\ &+\frac{(-3b\delta+3E)pv^2\left(\begin{array}{c}\left(-2p+\sqrt{-\Delta}\right)\times\\ \left(\tan\left(\frac{\sqrt{-\Delta}\xi}{4}\right)-\cot\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\right)\end{array}\right)}{\beta l(-2+\epsilon(\Delta)v^2)} \\ &+\frac{(-3b\delta+3E)pv^2\left(\begin{array}{c}(-2p+\sqrt{-\Delta})\times\\ \left(\tan\left(\frac{\sqrt{-\Delta}\xi}{4}\right)-\cot\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\right)\end{array}\right)^2}{4\beta l(-2+\epsilon(\Delta)v^2)}, \end{aligned} \tag{5.104}$$

$$\begin{aligned} U_{4,18} &= 12\frac{qrv^2(-b\delta+E)}{\beta l(-2+\epsilon(\Delta)v^2)} \\ &+\frac{(-6b\delta+6E)pv^2}{\beta l(-2+\epsilon(\Delta)v^2)}\left(-p+\frac{\begin{array}{c}\pm i\sqrt{(-A^2+B^2)(-\Delta)}\\ -A\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)\end{array}}{A\sin\left(\sqrt{-\Delta}\xi\right)+B}\right) \\ &+\frac{(-3b\delta+3E)v^2}{\beta l(-2+\epsilon(\Delta)v^2)}\left(-p+\frac{\begin{array}{c}\pm i\sqrt{(-A^2+B^2)(-\Delta)}\\ -A\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)\end{array}}{A\sin\left(\sqrt{-\Delta}\xi\right)+B}\right)^2, \end{aligned} \tag{5.105}$$

$$U_{4,19} = 12\frac{qrv^2(-b\delta+E)}{\beta l(-2+\epsilon(\Delta)v^2)} + \frac{(-6b\delta+6E)pv^2}{\beta l(-2+\epsilon(\Delta)v^2)}\left(-p+\frac{\pm i\sqrt{(-A^2+B^2)(-\Delta)} + A\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)}{\mathrm{A}\sin\left(\sqrt{-\Delta}\xi\right)+B}\right) + \frac{(-3b\delta+3E)v^2}{\beta l(-2+\epsilon(\Delta)v^2)}\left(-p+\frac{\pm i\sqrt{(-A^2+B^2)(-\Delta)} + A\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)}{\mathrm{A}\sin\left(\sqrt{-\Delta}\xi\right)+B}\right)^2. \quad (5.106)$$

Where two non-zero real constants A and B satisfies $A^2 - B^2 > 0$.

$$U_{4,20} = 12\frac{qrv^2(-b\delta+E)}{\beta l(-2+\epsilon(\Delta)v^2)} - \frac{(-24b\delta+24E)pqv^2\,\mathrm{r}\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)}{\beta l(-2+\epsilon(\Delta)v^2)\left(\sqrt{-\Delta}\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)+\mathrm{p}\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)} + \frac{(-48b\delta+48E)q^2v^2r^2\left(\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)^2}{\beta l(-2+\epsilon(\Delta)v^2)\left(\sqrt{-\Delta}\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)+\mathrm{p}\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)^2}, \quad (5.107)$$

$$U_{4,21} = 12\frac{qrv^2(-b\delta+E)}{\beta l(-2+\epsilon(\Delta)v^2)} + \frac{(-24b\delta+24E)pqv^2\,\mathrm{r}\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)}{\beta l(-2+\epsilon(\Delta)v^2)\left(\sqrt{-\Delta}\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right) - \mathrm{p}\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)} + \frac{(-48b\delta+48E)q^2v^2r^2\left(\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)^2}{\beta l(-2+\epsilon(\Delta)v^2)\left(\sqrt{-\Delta}\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right) - \mathrm{p}\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)^2}, \tag{5.108}$$

$$U_{4,22} = 12\frac{qrv^2(-b\delta+E)}{\beta l(-2+\epsilon(\Delta)v^2)} - \frac{(-24b\delta+24E)pqv^2\,\mathrm{r}\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)}{\beta l(-2+\epsilon(\Delta)v^2)\left(\sqrt{-\Delta}\sin\left(\sqrt{-\Delta}\xi\right) + p\cos\left(\sqrt{-\Delta}\xi\right) \pm \sqrt{-\Delta}\right)} + \frac{(-48b\delta+48E)q^2v^2r^2\left(\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)^2}{\beta l(-2+\epsilon(\Delta)v^2)\left(\sqrt{-\Delta}\sin\left(\sqrt{-\Delta}\xi\right) + p\cos\left(\sqrt{-\Delta}\xi\right) \pm \sqrt{-\Delta}\right)^2} \tag{5.109}$$

$$U_{4,23} = \frac{12qrv^2(-b\delta+E)}{\beta l(-2+\epsilon(\Delta)v^2)} + \frac{24(-b\delta+E)pqv^2\,\mathrm{r}\sin\left(\sqrt{-\Delta}\xi\right)}{\beta l(-2+\epsilon(\Delta)v^2)\left(\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right) - p\sin\left(\sqrt{-\Delta}\xi\right) \pm \sqrt{-\Delta}\right)} + \frac{48(-b\delta+E)q^2v^2r^2\left(\sin\left(\sqrt{-\Delta}\xi\right)\right)^2}{\beta l(-2+\epsilon(\Delta)v^2)\left(\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right) - p\sin\left(\sqrt{-\Delta}\xi\right) \pm \sqrt{-\Delta}\right)^2}, \tag{5.110}$$

$$U_{4,24} = 12\frac{qrv^2(-b\delta+E)}{\beta l(-2+\epsilon(\Delta)v^2)} + \frac{24(-b\delta+E)pqv^2\,\mathrm{r}\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)}{\beta l(-2+\epsilon(\Delta)v^2)\left(-p\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right) + 2\sqrt{-\Delta}\left(\cos\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\right)^2 - \sqrt{-\Delta}\right)} + \frac{48(-b\delta+E)q^2v^2r^2\left(\sin\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\right)^2}{\beta l(-2+\epsilon(\Delta)v^2)\left(-p\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right) + 2\sqrt{-\Delta}\left(\cos\left(\frac{\sqrt{-\Delta}\xi}{4}\right)\right)^2 - \sqrt{-\Delta}\right)^2}. \tag{5.111}$$

Family 3:

When $r=0$, and $pq \neq 0$, the hyperbolic function solutions are,

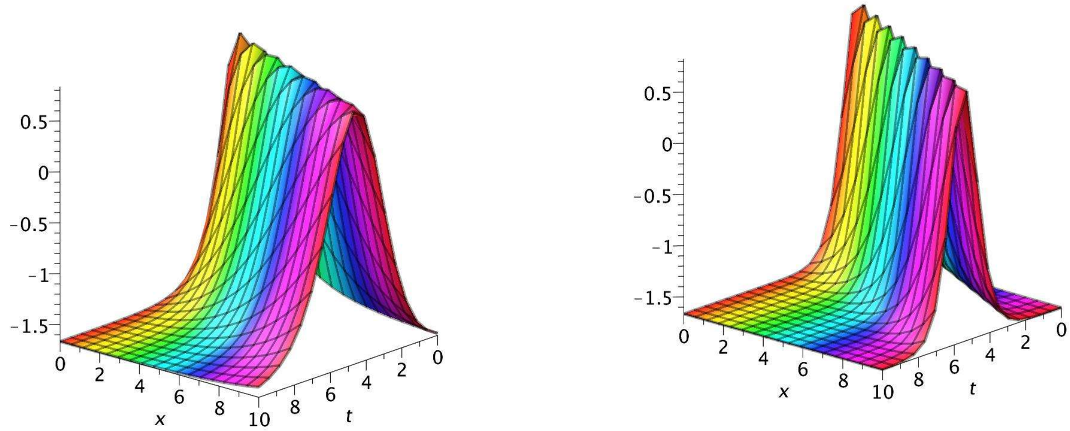
$$U_{4,25} = -\frac{(-12b\delta+12E)p^2qv^2\aleph}{(\epsilon p^2v^2-2)\beta lq(\aleph+\cosh(p\xi)-\sinh(p\xi))} + \frac{(-12b\delta+12E)q^2v^2p^2\aleph^2}{(\epsilon p^2v^2-2)\beta l(q(\aleph+\cosh(p\xi)-\sinh(p\xi)))^2}, \tag{5.112}$$

$$U_{4,26} = -\frac{(-12b\delta+12E)p^2qv^2(\cosh(p\xi)+\sinh(p\xi))}{(\epsilon p^2v^2-2)\beta lq(\aleph+\cosh(p\xi)+\sinh(p\xi))} + \frac{(-12b\delta+12E)q^2v^2p^2(\cosh(p\xi)+\sinh(p\xi))^2}{(\epsilon p^2v^2-2)\beta l(q(\aleph+\cosh(p\xi)+\sinh(p\xi)))^2}. \tag{5.113}$$

Where $\aleph$ is constant.

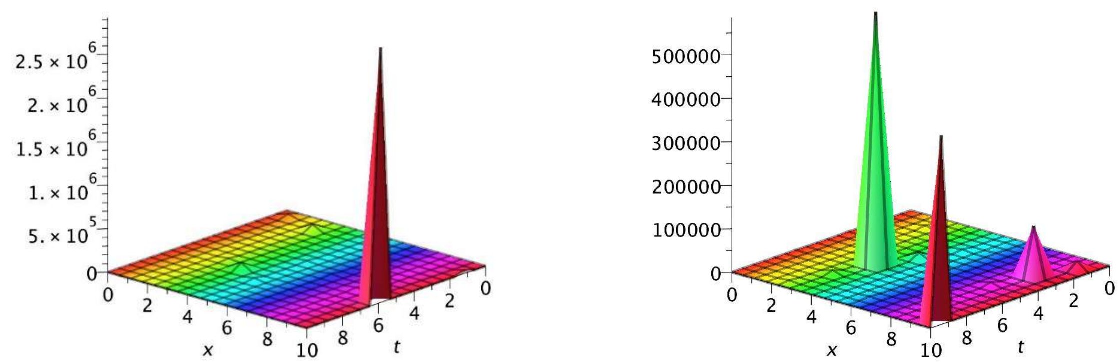
5.4 Graphical Explanation:

The doubly dispersive equation (DDE) is an important nonlinear model that can be used to define the nonlinear wave propagation in the elastic inhomogeneous circular cylinder Murnaghan's rod. These waves have become important for scientists and engineers in the study of seismology, and to determine the endurance of elastic materials and structures. These waves can be used in the studies for the development of non-destructive testing techniques especially for pipelines, and to understand the physical properties and internal structure of solids like brass, steel, glass and polymers [172]. It is worth mentioning that the solutions obtained in this study represent certain real-life situations. For example, the tan-hyperbolic solutions are useful in calculating the magnetic moment and rapidity of special relativity, cos-hyperbolic solutions represent the shape of hanging cable, cot-hyperbolic solutions appear in the Langevin function which arise in magnetic polarization, sec-hyperbolic solutions represent the laminar jet profile [173]. Similarly, exact solutions with the periodic functions exhibit periodic wave phenomena. It is significant to mention here that a lot of new solutions have been produced for Murnaghan's rod, and for the first time this equation has been solved for space-time fractional order. The reason of using fractional differential equation is that it is naturally related to physical phenomena with memory. Many well-known equations can be solved by space-time fractional differential equations to get variety of new solutions. Graphs of some obtained solutions has been discussed here for the better understanding of the solitary wave phenomenon. Figure 5.1 depicts 3D-graphs of dark soliton solution generated by $U_{1,1}$ with fractional order $\alpha = 0.7, 1$, with some given parameters $p = 3, q = 1, r = 2, b = 0.5, \beta = 1, E = 4, \epsilon = 0.1, l = 2, \upsilon = 2, \delta = 6$. Figure 5.2, 3D-graphs of solutions $U_{1,20}$ with fractional order $\alpha = 0.6, 1$, exhibits combined singular periodic wave solution by taking parameters $p = 2, q = 1, r = 2.5, b = 0.3, \beta = 1.5, E = 10, \epsilon = 0.5, l = 2, \upsilon = 3.5, \delta = 22.5$. Figure 5.3: 3D-graphs depicts dark singular solitons of $U_{2,5}$ with fractional order $\alpha = 0.5, 1$ by choosing parameters $p = 1, q = 5, r = 4, b = 0.5, \beta = 1, E = 4, \epsilon = 0.1, l = 2, \upsilon = 2, \delta = 6$.Figure 5.4,: 3D-graphsexhibits combined dark-bright soliton generated by $U_{3,3}$ with fractional order $\alpha = 0.4, 1$, by taking $p = 3, q = 2, r = 1, b = 0.3, \beta = 1.5, E = 10, \epsilon = 0.5, l = 2, \upsilon = 3.5, \delta = 22.5$.Figure 5.5 shows: 3D-graphs of hyperbolic solutions $U_{4,6}$ with fractional order $\alpha = 0.5, 1$ with parameters $p = 5, q = 3, r = 1, b = 0.9, \beta = 5, E = 11, \epsilon = 0.05, l = 2, \upsilon = 5, \delta = 33, A = 2, B = 4$.



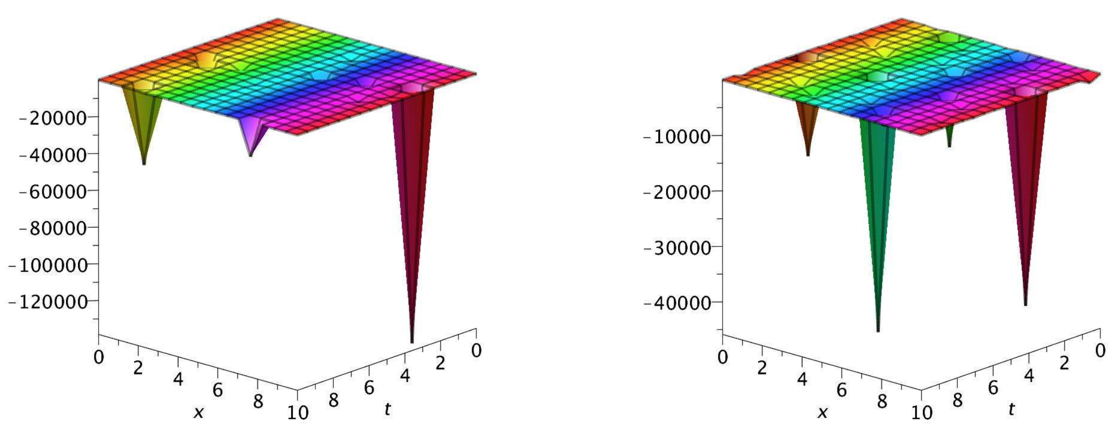
(a) **(b)**

Figure 5.1: 3D-graphs of $U_{1,1}$ with fractional order $\alpha = 0.7,\ 1$



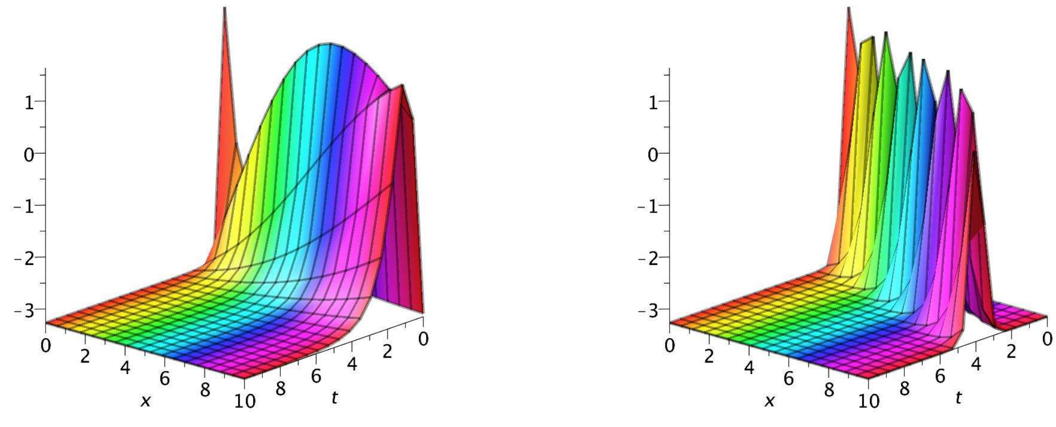
(a) **(b)**

Figure 5.2: 3D-graphs of $U_{1,20}$ with fractional order $\alpha = 0.6,\ 1$

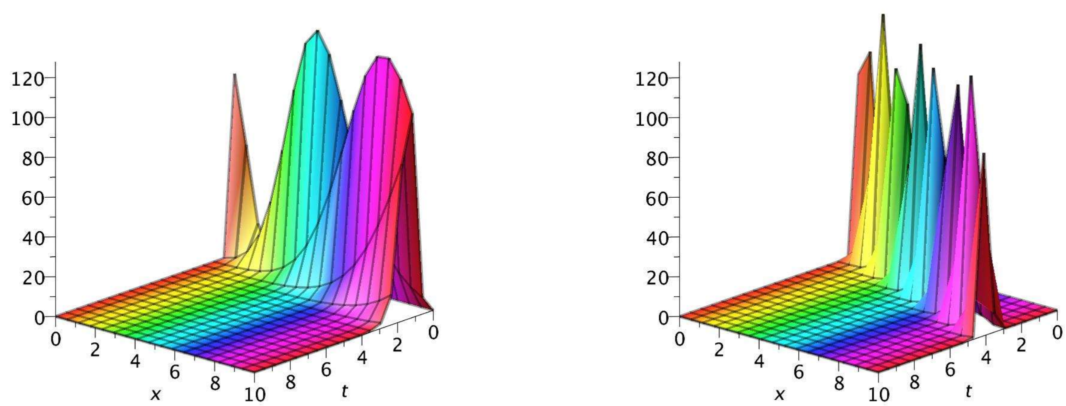


(a) **(b)**

Figure 5.3: 3D-graphs of $U_{2,5}$ with fractional order $\alpha = 0.5, 1$



(a) **(b)**
Figure 5.4: 3D-graphs of $U_{3,3}$ with fractional order $\alpha = 0.4, 1$



(a) **(b)**
Figure 5.5: 3D-graphs of $U_{4,6}$ with fractional order $\alpha = 0.5, 1$

5.5 Conclusions:

Improved generalized Riccati equation mapping method has been applied to secure exact traveling wave solutions to the space- time fractional Murnaghan's rod equation. As a result, some totally new solutions have been obtained. These solutions include several solitary wave solutions: dark, combined dark-bright, singular periodic wave, combined singular periodic wave solutions and one rational solution. A back substitution verifies the exactness of the solutions, and their overall behavior has been highlighted with the help of graphs. These new results might clarify the physical properties of brass, steel, and new elastic materials like polymers in the study of seismology.

5.6 Space-time conformable Telegraph equation:

In this section we will discuss space-time conformable telegraph equation commonly used to study to electrical signals in transmission lines [174].

$$D_{tt}^{2\alpha}u - D_{xx}^{2\alpha}u + D_t^{\alpha}u + \gamma u + \beta u^3 = 0, \tag{5.114}$$

where γ and β are arbitrary constants to be determined later by proposed method.

B. Gasmi, et. al. [175] implemented generalized Kudryashov method to derive various solitary wave solutions, in [176] have used Hirota bilinear method to generate N-solitons. Anjali Verma, et.al. have used tanh method, Mostafa M. A. Khater, et. al.[174] implemented five semi analytical and numerical simulations to compare results of analytical and approximate solutions. M. Mirzazadeh, et. al. [177] have applied first integral method in search of new exact solutions. C.Yue, et. al. [178] examined nonlinear time-space telegraph equation through three schemes.

Motived by a few of the above-mentioned works on telegraph equation we are using time-space conformable telegraph equation to derive new types of exact solutions using improved generalized Riccati equation mapping method. For this consider the following conformable fractional wave transformation,

$$u(x,t) = U(\xi\,), \tag{5.115}$$

where $\xi = \chi \frac{x^{\alpha}}{\Gamma(1+\alpha)} + \lambda \frac{t^{\alpha}}{\Gamma(1+\alpha)}$,

χ, λ are arbitrary constant, whereas $0 < \alpha \leq 1$, is the order of derivatives in conformable sense [39]. Using this transformation in Eq. (5.114), we get following non-linear ODE.

$$(\lambda^2 - \chi^2)\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}U(\xi) + \lambda \frac{d}{d\xi}U(\xi) + \gamma\, U(\xi) + \beta\, U(\xi)^3 = 0. \tag{5.116}$$

Now balancing the order between $\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}U(\xi)$ and $U(\xi)^3$ we get, $N + 2 = 3N \Rightarrow N = 1$, therefore series solution (1.67), takes the form,

$$U(\xi\,) = \frac{a_{-1}}{\phi(\xi)} + a_0 + a_1\phi(\xi). \tag{5.117}$$

Now, substituting Eq. (5.117) coupling with Eq. (1.68) into Eq. (5.116) after collecting coefficients of all terms with the same order in ϕ^i and ϕ^{-i}, where, $(i = 0,1,2, \ldots\ldots)$. and setting these coefficient to 0, we get a system of NL algebraic equations. Solving these equations with the aid of mathematical software we obtain following non-trivial solutions:

Set 1 :

$p^2 - 4rq = \Delta, \Omega = \frac{\sqrt{\gamma}}{4}\sqrt{9\gamma - 2}x$

$$a_0 = \frac{\sqrt{\gamma}\left((\Delta)\sqrt{(\Delta)^{-1}} - p\right)}{2\beta(\Delta)\sqrt{-(\beta\Delta)^{-1}}}, \qquad a_1 = 0, \qquad a_{-1} = r\sqrt{\gamma}\sqrt{-(\beta\Delta)^{-1}},$$

$$\chi = \frac{\sqrt{\gamma}\sqrt{9\gamma - 2}\sqrt{(\Delta)^{-1}}}{2}, \quad \lambda = \frac{3\gamma\sqrt{(\Delta)^{-1}}}{2}.$$

Under these conditions Eq. (5.117), takes the form,

$$U_1(\xi) = a_0 + \frac{a_{-1}}{\phi(\xi)}, \tag{5.118}$$

Set 2 :

$$a_0 = -\frac{\sqrt{\gamma}\left((\Delta)\sqrt{(\Delta)^{-1}} + p\right)}{2\beta(\Delta)\sqrt{-(\beta\Delta)^{-1}}}, \quad a_1 = q\sqrt{\gamma}\sqrt{-(\beta\Delta)^{-1}}$$

$$a_{-1} = 0, \quad \chi = \frac{\sqrt{\gamma}\sqrt{9\gamma - 2}\sqrt{(\Delta)^{-1}}}{2}, \quad \lambda = \frac{3\gamma\sqrt{(\Delta)^{-1}}}{2}.$$

Under these conditions Eq. (5.117), takes the form,

$$U_2(\xi) = a_0 + a_1\phi(\xi), \tag{5.119}$$

Set 3 :

$$a_0 = -\frac{\sqrt{\gamma}\,p}{\beta(\Delta)\sqrt{-(\beta\Delta)^{-1}}}, \quad a_1 = 0, \quad a_{-1} = 2r\sqrt{\gamma}\sqrt{-(\beta\Delta)^{-1}},$$

$$\chi = 2\sqrt{\gamma}\sqrt{(\Delta)^{-1}}, \qquad \lambda = 0,$$

Under these conditions Eq. (5.117), takes the form,

$$U_3(\xi) = a_0 + \frac{a_{-1}}{\phi(\xi)}, \tag{5.120}$$

Set 4 :

$$a_0 = -\frac{\sqrt{\gamma}\,p}{\beta(\Delta)\sqrt{-(\beta\Delta)^{-1}}}, \quad a_1 = 2q\sqrt{\gamma}\sqrt{-(\beta\Delta)^{-1}}, \quad a_{-1} = 0,$$

$$\chi = 2\sqrt{\gamma}\sqrt{(\Delta)^{-1}}, \qquad \lambda = 0.$$

Under these conditions Eq. (5.117), takes the form,

$$U_4(\xi) = a_0 + a_1\phi(\xi), \tag{5.121}$$

for the case 1, substituting the values of coefficients into Eq. (5.118) along with the Riccati equations solutions, we can get many different types of solutions including solitary wave solutions, periodic wave solutions and rational solutions.

Family 1:

For case 1, when $\Delta > 0$ and $pq \neq 0$ or $qr \neq 0$, the hyperbolic function solutions of Eq.(5.114) are as follows,

$$U_{1,1}(\xi) = -\frac{\left(\tanh\left(\Omega + \frac{3\gamma t}{4}\right) - 1\right)(\sqrt{\Delta}\,p - \Delta)\sqrt{\gamma}}{2\sqrt{\Delta}\beta\sqrt{-\beta^{-1}}\left(p + \sqrt{\Delta}\tanh\left(\Omega + \frac{3\gamma t}{4}\right)\right)}, \tag{5.122}$$

$$U_{1,2}(\xi) = -\frac{\left(\coth\left(\Omega + \frac{3\gamma t}{4}\right) - 1\right)(\sqrt{\Delta}\,p - \Delta)\sqrt{\gamma}}{2\sqrt{\Delta}\beta\sqrt{-\beta^{-1}}\left(p + \sqrt{\Delta}\coth\left(\Omega + \frac{3\gamma t}{4}\right)\right)}, \tag{5.123}$$

$$U_{1,3}(\xi) = \frac{-(\sqrt{\Delta}p - \Delta)\begin{pmatrix} i - \cosh\left(2\Omega + \frac{3\gamma t}{2}\right) \\ +\sinh\left(2\Omega + \frac{3\gamma t}{2}\right)\end{pmatrix}\sqrt{\gamma}}{2\sqrt{\Delta}\sqrt{-\beta^{-1}}\begin{pmatrix} \text{p}\cosh\left(2\Omega + \frac{3\gamma t}{2}\right) \\ +\sqrt{\Delta}\left(\sinh\left(2\Omega + \frac{3\gamma t}{2}\right) + i\right)\end{pmatrix}\beta}, \tag{5.124}$$

$$U_{1,4}(\xi)=\frac{-(\sqrt{\Delta}p-\Delta)\begin{pmatrix}\cosh\left(2\Omega+\frac{3\gamma t}{2}\right)\\-\sinh\left(2\Omega+\frac{3\gamma t}{2}\right)+1\end{pmatrix}\sqrt{\gamma}}{2\sqrt{\Delta}\sqrt{-\beta^{-1}}\begin{pmatrix}\mathrm{psinh}\left(2\Omega+\frac{3\gamma t}{2}\right)\\+\sqrt{\Delta}\left(\cosh\left(2\Omega+\frac{3\gamma t}{2}\right)+1\right)\end{pmatrix}\beta}, \tag{5.125}$$

$$U_{1,5}(\xi)=-\frac{\sqrt{\gamma}(\sqrt{\Delta}p-\Delta)}{2\sqrt{\Delta}\sqrt{-\beta^{-1}}\beta} \times\frac{\begin{pmatrix}\cosh\left(\frac{\Omega}{2}+\frac{3\gamma t}{8}\right)^2-1/2\\-\cosh\left(\frac{\Omega}{2}+\frac{3\gamma t}{8}\right)\sinh\left(\frac{\Omega}{2}+\frac{3\gamma t}{8}\right)\end{pmatrix}}{\begin{pmatrix}\sqrt{\Delta}(\cosh\left(\frac{\Omega}{2}+\frac{3\gamma t}{8}\right)^2-\frac{\sqrt{\Delta}}{2}\\+p\cosh\left(\frac{\Omega}{2}+\frac{3\gamma t}{8}\right)\sinh\left(\frac{\Omega}{2}+\frac{3\gamma t}{8}\right)\end{pmatrix}}, \tag{5.126}$$

here, A, B are real constants that satisfies $B^2-A^2>0$,

$$U_{1,6}(\xi)=-\frac{\sqrt{\gamma}(\sqrt{\Delta}p-\Delta)}{2\sqrt{\Delta}\sqrt{-\beta^{-1}}\beta} \times\frac{\left(-\sqrt{A^2+B^2}+\begin{pmatrix}\cosh\left(2\Omega+\frac{3\gamma t}{2}\right)\\-\sinh\left(\frac{\Omega}{2}+\frac{3\gamma t}{8}\right)\end{pmatrix}A-B\right)}{\begin{pmatrix}-\sqrt{\Delta}\sqrt{A^2+B^2}+\left(\mathrm{Asinh}\left(2\Omega+\frac{3\gamma t}{2}\right)+B\right)p\\+A\sqrt{\Delta}\cosh\left(\frac{\Omega}{2}+\frac{3\gamma t}{8}\right)\end{pmatrix}\beta}, \tag{5.127}$$

$$U_{1,7}(\xi)=\frac{\sqrt{\gamma}(\sqrt{\Delta}p-\Delta)}{2\sqrt{\Delta}\sqrt{-\beta^{-1}}\beta} \times\frac{\left(-\sqrt{B^2-A^2}+\begin{pmatrix}\cosh\left(2\Omega+\frac{3\gamma t}{2}\right)\\-\sinh\left(\frac{\Omega}{2}+\frac{3\gamma t}{8}\right)\end{pmatrix}A+B\right)}{\begin{pmatrix}\sqrt{\Delta}\sqrt{B^2-A^2}+\left(\mathrm{Acosh}\left(2\Omega+\frac{3\gamma t}{2}\right)+B\right)p\\+A\sqrt{\Delta}\sinh\left(\frac{\Omega}{2}+\frac{3\gamma t}{8}\right)\end{pmatrix}}, \tag{5.128}$$

$$U_{1,8}(\xi) = \frac{-\sqrt{\gamma}\left(\sinh\left(\Omega + \frac{3\gamma t}{4}\right) - \cosh\left(\Omega + \frac{3\gamma t}{4}\right)\right)}{2\sqrt{-\beta^{-1}}\beta\left(\cosh\left(\Omega + \frac{3\gamma t}{4}\right)\right)}, \tag{5.129}$$

$$U_{1,9}(\xi) = \frac{-\sqrt{\gamma}\sqrt{-\beta^{-1}}\left(\sinh\left(\Omega + \frac{3\gamma t}{4}\right) - \cosh\left(\Omega + \frac{3\gamma t}{4}\right)\right)}{2\left(\sinh\left(\Omega + \frac{3\gamma t}{4}\right)\right)}, \tag{5.130}$$

$$U_{1,10}(\xi) = \frac{-\sqrt{\gamma}\left(i - \cosh\left(2\Omega + \frac{3\gamma t}{2}\right) + \sinh\left(2\Omega + \frac{3\gamma t}{2}\right)\right)}{2\sqrt{-\beta^{-1}}\beta\left(\cosh\left(2\Omega + \frac{3\gamma t}{2}\right)\right)}, \tag{5.131}$$

$$U_{1,11}(\xi) = \frac{-\sqrt{\gamma}\sqrt{-\beta^{-1}}\left(\cosh\left(2\Omega + \frac{3\gamma t}{2}\right) - \sinh\left(2\Omega + \frac{3\gamma t}{2}\right) + 1\right)}{2\left(\sinh\left(2\Omega + \frac{3\gamma t}{2}\right)\right)}, \tag{5.132}$$

$$U_{1,12}(\xi) = \sqrt{\gamma}\sqrt{-\beta^{-1}}\frac{\begin{pmatrix} 2\cosh\left(\frac{\Omega}{2} + \frac{3\gamma t}{8}\right)\sinh\left(\frac{\Omega}{2} + \frac{3\gamma t}{8}\right) \\ \cosh\left(2\Omega + \frac{3\gamma t}{2}\right)^2 + 1 \end{pmatrix}}{4\cosh\left(\frac{\Omega}{2} + \frac{3\gamma t}{8}\right)\sinh\left(\frac{\Omega}{2} + \frac{3\gamma t}{8}\right)}. \tag{5.133}$$

Family 2:

When, $\Delta < 0$ and $pq \neq 0$ $(or\ qr \neq 0)$, we have the following trigonometric solutions for Eq. (5.114),

$$U_{1,13}(\xi) = -\frac{\left(\tanh\left(\Omega + \frac{3\gamma t}{4}\right) - 1\right)(\sqrt{\Delta}p - \Delta)\sqrt{\gamma}}{2\sqrt{\Delta}\beta\sqrt{-\beta^{-1}}\left(p + \sqrt{\Delta}\tanh\left(\Omega + \frac{3\gamma t}{4}\right)\right)}, \tag{5.134}$$

$$U_{1,14}(\xi) = -\frac{\left(\coth\left(\Omega + \frac{3\gamma t}{4}\right) - 1\right)(\sqrt{\Delta}p - \Delta)\sqrt{\gamma}}{2\sqrt{\Delta}\beta\sqrt{-\beta^{-1}}\left(p + \sqrt{\Delta}\coth\left(\Omega + \frac{3\gamma t}{4}\right)\right)}, \tag{5.135}$$

$$U_{1,15}(\xi)=\frac{\sqrt{-\beta^{-1}}\sqrt{\gamma}(\sqrt{\Delta}\,p-\Delta)\begin{pmatrix} i+\cosh\left(2\Omega+\frac{3\gamma t}{2}\right)\\ -\sinh\left(2\Omega+\frac{3\gamma t}{2}\right)\end{pmatrix}}{\sqrt{\Delta}\begin{pmatrix} -2p\cosh\left(2\Omega+\frac{3\gamma t}{2}\right)+\\ 2\left(i-\sinh\left(\Omega+\frac{3\gamma t}{4}\right)\right)\sqrt{\Delta}\end{pmatrix}}, \tag{5.136}$$

$$U_{1,16}(\xi)=\frac{\sqrt{-\beta^{-1}}\sqrt{\gamma}(\sqrt{\Delta}\,p-\Delta)\begin{pmatrix} 1+\cosh\left(2\Omega+\frac{3\gamma t}{2}\right)\\ -\sinh\left(2\Omega+\frac{3\gamma t}{2}\right)\end{pmatrix}}{\sqrt{\Delta}\begin{pmatrix} 2p\cosh\left(2\Omega+\frac{3\gamma t}{2}\right)+\\ 2\left(1+\cosh\left(\Omega+\frac{3\gamma t}{4}\right)\right)\sqrt{\Delta}\end{pmatrix}}, \tag{5.137}$$

$$U_{1,17}(\xi)=\frac{2\sqrt{-\beta^{-1}}\sqrt{\gamma}(\sqrt{\Delta}\,p-\Delta)}{\sqrt{\Delta}}\times\frac{\begin{pmatrix} \cosh\left(2\Omega+\frac{3\gamma t}{2}\right)^2-\frac{1}{2}\\ -\sinh\left(2\Omega+\frac{3\gamma t}{2}\right)\cosh\left(2\Omega+\frac{3\gamma t}{2}\right)\end{pmatrix}}{\begin{pmatrix} 4\sinh\left(2\Omega+\frac{3\gamma t}{2}\right)\cosh\left(2\Omega+\frac{3\gamma t}{2}\right)+\\ 2\left(2\cosh\left(2\Omega+\frac{3\gamma t}{2}\right)^2-1\right)\sqrt{\Delta}\end{pmatrix}}, \tag{5.138}$$

here two non-zero real constants A and B satisfies $A^2-B^2>0$.

$$U_{1,18}(\xi)=\frac{2\sqrt{-\beta^{-1}}\sqrt{\gamma}\left(\sqrt{\Delta}\,p-\Delta\right)}{\sqrt{\Delta}}\times\frac{\left(iA\cosh\left(2\Omega+\frac{3\gamma t}{2}\right)-i\sinh\left(2\Omega+\frac{3\gamma t}{2}\right)A-\sqrt{B^2-A^2}-B\right)}{\begin{pmatrix} 2iA\sqrt{\Delta}\cosh\left(2\Omega+\frac{3\gamma t}{2}\right)+2iA\sinh\left(2\Omega+\frac{3\gamma t}{2}\right)p\\ -2\sqrt{B^2-A^2}\sqrt{\Delta}+2Bp\end{pmatrix}}, \tag{5.139}$$

$$U_{1,19}(\xi)=\frac{\sqrt{-\beta^{-1}}\sqrt{\gamma}\left(\sqrt{\Delta}\,p-\Delta\right)}{} \tag{5.140}$$

$$\times \frac{\left(iA\cosh\left(2\Omega+\frac{3\gamma t}{2}\right)-i\sinh\left(2\Omega+\frac{3\gamma t}{2}\right)A+\sqrt{B^2-A^2}-B\right)}{\left(\begin{array}{c}2iA\sqrt{\Delta}\cosh\left(2\Omega+\frac{3\gamma t}{2}\right)+2iA\sinh\left(2\Omega+\frac{3\gamma t}{2}\right)p\\ +2\sqrt{B^2-A^2}\sqrt{\Delta}+2Bp\end{array}\right)},$$

$$U_{1,20}(\xi)=\frac{\sqrt{\gamma}\left(\sinh\left(\Omega+\frac{3\gamma t}{4}\right)-\cosh\left(\Omega+\frac{3\gamma t}{4}\right)\right)}{2\beta\sqrt{-\beta^{-1}}\cosh\left(\Omega+\frac{3\gamma t}{4}\right)}, \tag{5.141}$$

$$U_{1,21}(\xi)=\frac{-\sqrt{\gamma}\sqrt{-\beta^{-1}}\left(\sinh\left(\Omega+\frac{3\gamma t}{4}\right)-\cosh\left(\Omega+\frac{3\gamma t}{4}\right)\right)}{2\sinh\left(\Omega+\frac{3\gamma t}{4}\right)}, \tag{5.142}$$

$$U_{1,22}(\xi)=\frac{\sqrt{\gamma}\left(i+\cosh\left(2\Omega+\frac{3\gamma t}{2}\right)-\sinh\left(2\Omega+\frac{3\gamma t}{2}\right)\right)}{2\beta\sqrt{-\beta^{-1}}\cosh\left(2\Omega+\frac{3\gamma t}{2}\right)}, \tag{5.143}$$

$$U_{1,23}(\xi)=\frac{\sqrt{-\beta^{-1}}\sqrt{\gamma}\left(1+\cosh\left(2\Omega+\frac{3\gamma t}{2}\right)-\sinh\left(2\Omega+\frac{3\gamma t}{2}\right)\right)}{2\sinh\left(2\Omega+\frac{3\gamma t}{2}\right)}, \tag{5.144}$$

$$U_{1,24}(\xi)=\frac{-\sqrt{-\beta^{-1}}\sqrt{\gamma}\left(\begin{array}{c}1-2\cosh\left(\frac{\Omega}{2}+\frac{3\gamma t}{8}\right)^2+\\ 2\cosh\left(\frac{\Omega}{2}+\frac{3\gamma t}{8}\right)\sinh\left(\frac{\Omega}{2}+\frac{3\gamma t}{8}\right)\end{array}\right)}{4\cosh\left(\frac{\Omega}{2}+\frac{3\gamma t}{8}\right)\sinh\left(\frac{\Omega}{2}+\frac{3\gamma t}{8}\right)}. \tag{5.145}$$

Family 3:

When $r=0$, and $pq\neq 0$, we get soliton like solutions.

$$U_{1,25}(\xi)=\frac{\sqrt{\gamma}\left(\frac{dp^3\sqrt{p^{-2}}}{2}-\frac{dp^2}{2}\right)}{p^3 d\beta}\frac{1}{\sqrt{-\frac{1}{\beta p^2}}}, \tag{5.146}$$

$$U_{1,26}(\xi) = -\sqrt{\gamma}\sqrt{-\frac{1}{\beta p^2}} \tag{5.147}$$

$$\times \frac{\left(\begin{array}{c}\left(\frac{p^3}{2}\sqrt{p^{-2}} - \frac{p^2}{2}\right)\cosh\left(2\Omega + \frac{p\sqrt{p^{-2}}3\gamma t}{2}\right) + \\ \left(\frac{p^3}{2}\sqrt{p^{-2}} - \frac{p^2}{2}\right)\sinh\left(2\Omega + \frac{p\sqrt{p^{-2}}3\gamma t}{2}\right)\end{array}\right)}{p\left(\cosh\left(2\Omega + \frac{p\sqrt{p^{-2}}3\gamma t}{2}\right) + \sinh\left(2\Omega + \frac{p\sqrt{p^{-2}}3\gamma t}{2}\right)\right)}.$$

Here, d is arbitrary constant.

Family 2:

When, $\Delta < 0$ and $pq \neq 0$ $(or\ qr \neq 0)$, we have the following trigonometric solutions for Eq. (5.114),

$$U_{1,13}(\xi) = -\frac{\left(\tanh\left(\Omega + \frac{3\gamma t}{4}\right) - 1\right)(\sqrt{\Delta}\,p - \Delta)\sqrt{\gamma}}{2\sqrt{\Delta}\beta\sqrt{-\beta^{-1}}\left(p + \sqrt{\Delta}\tanh\left(\Omega + \frac{3\gamma t}{4}\right)\right)}, \tag{5.148}$$

$$U_{1,14}(\xi) = -\frac{\left(\coth\left(\Omega + \frac{3\gamma t}{4}\right) - 1\right)(\sqrt{\Delta}\,p - \Delta)\sqrt{\gamma}}{2\sqrt{\Delta}\beta\sqrt{-\beta^{-1}}\left(p + \sqrt{\Delta}\coth\left(\Omega + \frac{3\gamma t}{4}\right)\right)}, \tag{5.149}$$

$$U_{1,15}(\xi) = \frac{\sqrt{-\beta^{-1}}\sqrt{\gamma}(\sqrt{\Delta}\,p - \Delta)\left(\begin{array}{c} i + \cosh\left(2\Omega + \frac{3\gamma t}{2}\right) \\ -\sinh\left(\frac{\sqrt{\gamma}\sqrt{9\gamma - 2x}}{2} + \frac{3\gamma t}{2}\right)\end{array}\right)}{\sqrt{\Delta}\left(\begin{array}{c} -2p\cosh\left(2\Omega + \frac{3\gamma t}{2}\right) + \\ 2\left(i - \sinh\left(2\Omega + \frac{3\gamma t}{4}\right)\right)\sqrt{\Delta}\end{array}\right)}, \tag{5.150}$$

$$U_{1,16}(\xi) = \frac{\sqrt{-\beta^{-1}}\sqrt{\gamma}(\sqrt{\Delta}\,p - \Delta)\begin{pmatrix} 1 + \cosh\left(2\Omega + \frac{3\gamma t}{2}\right) \\ -\sinh\left(2\Omega + \frac{3\gamma t}{2}\right) \end{pmatrix}}{\sqrt{\Delta}\begin{pmatrix} 2p\cosh\left(2\Omega + \frac{3\gamma t}{2}\right) + \\ 2\left(1 + \cosh\left(\Omega + \frac{3\gamma t}{4}\right)\right)\sqrt{\Delta} \end{pmatrix}}, \tag{5.151}$$

$$U_{1,17}(\xi) = \frac{2\sqrt{-\beta^{-1}}\sqrt{\gamma}(\sqrt{\Delta}\,p - \Delta)}{\sqrt{\Delta}} \times \frac{\begin{pmatrix} \cosh\left(2\Omega + \frac{3\gamma t}{2}\right)^2 - \frac{1}{2} \\ -\sinh\left(2\Omega + \frac{3\gamma t}{2}\right)\cosh\left(2\Omega + \frac{3\gamma t}{2}\right) \end{pmatrix}}{\begin{pmatrix} 4\sinh\left(2\Omega + \frac{3\gamma t}{2}\right)\cosh\left(2\Omega + \frac{3\gamma t}{2}\right) + \\ 2\left(2\cosh\left(2\Omega + \frac{3\gamma t}{2}\right)^2 - 1\right)\sqrt{\Delta} \end{pmatrix}}, \tag{5.152}$$

here two non-zero real constants A and B satisfies $A^2 - B^2 > 0$.

$$U_{1,18}(\xi) = \frac{2\sqrt{-\beta^{-1}}\sqrt{\gamma}\left(\sqrt{\Delta}\,p - \Delta\right)}{\sqrt{\Delta}} \times \frac{\left(iA\cosh\left(2\Omega + \frac{3\gamma t}{2}\right) - i\sinh\left(2\Omega + \frac{3\gamma t}{2}\right)A - \sqrt{B^2 - A^2} - B\right)}{\begin{pmatrix} 2iA\sqrt{\Delta}\cosh\left(2\Omega + \frac{3\gamma t}{2}\right) + 2iA\sinh\left(2\Omega + \frac{3\gamma t}{2}\right)p \\ -2\sqrt{B^2 - A^2}\sqrt{\Delta} + 2Bp \end{pmatrix}}, \tag{5.153}$$

$$U_{1,19}(\xi) = \frac{\sqrt{-\beta^{-1}}\sqrt{\gamma}\left(\sqrt{\Delta}\,p - \Delta\right)}{\sqrt{\Delta}} \times \frac{\left(iA\cosh\left(2\Omega + \frac{3\gamma t}{2}\right) - i\sinh\left(2\Omega + \frac{3\gamma t}{2}\right)A + \sqrt{B^2 - A^2} - B\right)}{\begin{pmatrix} 2iA\sqrt{\Delta}\cosh\left(2\Omega + \frac{3\gamma t}{2}\right) + 2iA\sinh\left(2\Omega + \frac{3\gamma t}{2}\right)p \\ +2\sqrt{B^2 - A^2}\sqrt{\Delta} + 2Bp \end{pmatrix}}, \tag{5.154}$$

$$U_{1,20}(\xi) = \frac{\sqrt{\gamma}\left(\sinh\left(\Omega + \frac{3\gamma t}{4}\right) - \cosh\left(\Omega + \frac{3\gamma t}{4}\right)\right)}{2\beta\sqrt{-\beta^{-1}}\cosh\left(\Omega + \frac{3\gamma t}{4}\right)}, \tag{5.155}$$

$$U_{1,21}(\xi) = \frac{-\sqrt{\gamma}\sqrt{-\beta^{-1}}\left(\sinh\left(\Omega + \frac{3\gamma t}{4}\right) - \cosh\left(\Omega + \frac{3\gamma t}{4}\right)\right)}{2\sinh\left(\Omega + \frac{3\gamma t}{4}\right)}, \tag{5.156}$$

$$U_{1,22}(\xi) = \frac{\sqrt{\gamma}\left(i + \cosh\left(2\Omega + \frac{3\gamma t}{2}\right) - \sinh\left(2\Omega + \frac{3\gamma t}{2}\right)\right)}{2\beta\sqrt{-\beta^{-1}}\cosh\left(2\Omega + \frac{3\gamma t}{2}\right)}, \tag{5.157}$$

$$U_{1,23}(\xi) = \frac{\sqrt{-\beta^{-1}}\sqrt{\gamma}\left(1 + \cosh\left(2\Omega + \frac{3\gamma t}{2}\right) - \sinh\left(2\Omega + \frac{3\gamma t}{2}\right)\right)}{2\sinh\left(2\Omega + \frac{3\gamma t}{2}\right)}, \tag{5.158}$$

$$U_{1,24}(\xi) = \frac{-\sqrt{-\beta^{-1}}\sqrt{\gamma}\begin{pmatrix} 1 - 2\cosh\left(\frac{\Omega}{2} + \frac{3\gamma t}{8}\right)^2 + \\ 2\cosh\left(\frac{\Omega}{2} + \frac{3\gamma t}{8}\right)\sinh\left(\frac{\Omega}{2} + \frac{3\gamma t}{8}\right)\end{pmatrix}}{4\cosh\left(\frac{\Omega}{2} + \frac{3\gamma t}{8}\right)\sinh\left(\frac{\Omega}{2} + \frac{3\gamma t}{8}\right)}. \tag{5.159}$$

Family 3:

When $r = 0$, and $pq \neq 0$, we get soliton like solutions.

$$U_{1,25}(\xi) = \frac{\sqrt{\gamma}\left(\frac{dp^3\sqrt{p^{-2}}}{2} - \frac{dp^2}{2}\right)}{p^3 d\beta}\frac{1}{\sqrt{-\frac{1}{\beta p^2}}}, \tag{5.160}$$

$$U_{1,26}(\xi) = \sqrt{\gamma} \tag{5.161}$$

$$\times \frac{\begin{pmatrix}\left(\frac{p^3}{2}\sqrt{p^{-2}}-\frac{p^2}{2}\right)\cosh\left(2\Omega+\frac{p\sqrt{p^{-2}}+3\gamma t}{2}\right)+\\ \left(\frac{p^3}{2}\sqrt{p^{-2}}-\frac{p^2}{2}\right)\sinh\left(2\Omega+\frac{p\sqrt{p^{-2}}3\gamma t}{2}\right)\end{pmatrix}}{p\left(\cosh\left(2\Omega+\frac{p\sqrt{p^{-2}}3\gamma t}{2}\right)+\sinh\left(2\Omega+\frac{p\sqrt{p^{-2}}3\gamma t}{2}\right)\right)}.$$

Here, d is arbitrary constant.

Family 1:

For case 3, when $p^2-4rq>0$ and $pq\neq 0$ or $qr\neq 0$, the hyperbolic function solutions of Eq. (5.114) are as follows,

$$U_{3,1}(\xi)=-\frac{\left(i\sqrt{\Delta}\tan\left(\frac{\sqrt{2}\sqrt{\gamma}}{2}x\right)p+\Delta\right)\sqrt{\gamma}}{\left(p+i\sqrt{\Delta}\tan\left(\frac{\sqrt{2}\sqrt{\gamma}}{2}x\right)\right)\sqrt{\Delta}\,\beta}\frac{1}{\sqrt{-\beta^{-1}}}, \tag{5.162}$$

$$U_{3,2}(\xi)=-\frac{\left(i\sqrt{\Delta}\cot\left(\frac{\sqrt{2}\sqrt{\gamma}}{2}x\right)p-\Delta\right)\sqrt{\gamma}}{\left(i\sqrt{\Delta}\cot\left(\frac{\sqrt{2}\sqrt{\gamma}}{2}x\right)-p\right)\sqrt{\Delta}\,\beta}\frac{1}{\sqrt{-\beta^{-1}}}, \tag{5.163}$$

$$U_{3,3}(\xi)=-\frac{\left(ip\left(\sin\left(\sqrt{2}\sqrt{\gamma}x\right)+1\right)\sqrt{\Delta}+\cos\left(\sqrt{2}\sqrt{\gamma}x\right)(\Delta)\right)\sqrt{\gamma}}{\sqrt{\Delta}\beta\left(\left(i\sin\left(\sqrt{2}\sqrt{\gamma}x\right)+i\right)\sqrt{\Delta}+p\cos\left(\sqrt{2}\sqrt{\gamma}x\right)\right)}\frac{1}{\sqrt{-\beta^{-1}}}, \tag{5.164}$$

$$U_{3,4}(\xi)=-\frac{\left(ip\left(\cos\left(\sqrt{2}\sqrt{\gamma}x\right)+1\right)\sqrt{\Delta}-\sin\left(\sqrt{2}\sqrt{\gamma}x\right)(\Delta)\right)\sqrt{\gamma}}{\sqrt{\Delta}\beta\left(\left(i\,cos\left(\sqrt{2}\sqrt{\gamma}x\right)+i\right)\sqrt{\Delta}-p\sin\left(\sqrt{2}\sqrt{\gamma}x\right)\right)}\frac{1}{\sqrt{-\beta^{-1}}}, \tag{5.165}$$

$$U_{3,5}(\xi) = -\frac{\begin{pmatrix} i\left(\cos\left(\frac{\sqrt{2}\sqrt{\gamma}x}{4}\right)^2 - \frac{1}{2}\right)p\sqrt{\Delta} \\ -\cos\left(\frac{\sqrt{2}\sqrt{\gamma}x}{4}\right)\sin\left(\frac{\sqrt{2}\sqrt{\gamma}x}{4}\right)(\Delta) \end{pmatrix}\sqrt{\gamma}}{\sqrt{\Delta}\begin{pmatrix} i\left(\cos\left(\frac{\sqrt{2}\sqrt{\gamma}x}{4}\right)^2 - \frac{i}{2}\right)\sqrt{\Delta} \\ -p\cos\left(\frac{\sqrt{2}\sqrt{\gamma}x}{4}\right)\sin\left(\frac{\sqrt{2}\sqrt{\gamma}x}{4}\right) \end{pmatrix}\beta}\frac{1}{\sqrt{-\beta^{-1}}}, \tag{5.166}$$

here, A, B are real constants that satisfies $B^2 - A^2 > 0$,

$$U_{3,6}(\xi) = -\frac{\sqrt{\gamma}\left(-\sqrt{\Delta}\sqrt{A^2+B^2}p + A\sqrt{\Delta}\cos\left(\sqrt{2}\sqrt{\gamma}x\right)p + \left(iA\sin\left(\sqrt{2}\sqrt{\gamma}x\right) + B\right)(\Delta)\right)}{\sqrt{\Delta}\,\beta\left(-\sqrt{\Delta}\sqrt{A^2+B^2} + A\sqrt{\Delta}\cos\left(\sqrt{2}\sqrt{\gamma}x\right) + \left(iA\sin\left(\sqrt{2}\sqrt{\gamma}x\right) + B\right)p\right)}\frac{1}{\sqrt{-\beta^{-1}}}, \tag{5.167}$$

$$U_{3,7}(\xi) = -\frac{\sqrt{\gamma}\left(-\sqrt{\Delta}\sqrt{-A^2+B^2}p + iA\sqrt{\Delta}\sin\left(\sqrt{2}\sqrt{\gamma}x\right)p + \left(A\cos\left(\sqrt{2}\sqrt{\gamma}x\right) + B\right)(\Delta)\right)}{\sqrt{\Delta}\,\beta\left(-\sqrt{\Delta}\sqrt{-A^2+B^2} + iA\sqrt{\Delta}\sin\left(\sqrt{2}\sqrt{\gamma}x\right) + \left(Acos\left(\sqrt{2}\sqrt{\gamma}x\right) + B\right)p\right)}\frac{1}{\sqrt{-\beta^{-1}}}, \tag{5.168}$$

$$U_{3,8}(\xi) = \frac{-i\sin\left(\sqrt{2}\sqrt{\gamma}x\right)\sqrt{\gamma}}{\beta\sqrt{-\beta^{-1}}cos\left(\sqrt{2}\sqrt{\gamma}x\right)}, \tag{5.169}$$

$$U_{3,9}(\xi) = \frac{-i\sin\left(\sqrt{2}\sqrt{\gamma}x\right)\sqrt{\gamma}\sqrt{-\beta^{-1}}}{cos\left(\sqrt{2}\sqrt{\gamma}x\right)}, \tag{5.170}$$

$$U_{3,10}(\xi) = \frac{-i\left(\sin\left(\sqrt{2}\sqrt{\gamma}x\right) + 1\right)\sqrt{\gamma}}{\beta\sqrt{-\beta^{-1}}cos\left(\sqrt{2}\sqrt{\gamma}x\right)}, \tag{5.171}$$

$$U_{3,11}(\xi) = \frac{-i\left(\cos\left(\sqrt{2}\sqrt{\gamma}x\right) + 1\right)\sqrt{\gamma}\sqrt{-\beta^{-1}}}{\sin\left(\sqrt{2}\sqrt{\gamma}x\right)}, \tag{5.172}$$

$$U_{3,12}(\xi) = \frac{-\frac{i}{2}\left(2\cos\left(\frac{\sqrt{2}\sqrt{\gamma}x}{4}\right)^2 - 1\right)\sqrt{\gamma}\sqrt{-\beta^{-1}}}{\sin\left(\frac{\sqrt{2}\sqrt{\gamma}x}{4}\right)\cos\left(\frac{\sqrt{2}\sqrt{\gamma}x}{4}\right)}, \tag{5.173}$$

$$U_{3,13}(\xi)=-\frac{\left(i\sqrt{\Delta}\tan\left(\frac{\sqrt{2}\sqrt{\gamma}x}{2}\right)p+\Delta\right)\sqrt{\gamma}}{\left(p+i\sqrt{\Delta}\tan\left(\frac{\sqrt{2}\sqrt{\gamma}x}{2}\right)\right)\sqrt{\Delta}\beta}\frac{1}{\sqrt{-\beta^{-1}}}. \tag{5.174}$$

Family 2:

When, <0 and $pq\neq 0$ $(or\ qr\neq 0)$, we have the following trigonometric solutions for Eq. (5.114).

$$U_{3,14}(\xi)=-\frac{(i\sqrt{\Delta}co\,t\left(\frac{\sqrt{2}\sqrt{\gamma}x}{2}\right)p-\Delta)\sqrt{\gamma}}{\left(i\sqrt{\Delta}co\,t\left(\frac{\sqrt{2}\sqrt{\gamma}x}{2}\right)-p\right)\sqrt{\Delta}\beta}\frac{1}{\sqrt{-\beta^{-1}}}, \tag{5.175}$$

$$U_{3,15}(\xi)=-\frac{\left(ip(\ sin(\sqrt{2}\sqrt{\gamma}x)-1)\sqrt{\Delta}+cos(\sqrt{2}\sqrt{\gamma}x)\Delta\right)\sqrt{\gamma}\sqrt{-\beta^{-1}}}{\left((i\ sin(\sqrt{2}\sqrt{\gamma}x)-i)\sqrt{\Delta}+pcos(\sqrt{2}\sqrt{\gamma}x)\right)\sqrt{\Delta}}, \tag{5.176}$$

$$U_{3,16}(\xi)=-\frac{\left((cos(\sqrt{2}\sqrt{\gamma}x)+1)ip\sqrt{\Delta}-sin(\sqrt{2}\sqrt{\gamma}x)\Delta\right)\sqrt{\gamma}\sqrt{-\beta^{-1}}}{\left((i\ sin(\sqrt{2}\sqrt{\gamma}x)-i)\sqrt{\Delta}+pcos(\sqrt{2}\sqrt{\gamma}x)\right)\sqrt{\Delta}}, \tag{5.177}$$

$$U_{3,17}(\xi)=\frac{\left(2i\left(\cos\left(\frac{\sqrt{2}\sqrt{\gamma}x}{4}\right)^2-\frac{1}{2}\right)p\sqrt{\Delta}-\sin\left(\frac{\sqrt{2}\sqrt{\gamma}x}{2}\right)(\Delta)\right)\sqrt{\gamma}}{\sqrt{\Delta}\left(\left(2i\cos\left(\frac{\sqrt{2}\sqrt{\gamma}x}{4}\right)^2-i\right)\sqrt{\Delta}-p\sin\left(\frac{\sqrt{2}\sqrt{\gamma}x}{2}\right)\right)}\sqrt{-\beta^{-1}}, \tag{5.178}$$

here, A, B are real constants that satisfies $B^2-A^2>0$.

$$U_{3,18}(\xi) =\frac{\sqrt{\gamma}\left(-\sqrt{-A^2+B^2}\sqrt{\Delta}p+iA\sqrt{\Delta}\cos(\sqrt{2}\sqrt{\gamma}x)p-(\Delta)(A\sin(\sqrt{2}\sqrt{\gamma}x)-B)\right)}{\sqrt{\Delta}\left(-\sqrt{-A^2+B^2}\sqrt{\Delta}+iA\sqrt{\Delta}\cos(\sqrt{2}\sqrt{\gamma}x)-p(A\sin(\sqrt{2}\sqrt{\gamma}x)-B)\right)}\sqrt{-\beta^{-1}}, \tag{5.179}$$

$$U_{3,19}(\xi) = \frac{\sqrt{\gamma}\left(\sqrt{-A^2+B^2}\sqrt{\Delta}p + iA\sqrt{\Delta}\cos\left(\sqrt{2}\sqrt{\gamma}x\right)p - (\Delta)\left(A\sin\left(\sqrt{2}\sqrt{\gamma}x\right) - B\right)\right)}{\sqrt{\Delta}\left(\sqrt{-A^2+B^2}\sqrt{\Delta} + iA\sqrt{\Delta}\cos\left(\sqrt{2}\sqrt{\gamma}x\right) - p\left(A\sin\left(\sqrt{2}\sqrt{\gamma}x\right) - B\right)\right)}\sqrt{-\beta^{-1}}, \tag{5.180}$$

$$U_{3,20}(\xi) = \frac{-icos\left(\frac{\sqrt{2}\sqrt{\gamma}x}{2}\right)\sqrt{\gamma}\sqrt{-\beta^{-1}}}{\sin\left(\frac{\sqrt{2}\sqrt{\gamma}x}{2}\right)}, \tag{5.181}$$

$$U_{3,21}(\xi) = \frac{-i\left(\sin\left(\sqrt{2}\sqrt{\gamma}x\right) - 1\right)\sqrt{\gamma}}{\beta\sqrt{-\beta^{-1}}cos\left(\sqrt{2}\sqrt{\gamma}x\right)}, \tag{5.182}$$

$$U_{3,22}(\xi) = \frac{-i\left(\cos\left(\sqrt{2}\sqrt{\gamma}x\right) + 1\right)\sqrt{-\beta^{-1}}\sqrt{\gamma}}{\sin\left(\sqrt{2}\sqrt{\gamma}x\right)}. \tag{5.183}$$

Family 3:

When $r = 0$, and $pq \neq 0$, we get soliton like solutions.

$$U_{3,23}(\xi) = \frac{\sqrt{\gamma}}{p\beta}\left(\sqrt{-\frac{1}{\beta p^2}}\right)^{-1}, \tag{5.184}$$

$$U_{3,24}(\xi) = \frac{\sqrt{\gamma}}{p}\frac{\left(p^2\cosh\left(p\sqrt{2}\sqrt{\gamma}\sqrt{-p^{-2}}x\right) + p^2\sinh\left(p\sqrt{2}\sqrt{\gamma}\sqrt{-p^{-2}}x\right)\right)}{\left(\cosh\left(p\sqrt{2}\sqrt{\gamma}\sqrt{-p^{-2}}x\right) + \sinh\left(p\sqrt{2}\sqrt{\gamma}\sqrt{-p^{-2}}x\right)\right)}\sqrt{-\frac{1}{\beta p^2}}. \tag{5.185}$$

Family 1:

For case 4, when $\Delta > 0$ and $pq \neq 0$ or $qr \neq 0$, the hyperbolic function solutions of Eq. (5.114) are as follows,

$$U_{4,1}(\xi) = \frac{i\tan\left(\frac{\sqrt{2}\sqrt{\gamma}}{2}x\right)\sqrt{\gamma}}{\beta\sqrt{-\beta^{-1}}}, \tag{5.186}$$

$$U_{4,2}(\xi) = -\frac{i\cot\left(\frac{\sqrt{2}\sqrt{\gamma}}{2}x\right)\sqrt{\gamma}}{\beta\sqrt{-\beta^{-1}}}, \tag{5.187}$$

$$U_{4,3}(\xi) = \frac{i\left(\sin\left(\sqrt{2}\sqrt{\gamma}x\right)+1\right)\sqrt{\gamma}}{\beta\sqrt{-\beta^{-1}}\cos\left(\sqrt{2}\sqrt{\gamma}x\right)}, \tag{5.188}$$

$$U_{4,4}(\xi) = \frac{i\left(\cos\left(\sqrt{2}\sqrt{\gamma}x\right)+1\right)\sqrt{-\beta^{-1}}\sqrt{\gamma}}{\sin\left(\sqrt{2}\sqrt{\gamma}x\right)}, \tag{5.189}$$

$$U_{4,5}(\xi) = \frac{\frac{i}{2}\left(2\cos\left(\frac{\sqrt{2}\sqrt{\gamma}x}{4}\right)^2 - 1\right)\sqrt{-\beta^{-1}}\sqrt{\gamma}}{\sin\left(\frac{\sqrt{2}\sqrt{\gamma}x}{4}\right)\cos\left(\frac{\sqrt{2}\sqrt{\gamma}x}{4}\right)}, \tag{5.190}$$

here, A, B are real constants that satisfies $B^2 - A^2 > 0$,

$$U_{4,6}(\xi) = \frac{\sqrt{\gamma}\left(A\cos\left(\sqrt{2}\sqrt{\gamma}x\right) - \sqrt{A^2+B^2}\right)}{\beta\left(iA\sin\left(\sqrt{2}\sqrt{\gamma}x\right)+B\right)}\frac{1}{\sqrt{-\beta^{-1}}}, \tag{5.191}$$

$$U_{4,8}(\xi) = \frac{\sqrt{\gamma}\left(iA\sin\left(\sqrt{2}\sqrt{\gamma}x\right) + \sqrt{-A^2+B^2}\right)}{\beta\left(Acos\left(\sqrt{2}\sqrt{\gamma}x\right)+B\right)}\frac{1}{\sqrt{-\beta^{-1}}}, \tag{5.192}$$

$$U_{4,9}(\xi) = \frac{-\left(-i\sin\left(\frac{\sqrt{2}\sqrt{\gamma}x}{2}\right)\sqrt{\Delta}p + \cos\left(\frac{\sqrt{2}\sqrt{\gamma}x}{2}\right)p^2 - 4\cos\left(\frac{\sqrt{2}\sqrt{\gamma}x}{2}\right)qr\right)\sqrt{\gamma}}{\left(-i\sin\left(\frac{\sqrt{2}\sqrt{\gamma}x}{2}\right)\sqrt{\Delta} + p\cos\left(\frac{\sqrt{2}\sqrt{\gamma}x}{2}\right)\right)\sqrt{\Delta}\beta\sqrt{-\beta^{-1}}}, \tag{5.193}$$

$$U_{4,10}(\xi) = \frac{-\left(-cos\left(\frac{\sqrt{2}\sqrt{\gamma}x}{2}\right)\sqrt{\Delta}p + i\Delta\sin\left(\frac{\sqrt{2}\sqrt{\gamma}x}{2}\right)\right)\sqrt{\gamma}}{\left(-cos\left(\frac{\sqrt{2}\sqrt{\gamma}x}{2}\right)\sqrt{\Delta} + ip\sin\left(\frac{\sqrt{2}\sqrt{\gamma}x}{2}\right)\right)\sqrt{\Delta}\beta\sqrt{-\beta^{-1}}}, \tag{5.194}$$

$$U_{4,11}(\xi) = -\frac{\sqrt{\gamma}\left(ip\left((\sin\left(\sqrt{2}\sqrt{\gamma}x\right)+1\right)\sqrt{\Delta}-\cos\left(\sqrt{2}\sqrt{\gamma}x\right)(\Delta)\right)}{\sqrt{\Delta}\left(\left(i\sin\left(\sqrt{2}\sqrt{\gamma}x\right)+i\right)\sqrt{\Delta}-p\cos\left(\sqrt{2}\sqrt{\gamma}x\right)\right)\beta}\frac{1}{\sqrt{-\beta^{-1}}}, \tag{5.195}$$

$$U_{4,12}(\xi) = -\frac{\sqrt{\gamma}\left(-p\left((\cos\left(\sqrt{2}\sqrt{\gamma}x\right)+1\right)\sqrt{\Delta}+\mathrm{i}\sin\left(\sqrt{2}\sqrt{\gamma}x\right)(\Delta)\right)}{\sqrt{\Delta}\left(\left(-\cos\left(\sqrt{2}\sqrt{\gamma}x\right)-1\right)\sqrt{\Delta}+ip\sin\left(\sqrt{2}\sqrt{\gamma}x\right)\right)\beta}\frac{1}{\sqrt{-\beta^{-1}}}. \tag{5.196}$$

Family 2:

When, $\Delta < 0$ and $pq \neq 0$ $(or\ qr \neq 0)$, we have the following trigonometric solutions for Eq. (5.114),

$$U_{4,13}(\xi) = -\frac{2\sqrt{\gamma}\left(\left(-\cos\left(\frac{\sqrt{2}\sqrt{\gamma}x}{4}\right)^2 p + p/2\right)\sqrt{\Delta}+i(\Delta)\sin\left(\frac{\sqrt{2}\sqrt{\gamma}x}{2}\right)\right)}{\sqrt{\Delta}\left(ip\sin\left(\frac{\sqrt{2}\sqrt{\gamma}x}{2}\right)-2\sqrt{\Delta}\cos\left(\frac{\sqrt{2}\sqrt{\gamma}x}{4}\right)^2+\sqrt{\Delta}\right)\beta\sqrt{-\beta^{-1}}}, \tag{5.197}$$

$$U_{4,14}(\xi) = \frac{i\left(\sin\left(\sqrt{2}\sqrt{\gamma}x\right)-1\right)\sqrt{\gamma}}{\beta\sqrt{-\beta^{-1}}\cos\left(\sqrt{2}\sqrt{\gamma}x\right)}, \tag{5.198}$$

here, A, B are real constants that satisfies $B^2 - A^2 > 0$,

$$U_{4,15}(\xi) = \frac{\sqrt{\gamma}\left(iA\cos\left(\sqrt{2}\sqrt{\gamma}x\right)-\sqrt{A^2+B^2}\right)}{\beta\left(A\sin\left(\sqrt{2}\sqrt{\gamma}x\right)-B\right)}\frac{1}{\sqrt{-\beta^{-1}}}, \tag{5.199}$$

$$U_{4,16}(\xi) = -\frac{\sqrt{\gamma}\left(iA\cos\left(\sqrt{2}\sqrt{\gamma}x\right)+\sqrt{A^2+B^2}\right)}{\beta\left(A\sin\left(\sqrt{2}\sqrt{\gamma}x\right)-B\right)}\frac{1}{\sqrt{-\beta^{-1}}}, \tag{5.200}$$

$$U_{4,17}(\xi) = \frac{\left(i\sin\left(\frac{\sqrt{2}\sqrt{\gamma}x}{2}\right)\sqrt{\Delta}p-\cos\left(\frac{\sqrt{2}\sqrt{\gamma}x}{2}\right)\Delta\right)\sqrt{\gamma}\sqrt{-\beta^{-1}}}{\left(-i\sin\left(\frac{\sqrt{2}\sqrt{\gamma}x}{2}\right)\sqrt{\Delta}-p\cos\left(\frac{\sqrt{2}\sqrt{\gamma}x}{2}\right)\right)\sqrt{\Delta}}, \tag{5.201}$$

$$U_{4,18}(\xi)=\frac{\left(i\cos\left(\frac{\sqrt{2}\sqrt{\gamma}x}{2}\right)\sqrt{\Delta}p+\sin\left(\frac{\sqrt{2}\sqrt{\gamma}x}{2}\right)\Delta\right)\sqrt{\gamma}\sqrt{-\beta^{-1}}}{\left(i\cos\left(\frac{\sqrt{2}\sqrt{\gamma}x}{2}\right)\sqrt{\Delta}+p\cos\left(\frac{\sqrt{2}\sqrt{\gamma}x}{2}\right)\right)\sqrt{\Delta}}, \tag{5.202}$$

$$U_{4,19}(\xi)=\frac{\sqrt{\gamma}\left(ip\left((\sin\left(\sqrt{2}\sqrt{\gamma}x\right)-1\right)\sqrt{\Delta}-\cos\left(\sqrt{2}\sqrt{\gamma}x\right)(\Delta)\right)}{\sqrt{\Delta}\left(\left(i\sin\left(\sqrt{2}\sqrt{\gamma}x\right)-i\right)\sqrt{\Delta}-p\cos\left(\sqrt{2}\sqrt{\gamma}x\right)\right)\beta}\frac{1}{\sqrt{-\beta^{-1}}}, \tag{5.203}$$

$$U_{4,20}(\xi)=\frac{\sqrt{\gamma}\left(ip\left((\cos\left(\sqrt{2}\sqrt{\gamma}x\right)+1\right)\sqrt{\Delta}+\sin\left(\sqrt{2}\sqrt{\gamma}x\right)(\Delta)\right)}{\sqrt{\Delta}\left(\left(i\cos\left(\sqrt{2}\sqrt{\gamma}x\right)+i\right)\sqrt{\Delta}+p\sin\left(\sqrt{2}\sqrt{\gamma}x\right)\right)\beta}\frac{1}{\sqrt{-\beta^{-1}}}, \tag{5.204}$$

$$U_{4,21}(\xi)=\frac{2\sqrt{\gamma}\left(i\left(\cos\left(\frac{\sqrt{2}\sqrt{\gamma}x}{4}\right)^{2}-1/2\right)p\sqrt{\Delta}+\sin\left(\frac{\sqrt{2}\sqrt{\gamma}x}{2}\right)(\Delta)\right)}{\sqrt{\Delta}\left(\left(2i\cos\left(\frac{\sqrt{2}\sqrt{\gamma}x}{4}\right)^{2}-i\right)\sqrt{\Delta}+p\sin\left(\frac{\sqrt{2}\sqrt{\gamma}x}{2}\right)\right)}\sqrt{-\beta^{-1}}, \tag{5.205}$$

$$U_{4,22}(\xi)=\sqrt{\gamma}p\left(\cosh\left(p\sqrt{2}\sqrt{\gamma}\sqrt{-p^{-2}}x\right)-\sinh\left(p\sqrt{2}\sqrt{\gamma}\sqrt{-p^{-2}}x\right)-d\right)$$
$$\sqrt{-\frac{1}{\beta p^{2}}}\left(d+\cosh\left(p\sqrt{2}\sqrt{\gamma}\sqrt{-p^{-2}}x\right)-\sinh\left(p\sqrt{2}\sqrt{\gamma}\sqrt{-p^{-2}}x\right)\right)^{-1}. \tag{5.206}$$

Family 3:

When $r=0$, and $pq\neq 0$, we get soliton like solutions.

$$U_{4,23}(\xi)=\frac{\left(\cosh\left(p\sqrt{2}\sqrt{\gamma}\sqrt{-p^{-2}}x\right)-\sinh\left(p\sqrt{2}\sqrt{\gamma}\sqrt{-p^{-2}}x\right)-d\right)}{\left(\cosh\left(p\sqrt{2}\sqrt{\gamma}\sqrt{-p^{-2}}x\right)-\sinh\left(p\sqrt{2}\sqrt{\gamma}\sqrt{-p^{-2}}x\right)+d\right)}p\sqrt{\gamma}\sqrt{-\frac{1}{\beta p^{2}}}. \tag{5.207}$$

5.7 Graphical Explanation:

In this section we discuss graphical simulation of some of exact solutions of space-time conformable telegraph equation. 3-Dimensional and 2-Dimensional graphs of various solutions have been examined by choosing appropriate values of fractional order operator α. In Figure 5.6-

Figure 5.9, it is obvious that for smaller value of fractional order operator α we get shock waves and by increasing the value of α equals to 1 we get solitary wave.

Figure 5.6: Represents graphical simulation of kink wave soliton for $U_{1,1}$ expressed in Eq. (5.122) by choosing parameters, $p = 3, q = 1, r = 2, \beta = 2.5, \gamma = 1$. Fig (a)-(c) depicts 3D graphs of abs($U_{1,1}$) for $\alpha = 0.1, 0.6, 1$, while Fig(d) depicts 2D graph for abs($U_{1,1}$) for $\alpha = 0.1, 0.6, 1$ respectively in the range of $-10 \leq x \leq 10, t = 1$.

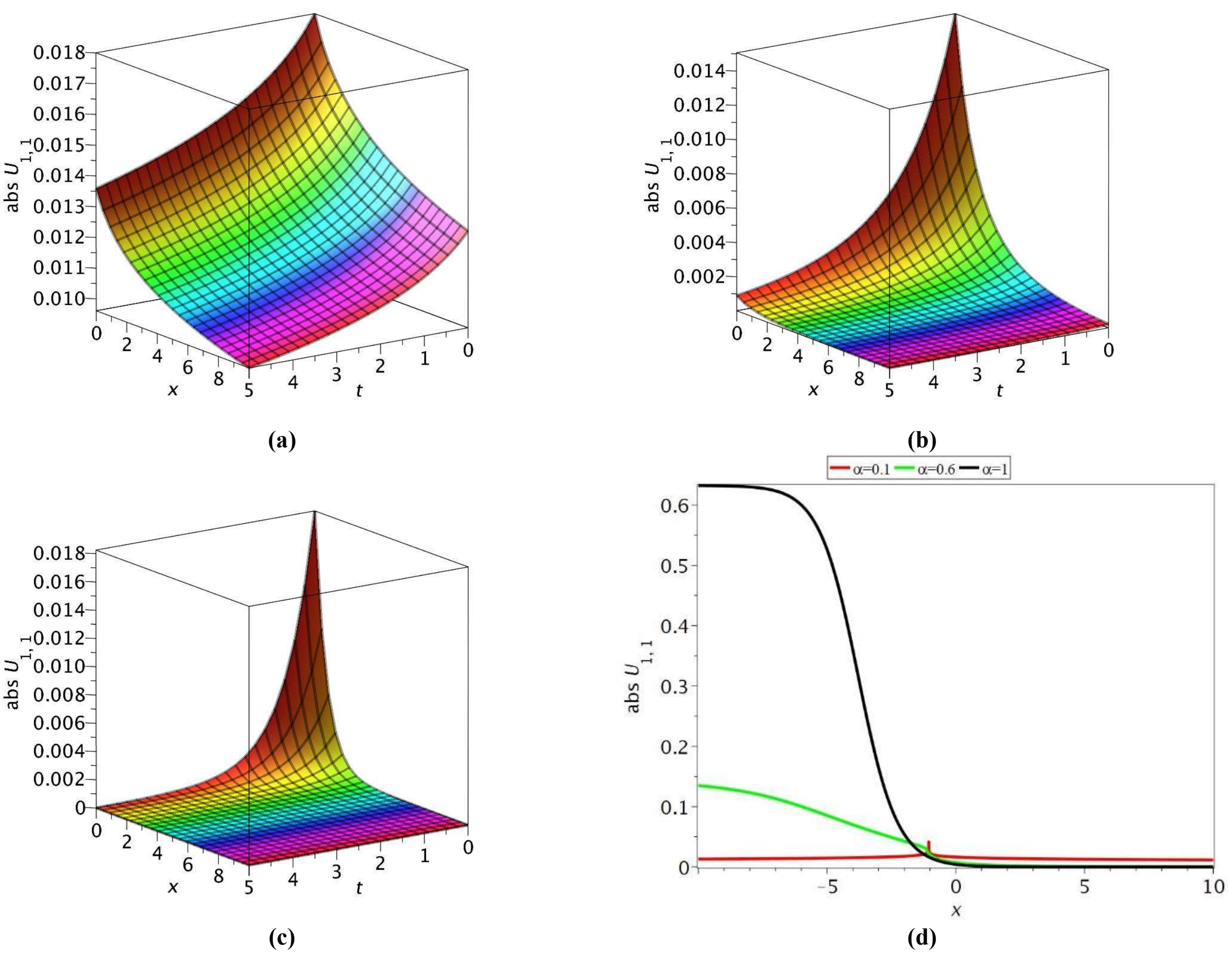
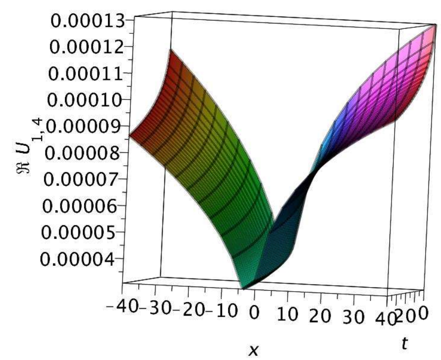
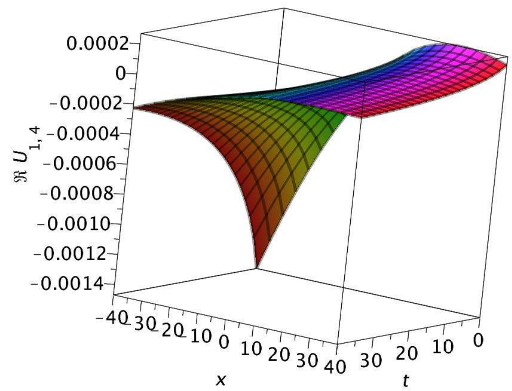


Figure 5.6: Graphical representation of kink wave soliton for $U_{1,1}$.

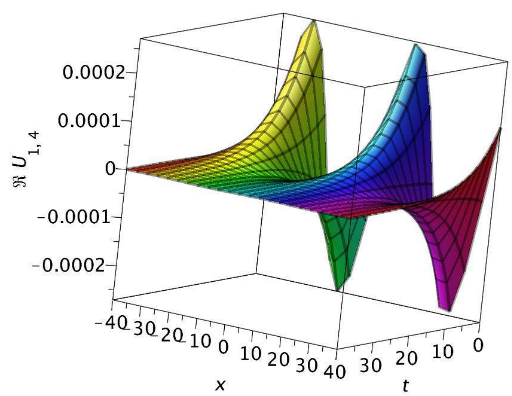
Figure 5.8: Graphical representation of periodic wave solution for $\boldsymbol{U_{3,1}}$ Figure 5.7:: Represents graphical simulation of periodic wave solution for $U_{1,4}$ expressed in Eq.(5.125) by choosing parameters, $p = 3, q = 0.1, r = 0.2, \beta = 5, \gamma = 1$. Fig (a)-(c) depicts 3D graphs of Re($U_{1,4}$) for $\alpha = 0.3, 0.7, 1$, while Fig(d) depicts 2D graph for Re($U_{1,4}$) for $\alpha = 0.3, 0.7, 1$ respectively in the range of $-40 \leq x \leq 40, t = 2$.



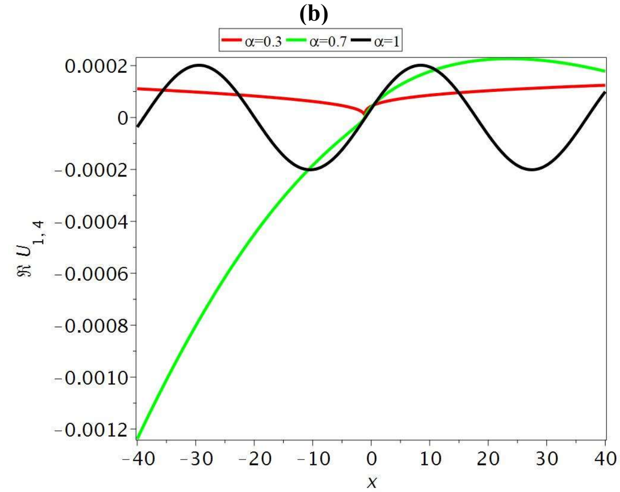
(a)



(b)



(c)



(d)

Figure 5.7: Graphical representation of periodic wave solution for $U_{1,4}$

Figure 5.8:Depicts graphical simulation of periodic wave solution for $U_{3,1}$ of Eq.(5.162) by choosing parameters, $p = 4, q = 1, r = 2, \beta = 2.5, \gamma = 1$. Fig (a)-(c) depicts 3D graphs of $\mathrm{Re}(U_{3,1})$ for $\alpha = 0.1, 0.5, 1$, while Fig(d) depicts 2D graph for $\mathrm{Re}(U_{3,1})$ for $\alpha = 0.1, 0.5, 1$ respectively in the range of $-10 \le x \le 10$.

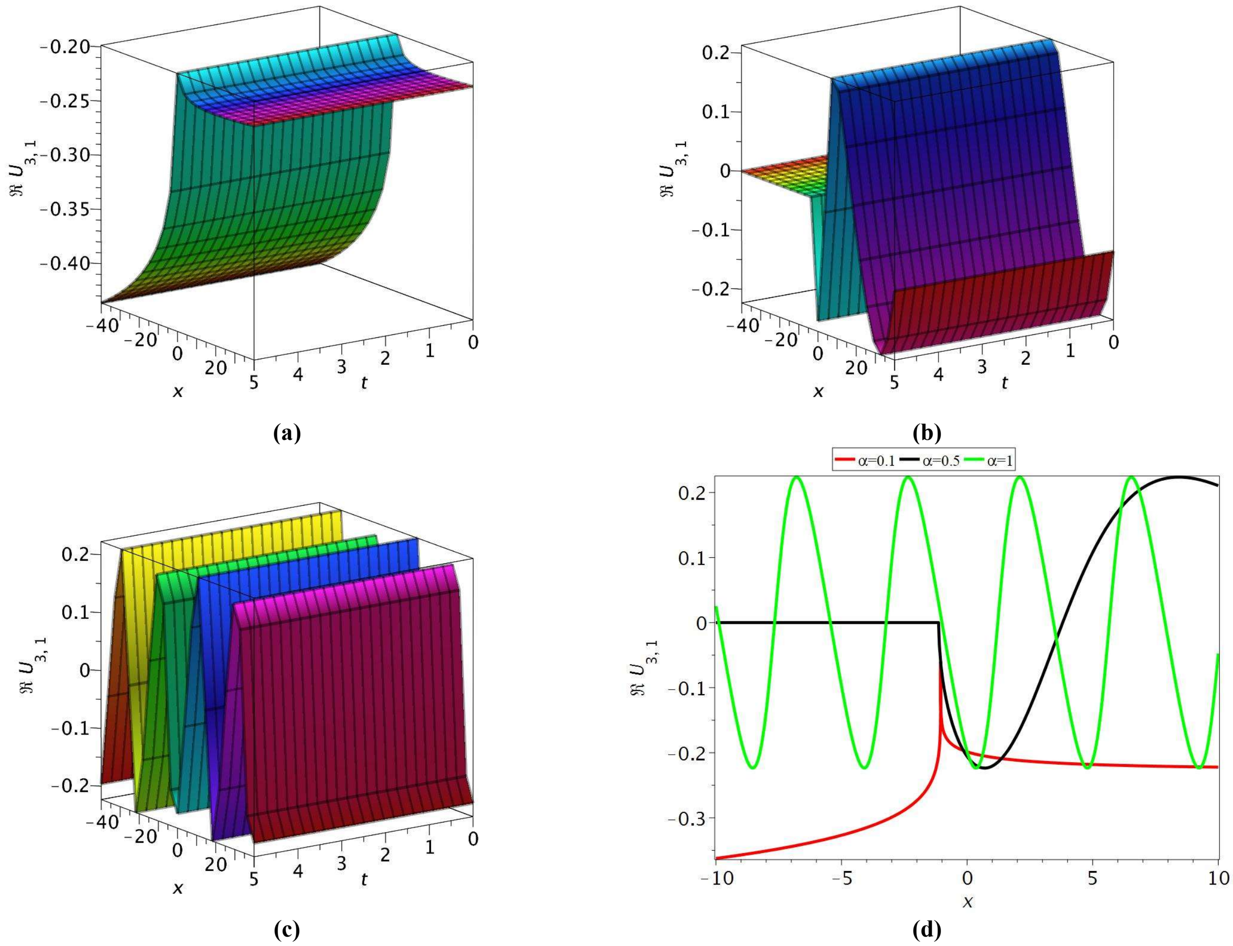


Figure 5.8: Graphical representation of periodic wave solution for $U_{3,1}$

Figure 5.9: Depicts graphical representation of periodic wave solution for $U_{4,22}$, expressed in Eq.(5.206) by choosing parameters, $p = 1, d = 4, \beta = 1, \gamma = 7$. Fig (a)-(c) depicts 3D graphs of $\text{Im}(U_{1,4})$ for $\alpha = 0.4, 0.6, 1$, while Fig(d) depicts 2D graph for $\text{Im}(U_{4,22})$ for $\alpha = 0.4, 0.6, 1$ respectively in the range of $-10 \leq x \leq 10$.

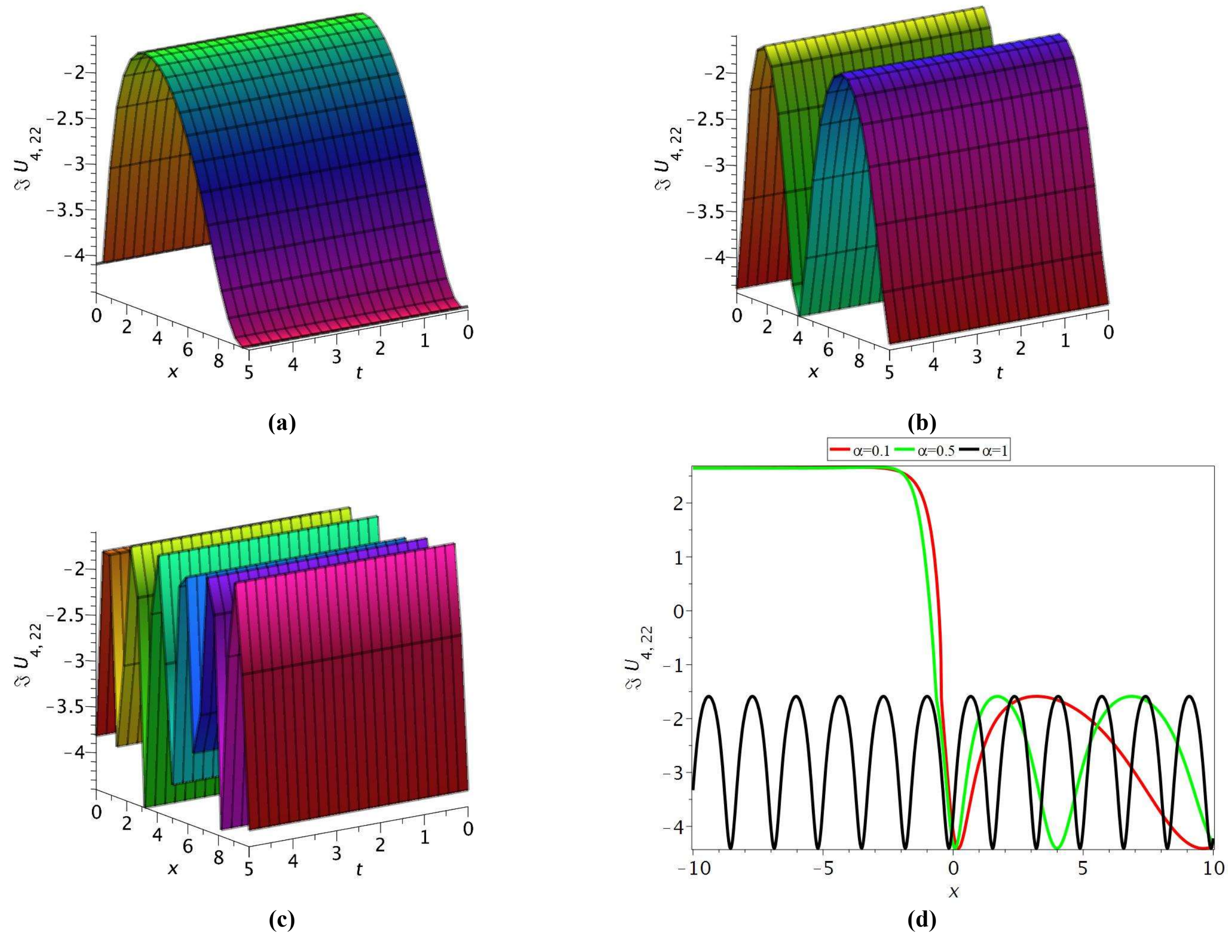


Figure 5.9: Graphical representation of periodic wave solution for $U_{4,22}$

5.8 Conclusion:

Improved generalized Riccati equation mapping method has been used to extract exact traveling wave solutions to the space- time fractional telegraph equation. Numerous travelling wave solutions have been generated in the form of hyperbolic, periodic wave and rational solutions. Wave behavior have been studied through 3-D and 2-D graphs by choosing suitable values of α and free parameters involved. These results might be helpful in the study of electrical signals in transmission lines.

5.9 Space-time fractional (2+1)-dimensional Heisenberg Ferromagnet Model:

Another important equation we have considered here is the newly derived variant of Nonlinear Schrödinger Equation (NLSE) that describes space-time fractional (2+1)-dimensional Heisenberg ferromagnetic spin chains with bilinear and anisotropic interactions in the semi classical limit derive by M. Latha and C. Vasanthi [179].

$$i\,{}_0^A D_t^{\alpha} u + \alpha_1\,{}_0^A D_x^{2\alpha} u + \alpha_2\,{}_0^A D_y^{2\alpha} u + \alpha_3\,{}_0^A D_{xy}^{2\alpha} u - \alpha_4 u|u|^2 = 0, \quad (5.208)$$

where $u = u(x, y, t)$, ${}_0^A D_t^{\alpha}$, ${}_0^A D_x^{\alpha}$, ${}_0^A D_y^{\alpha}$ are Atangana's conformable derivatives [40], $\Psi_1 = \gamma^4(J + J_2)$, $\Psi_2 = \gamma^4(J_1 + J_2)$, $\Psi_3 = 2\gamma^4 J_2$, $\Psi_4 = 2\gamma^4 A$, parameter γ is lattice parameter, J, J_1 represents bilinear exchange interaction coefficients with respect to x and y respectively. J_2 is the neighboring interaction on the diagonal, whereas uniaxial crystal field anisotropy parameter is denoted by A [33]. Heisenberg ferromagnet model (HFM) is an interesting nonlinear model that exhibits magnetic solitons and, also very important to study magnetic behavior in magnetic materials [33]. Finding the new exact solutions for this model will help scientists to understand nonlinear behaviour of ferromagnetic substances. Now a days the new technology magneto-optical recording is gaining popularity for higher storage and fast reading [180]. Also, the magnetization reversal in ferromagnetic medium due to the occurrence of spin soliton flipping has an application in magnetic memories and recording [181]. Baskonus et. al [182, 183] studied (2+1)-dimensional Heisenberg ferromagnetic spin chains and construct dark, bright, combined dark-bright, singular, and combined singular soliton solutions**.** H. Triki and M. Wazwaz [33] find out bright and dark solitons and periodic wave solutions for this equation. Liu et al [184, 185] studied bright and dark soliton for Heisenberg model. Baleanu et al. [186] studied optical soliton for this model. A. Kundu et al. [187] applied modified Kudryashov method on (2 + 1)-dimensional Heisenberg ferromagnetic spin chain equation. In [188] authors investigate Heisenberg model with the of modified extended tanh expansion method using Riccati equation.

To solve Eq. (5.208) we use the following transformation:

$$u(x, y, t) = U(\xi)e^{i\psi}. \quad (5.209)$$

where,

$$\xi = \frac{\chi_1}{\alpha}\left(x + \frac{1}{\Gamma(\alpha)}\right)^{\alpha} + \frac{\chi_2}{\alpha}\left(y + \frac{1}{\Gamma(\alpha)}\right)^{\alpha} - \frac{\lambda}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha},$$

$$\psi = \frac{\Upsilon_1}{\alpha}\left(x + \frac{1}{\Gamma(\alpha)}\right)^{\alpha} + \frac{\Upsilon_2}{\alpha}\left(y + \frac{1}{\Gamma(\alpha)}\right)^{\alpha} + \frac{\omega}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}. \quad (5.210)$$

where ψ is the phase component, Υ represents wave number, ω is the soliton frequency, λ is the velocity of soliton and χ is the width of soliton. Now substituting Eq. (5.209) and Eq. (5.210) into Eq. (5.208) and separating the obtained ODE into real and imaginary components we get real part as:

$$({\chi_1}^2\alpha_1 + \chi_1\chi_2\alpha_3 + {\chi_2}^2\alpha_2)\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}U(\xi) - U(\xi)\left({\Upsilon_2}^2\alpha_2 + \Upsilon_2\Upsilon_1\alpha_3 + {\Upsilon_1}^2\alpha_1 + \omega\right) \quad (5.211)$$

$$-\alpha_4\left(U(\xi)\right)^3 = 0,$$

and imaginary component gives:

$$\lambda = \chi_1\Upsilon_2\alpha_3 + 2\alpha_1\chi_1\Upsilon_1 + 2\alpha_2\chi_2\Upsilon_2 + \chi_2\Upsilon_1\alpha_3. \quad (5.212)$$

Eq. (5.211) and Eq. (5.212) obtained by applying the properties of Atangana's conformable derivative explained in Eq. (1.25) - (1.32). By using homogeneous balance principle between the highest order derivative and nonlinearity yields $M = 1$. Therefore, Eq. (1.67) has a solution.

$$U(\xi) = \frac{b_{-1}}{\phi(\xi)} + b_0 + b_1\phi(\xi). \quad (5.213)$$

Now, substituting Eq. (5.213) along with Eq. (1.68) into Eq. (5.211) after collecting all terms with the same order in ϕ^i and ϕ^{-i}, where, $(i = 0,1,2, \ldots\ldots)$. and equating each coefficient to 0, we obtain a system of NL algebraic equations. Solving these equations yields following cases and non-trivial solutions:

Set 1 :

$$b_1 = 0, \quad b_0 = \frac{\sqrt{2}\sqrt{\frac{{\chi_1}^2\alpha_1 + \chi_1\chi_2\alpha_3 + {\chi_2}^2\alpha_2}{\alpha_4}}\, l}{2},$$

$$b_{-1} = k\sqrt{2}\sqrt{\frac{{\chi_1}^2\alpha_1 + \chi_1\chi_2\alpha_3 + {\chi_2}^2\alpha_2}{\alpha_4}}, \quad (5.214)$$

$$\omega = \begin{array}{c} -\frac{\alpha_2(l^2-4mk)\chi_2{}^2}{2} - \frac{\alpha_3\chi_1(l^2-4mk)\chi_2}{2} - \frac{\alpha_1(l^2-4mk)\chi_1{}^2}{2}, \\ -\Upsilon_1\Upsilon_2\alpha_3 - \Upsilon_2{}^2\alpha_2 - \Upsilon_1{}^2\alpha_1 \end{array}$$

$$U_1(\xi) = b_0 + \frac{b_{-1}}{\phi(\xi)}. \tag{5.215}$$

Set 2 :

$$b_0 = \frac{l(\chi_1{}^2\alpha_1 + \chi_1\chi_2\alpha_3 + \chi_2{}^2\alpha_2)\sqrt{2}}{2 \times \sqrt{\frac{\chi_1{}^2\alpha_1 + \chi_1\chi_2\alpha_3 + \chi_2{}^2\alpha_2}{\alpha_4}}\alpha_4},$$

$$b_1 = \sqrt{2}\sqrt{\frac{\chi_1{}^2\alpha_1 + \chi_1\chi_2\alpha_3 + \chi_2{}^2\alpha_2}{\alpha_4}}m, \quad b_{-1} = 0, \tag{5.216}$$

$$\omega = \begin{array}{c} -\frac{\alpha_2(l^2-4mk)\chi_2{}^2}{2} - \frac{\alpha_3\chi_1(l^2-4mk)\chi_2}{2} - \frac{\alpha_1(l^2-4mk)\chi_1{}^2}{2}, \\ -\Upsilon_1\Upsilon_2\alpha_3 - \Upsilon_2{}^2\alpha_2 - \Upsilon_1{}^2\alpha_1, \end{array}$$

$$U_2(\xi) = b_0 + b_1\phi(\xi). \tag{5.217}$$

Please note, the following substitutions have been made in the following solutions to make the results more elegant.

$\Omega = \sqrt{l^2 - 4mk}, \Omega' = \sqrt{4mk - l^2},$

with

$$\xi = \frac{\chi_1}{\alpha}\left(x + \frac{1}{\Gamma(\alpha)}\right)^{\alpha} + \frac{\chi_2}{\alpha}\left(y + \frac{1}{\Gamma(\alpha)}\right)^{\alpha} - \frac{\lambda}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha},$$

$$\lambda = \chi_1\Upsilon_2\alpha_3 + 2\alpha_1\chi_1\Upsilon_1 + 2\alpha_2\chi_2\Upsilon_2 + \chi_2\Upsilon_1\alpha_3,$$

$$\psi = \frac{\Upsilon_1}{\alpha}\left(x + \frac{1}{\Gamma(\alpha)}\right)^{\alpha} + \frac{\Upsilon_2}{\alpha}\left(y + \frac{1}{\Gamma(\alpha)}\right)^{\alpha} + \frac{\omega}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}.$$

For the **case 1**, substituting the values from Eq. (5.214) into Eq.(5.215) along with the Riccati equations solutions, we get.

Family 1:

When $l^2 - 4mk > 0$ and $lm \neq 0$ or $mk \neq 0$, the hyperbolic function solutions of Eq.(5.208) are as follows:

$$U_{1,1}=\left(b_0-\frac{2m(b_{-1})}{(l+\Omega)\tanh\left(\frac{\Omega\xi}{2}\right)}\right)e^{i\psi}\ , \tag{5.218}$$

$$U_{1,2}=\left(b_0-\frac{2m(b_{-1})}{(l+\Omega)\coth\left(\frac{\Omega\xi}{2}\right)}\right)e^{i\psi}\ , \tag{5.219}$$

$$U_{1,3}=\left(b_0-\frac{2m(b_{-1})}{(l+\Omega)\big(\tanh(\Omega\xi)\pm isech(\Omega\xi)\big)}\right)e^{i\psi}, \tag{5.220}$$

$$U_{1,4}=\left(b_0-\frac{2m(b_{-1})}{(l+\Omega)\big(coth(\Omega\xi)\pm csch(\Omega\xi)\big)}\right)e^{i\psi}\ , \tag{5.221}$$

$$U_{1,5}=\left(b_0-\frac{4m(b_{-1})}{(2l+\Omega)\left(2coth\left(\frac{\Omega\xi}{2}\right)\right)}\right)e^{i\psi}, \tag{5.222}$$

$$U_{1,6}=\left(b_0+\frac{2m(b_{-1})}{-l+\frac{\pm\sqrt{(A^2+B^2)}\Omega-A\Omega\, cosh(\Omega\xi)}{A\sinh(\Omega\xi)+B}}\right)e^{i\psi}, \tag{5.223}$$

$$U_{1,7}=\left(b_0+\frac{2m(b_{-1})}{-l-\frac{\pm\sqrt{(-A^2+B^2)}\Omega+A\Omega\, sinh(\Omega\xi)}{A\cosh(\Omega\xi)+B}}\right)e^{i\psi}, \tag{5.224}$$

where two non-zero real constants A and B satisfies $A^2-B^2>0$.

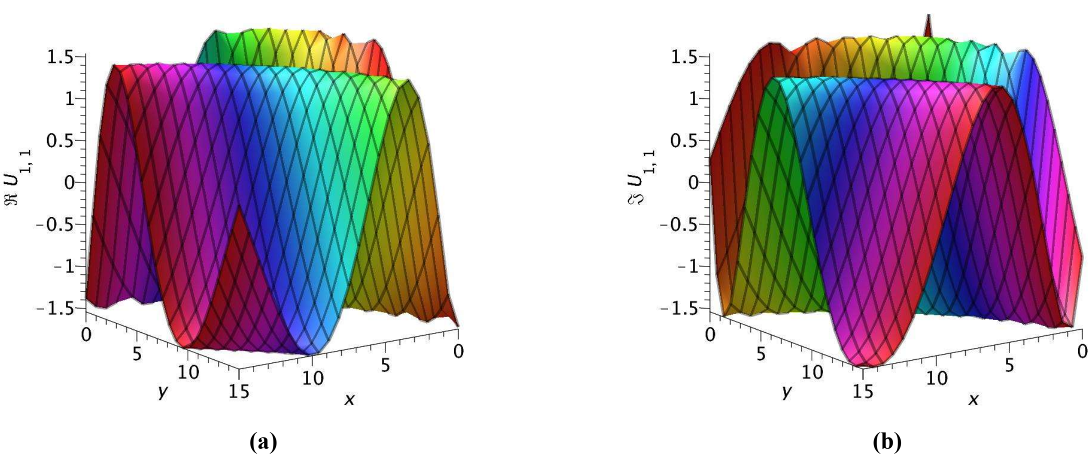
$$U_{1,8}=\left(b_0+\frac{\left(\frac{b_{-1}}{k}\right)\left(\Omega sinh\left(\frac{\Omega\xi}{2}\right)-lcosh\left(\frac{\Omega\xi}{2}\right)\right)}{2\ cosh\left(\frac{\Omega\xi}{2}\right)}\right)e^{i\psi}, \tag{5.225}$$

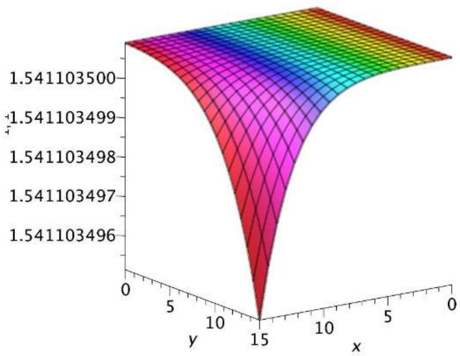
$$U_{1,9} = \left(b_0 - \frac{\left(\frac{b_{-1}}{k}\right)\left(-\Omega cosh\left(\frac{\Omega\xi}{2}\right) + lsinh\left(\frac{\Omega\xi}{2}\right)\right)}{2\ sinh\left(\frac{\Omega\xi}{2}\right)} \right) e^{i\psi}, \tag{5.226}$$

$$U_{1,10} = \left(b_0 + \frac{(\frac{b_{-1}}{k})(\Omega sinh(\Omega\xi) - l\ cosh(\Omega\xi) \pm i\Omega)}{2\ cosh(\Omega\xi)} \right) e^{i\psi}, \tag{5.227}$$

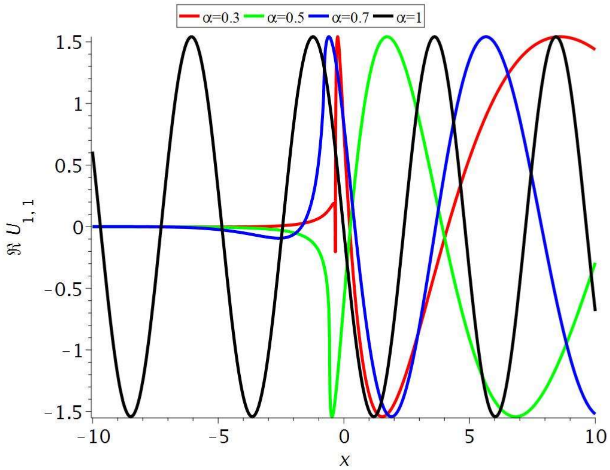
$$U_{1,11} = \left(b_0 + \frac{(\frac{b_{-1}}{k})(\Omega cosh(\Omega\xi) - lsinh(\Omega\xi) \pm \Omega)}{2\ sinh(\Omega\xi)} \right) e^{i\psi}, \tag{5.228}$$

$$U_{1,12} = \left(b_0 + \frac{\left(\frac{b_{-1}}{k}\right)\left(-lsinh\left(\frac{\Omega\xi}{2}\right) + 2\Omega cosh^2\left(\frac{\Omega\xi}{4}\right) - \Omega\right)}{2\sinh\left(\frac{\Omega\xi}{2}\right)} \right) e^{i\psi}, \tag{5.229}$$





(c)



(d)

Figure 5.10: (a)-(c) 3D illustration of $Re(U_{1,1}), Im(U_{1,1}), abs(U_{1,1})$ with $l = 3, m = 1, k = 2, \alpha_1 = 1.5, \alpha_2 = 1.5, \alpha_3 = 1.5, \alpha_4 = 1.5, \Upsilon_1 = 1.3, \Upsilon_2 = 1.2, \chi_1 = 1.5, \chi_2 = 1, \alpha = 0.5, t = 1.5, x = 0..15, y = 0..15$, and, (d) 2D illustration of $Re(U_{1,1})$with , $\alpha = 0.3, 0.5, 0.7, 1$ at $x = -10...10, y = 3, t = 1.5$

Family 2:

If $l^2 - 4mk < 0$ and $lm \neq 0$ $(or\ mk \neq 0)$, we have the following trigonometric solutions for Eq. (5.208),

$$U_{1,13} = \left(b_0 + \frac{2m(b_{-1})}{(-l+\Omega')tan\left(\frac{\Omega'\xi}{2}\right)} \right) e^{i\psi}, \tag{5.230}$$

$$U_{1,14} = \left(b_0 - \frac{2m(b_{-1})}{(l+\Omega')cot\left(\frac{\Omega'\xi}{2}\right)} \right) e^{i\psi}, \tag{5.231}$$

$$U_{1,15} = \left(b_0 + \frac{2m(b_{-1})}{(-l+\Omega')\left(\tan(\Omega'\xi) \pm sec(\Omega'\xi)\right)} \right) e^{i\psi}, \tag{5.232}$$

$$U_{1,16} = \left(b_0 - \frac{2m(b_{-1})}{(l+\Omega')\left(cot(\Omega'\xi) \pm csc(\Omega'\xi)\right)} \right) e^{i\psi}, \tag{5.233}$$

$$U_{1,17} = \left(b_0 + \frac{4m(b_{-1})}{(-2l+\Omega')\left(-2cot\left(\frac{\Omega'\xi}{2}\right)\right)} \right) e^{i\psi}, \tag{5.234}$$

$$U_{1,18} = \left(b_0 + \frac{2m(b_{-1})}{-l + \frac{\pm i\sqrt{(-A^2 + B^2)}\Omega' - A\Omega' \cos(\Omega'\xi)}{A\sin(\Omega'\xi) + B}} \right) e^{i\psi}, \tag{5.235}$$

$$U_{1,19} = \left(b_0 + \frac{2m(b_{-1})}{-l - \frac{\pm i\sqrt{(-A^2 + B^2)}\Omega' + A\Omega' \cos(\Omega'\xi)}{A\sin(\Omega'\xi) + B}} \right) e^{i\psi}, \tag{5.236}$$

where two non-zero real constants A and B satisfies $A^2 - B^2 > 0$.

$$U_{1,20} = \left(b_0 - \frac{\left(\frac{b_{-1}}{k}\right)\left(\Omega' \sin\left(\frac{\Omega'\xi}{2}\right) + l\cos\left(\frac{\Omega'\xi}{2}\right)\right)}{2\cos\left(\frac{\Omega'\xi}{2}\right)} \right) e^{i\psi}, \tag{5.237}$$

$$U_{1,21} = \left(b_0 + \frac{\left(\frac{b_{-1}}{k}\right)\left(\Omega' \cos\left(\frac{\Omega'\xi}{2}\right) - l\sin\left(\frac{\Omega'\xi}{2}\right)\right)}{2\sin\left(\frac{\Omega'\xi}{2}\right)} \right) e^{i\psi}, \tag{5.238}$$

$$U_{1,22} = \left(b_0 - \frac{\left(\frac{b_{-1}}{k}\right)(\Omega' \sin(\Omega'\xi) + l\cos(\Omega'\xi) \pm \Omega')}{2\cos(\Omega'\xi)} \right) e^{i\psi}, \tag{5.239}$$

$$U_{1,23} = \left(b_0 + \frac{\left(\frac{b_{-1}}{k}\right)(\Omega' \cos(\Omega'\xi) - l\sin(\Omega'\xi) \pm \Omega')}{2\sin(\Omega'\xi)} \right) e^{i\psi}, \tag{5.240}$$

$$U_{1,24} = \left(b_0 + \frac{\left(\frac{b_{-1}}{k}\right)\left(-l\sin\left(\frac{\Omega'\xi}{2}\right) + 2\Omega' \cos^2\left(\frac{\Omega'\xi}{4}\right) - \Omega'\right)}{2\sin\left(\frac{\Omega'\xi}{2}\right)} \right) e^{i\psi}, \tag{5.241}$$

where,

$$\Omega = \sqrt{l^2 - 4mk}, \Omega' = \sqrt{4mk - l^2},$$

$$\xi = \frac{\chi_1}{\alpha}\left(x + \frac{1}{\Gamma(\alpha)}\right)^{\alpha} + \frac{\chi_2}{\alpha}\left(y + \frac{1}{\Gamma(\alpha)}\right)^{\alpha} - \frac{\lambda}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha},$$

$$\lambda = \chi_1 \Upsilon_2 \alpha_3 + 2\alpha_1 \chi_1 \Upsilon_1 + 2\alpha_2 \chi_2 \Upsilon_2 + \chi_2 \Upsilon_1 \alpha_3 \,,$$

$$\psi = \frac{\Upsilon_1}{\alpha}\left(x + \frac{1}{\Gamma(\alpha)}\right)^{\alpha} + \frac{\Upsilon_2}{\alpha}\left(y + \frac{1}{\Gamma(\alpha)}\right)^{\alpha} + \frac{\omega}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}.$$

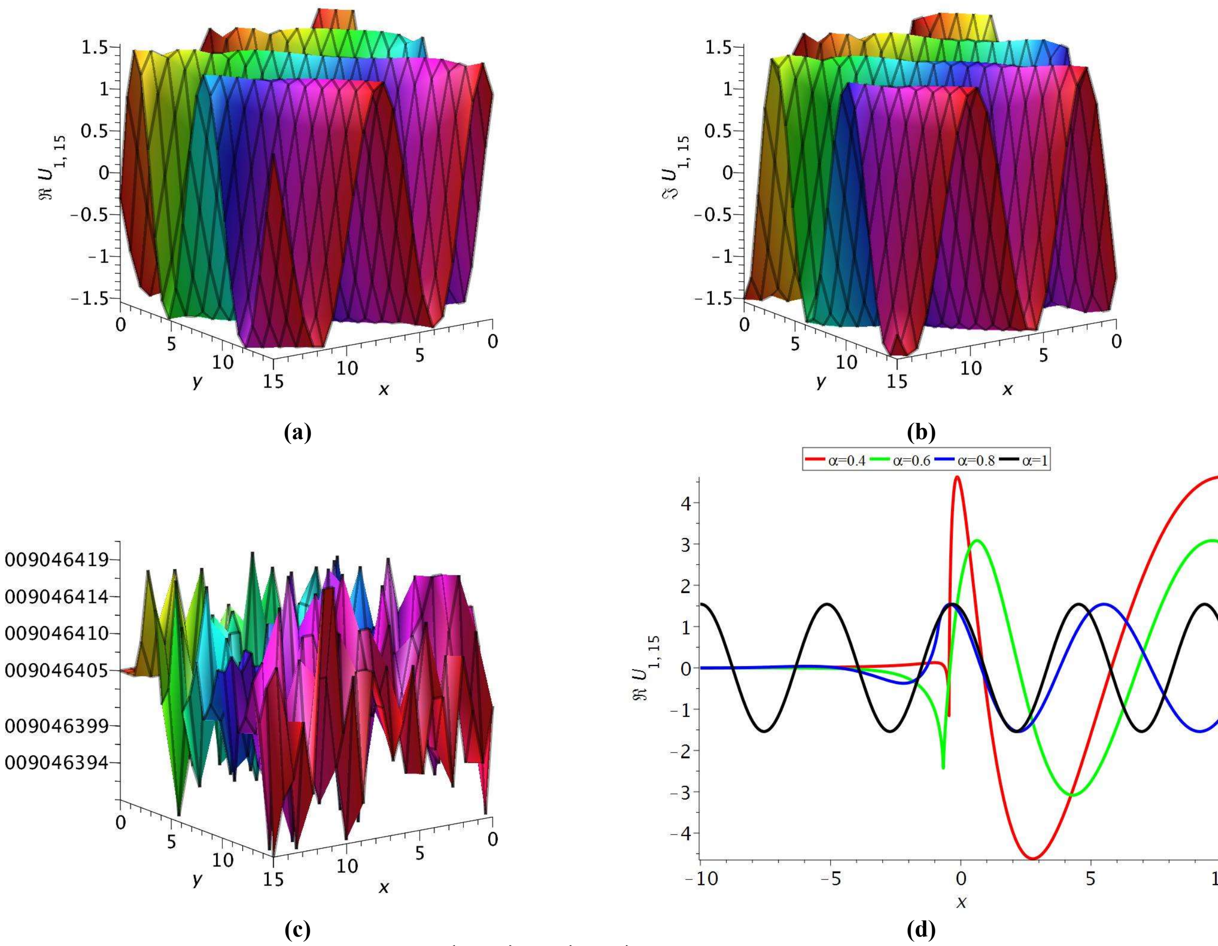


Figure 5.11: (a)-(c) 3D illustration of $Re(U_{1,15}), Im(U_{1,15}), abs(U_{1,15})$ with $p = l, m = 1, k = 2, \alpha_1 = 1.5, \alpha_2 = 1.5, \alpha_3 = 1.5, \alpha_4 = 1.5, \Upsilon_1 = 1.3, \Upsilon_2 = 1.2, \chi_1 = 1.5, \chi_2 = 1, \alpha = 0.8, t = 1.5, x = 0..15, y = 0..15$, and (d) 2D illustration of $Re(U_{1,15})$by choosing , $\alpha = 0.4, 0.6, 0.8, 1$ at $-10 \le x \le 10, t = 1.5, y = 2$

Family 3:

When $k = 0$, and $lm \neq 0$ the solutions of Eq. (5.208) are as follows:

$$U_{1,25} = (b_0)e^{i\psi}, \qquad \text{here} \tag{5.242}$$

$$\omega = -\frac{\alpha_2(l^2){\chi_2}^2}{2} - \frac{\alpha_3\chi_1(l^2)\chi_2}{2} - \frac{\alpha_1(l^2){\chi_1}^2}{2} - \Upsilon_1\Upsilon_2\alpha_3 - {\Upsilon_2}^2\alpha_2 - {\Upsilon_1}^2\alpha_1 \,,$$

In **case 2**, we have following families of solutions:

Family1:

When $l^2 - 4mk > 0$ and $lm \neq 0$ or $mk \neq 0$, the hyperbolic function solutions for Eq. (5.208) are as follows:

$$U_{2,1} = \left(b_0 - \left(\frac{b_1}{2m}\right)\left((l + \Omega)\tanh\left(\frac{\Omega\xi}{2}\right)\right)\right)e^{i\psi}, \tag{5.243}$$

$$U_{2,2} = \left(b_0 - \left(\frac{b_1}{2m}\right)\left((l + \Omega)coth\left(\frac{\Omega\xi}{2}\right)\right)\right)e^{i\psi}, \tag{5.244}$$

$$U_{2,3} = \left(b_0 - \left(\frac{b_1}{2m}\right)(l + \Omega)(\tanh(\Omega\xi) \pm isech(\Omega\xi))\right)e^{i\psi}, \tag{5.245}$$

$$U_{2,4} = \left(b_0 - \left(\frac{b_1}{2m}\right)(l + \Omega)(\coth(\Omega\xi) \pm csch(\Omega\xi))\right)e^{i\psi}, \tag{5.246}$$

$$U_{2,5} = \left(b_0 - \left(\frac{b_1}{4m}\right)\left(2(2l + \Omega)coth\left(\frac{\Omega\xi}{2}\right)\right)\right)e^{i\psi}, \tag{5.247}$$

$$U_{2,6} = \left(b_0 - \left(\frac{b_1}{2m}\right)\left(-l + \frac{\pm\sqrt{(A^2 + B^2)}\Omega - A\Omega\, cosh(\Omega\xi)}{A\sinh(\Omega\xi) + B}\right)\right)e^{i\psi}, \tag{5.248}$$

$$U_{2,7} = \left(b_0 - \left(\frac{b_1}{2m}\right)\left(-l - \frac{\pm\sqrt{(-A^2 + B^2)}\Omega + A\Omega sinh(\Omega\xi)}{Acosh(\Omega\xi) + B}\right)\right)e^{i\psi}, \tag{5.249}$$

where two non-zero real constants *A* and *B* satisfies $A^2 - B^2 > 0$.

$$U_{2,8} = \left(b_0 + \frac{2b_1kcosh\left(\frac{\Omega\xi}{2}\right)}{\Omega sinh\left(\frac{\Omega\xi}{2}\right) - lcosh\left(\frac{\Omega\xi}{2}\right)}\right)e^{i\psi}, \tag{5.250}$$

$$U_{2,9} = \left(b_0 - \frac{2b_1ksinh\left(\frac{\Omega\xi}{2}\right)}{-\Omega cosh\left(\frac{\Omega\xi}{2}\right) + lsinh\left(\frac{\Omega\xi}{2}\right)}\right)e^{i\psi}, \tag{5.251}$$

$$U_{2,10} = \left(b_0 + \frac{2b_1 k cosh(\Omega\xi)}{\Omega sinh(\Omega\xi) - l cosh(\Omega\xi) \pm i\Omega} \right) e^{i\psi}, \tag{5.252}$$

$$U_{2,11} = \left(b_0 + \frac{2b_1 k sinh(\Omega\xi)}{\Omega cosh(\Omega\xi) - l sinh(\Omega\xi) \pm \Omega} \right) e^{i\psi}, \tag{5.253}$$

$$U_{2,12} = \left(b_0 + \frac{2b_1 k sinh\left(\frac{\Omega\xi}{2}\right)}{-l sinh\left(\frac{\Omega\xi}{2}\right) + 2\Omega cosh\left(\frac{\Omega\xi}{4}\right)^2 - \Omega} \right) e^{i\psi}, \tag{5.254}$$

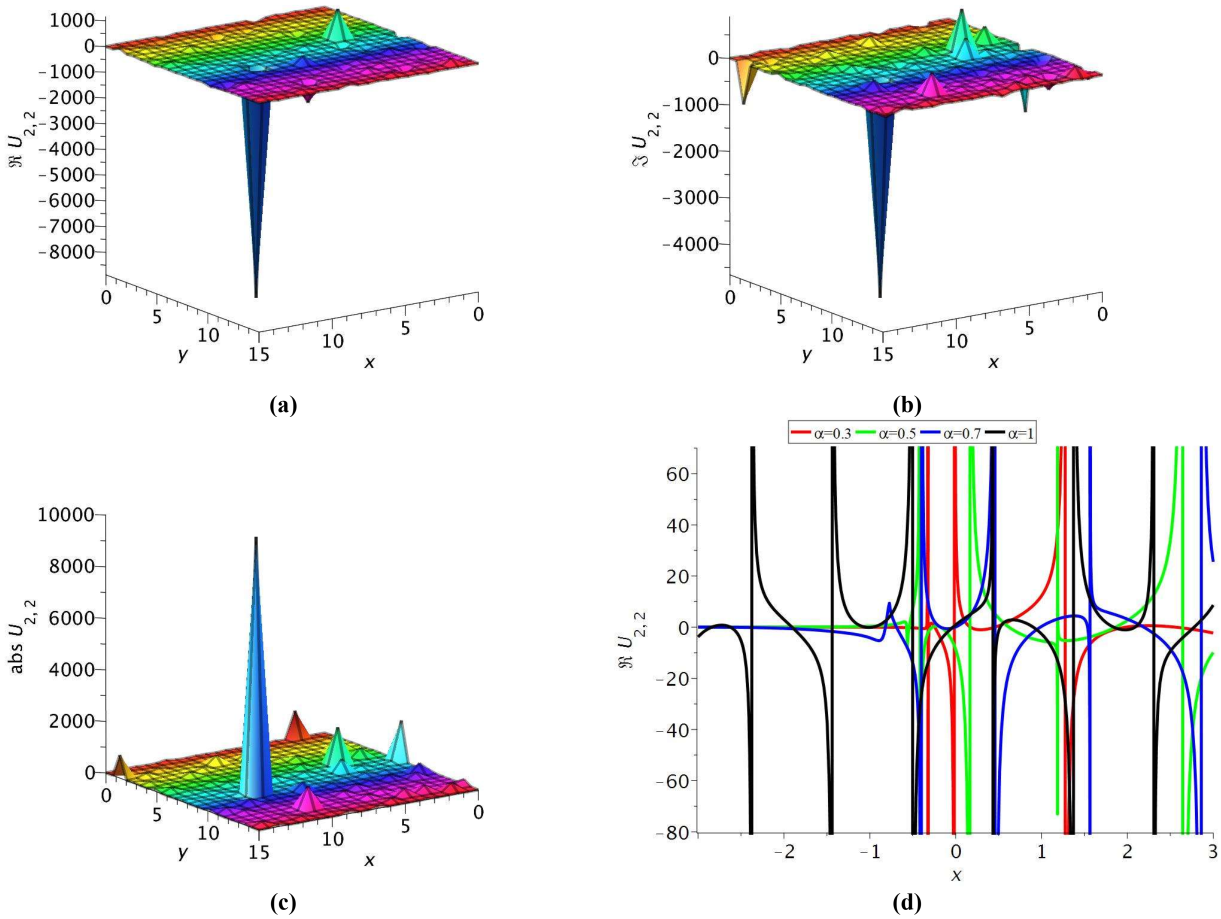


Figure 5.12: (a)-(c) 3D illustration of $Re(U_{2,2})$, $Im(U_{2,2})$, $abs(U_{2,2})$ with arbitrary parameters $l = 2, m = 3, k = 2, \alpha_1 = 1.5, \alpha_2 = 1.5, \alpha_3 = 1.5, \alpha_4 = 1.5, \rho = 2,\ \mu = 1.2,\ \beta = 1.5, \gamma = 2, \alpha = 0.5, t = 0.5, x = 0..15, y = 0..15$, and (d) 2D illustraion of $Re(U_{2,2})$ with , $\alpha = 0.3, 0.5, 0.7, 1$ at $x = -3..3,\ t = 0.5, y = 3$.

Family2:

If $l^2 - 4mk < 0$ and $lm \neq 0$ $(or\ mk \neq 0)$, we have the following trigonometric solutions for Eq. (5.208):

$$U_{2,13} = \left(b_0 + \left(\frac{b_1}{2m}\right)\left((-l + \Omega')\tan\left(\frac{\Omega'\xi}{2}\right)\right)\right)e^{i\psi}, \tag{5.255}$$

$$U_{2,14} = \left(b_0 - \left(\frac{b_1}{2m}\right)\left((l + \Omega')cot\left(\frac{\Omega'\xi}{2}\right)\right)\right)e^{i\psi}, \tag{5.256}$$

$$U_{2,15} = \left(b_0 + \left(\frac{b_1}{2m}\right)(-l + \Omega')(\tan(\Omega'\xi) \pm sec(\Omega'\xi))\right)e^{i\psi}, \tag{5.257}$$

$$U_{2,16} = \left(b_0 - \left(\frac{b_1}{2m}\right)(l + \Omega')(\cot(\Omega'\xi) \pm csc(\Omega'\xi))\right)e^{i\psi}, \tag{5.258}$$

$$U_{2,17} = \left(b_0 + \left(\frac{b_1}{4m}\right)(-2l + \Omega')\left(-2cot\left(\frac{\Omega'\xi}{2}\right)\right)\right)e^{i\psi}, \tag{5.259}$$

$$U_{2,18} = \left(b_0 + \left(\frac{b_1}{2m}\right)\left(-l + \frac{\pm\sqrt{(-A^2 + B^2)}\Omega' - A\Omega' cos(\Omega'\xi)}{A\sin(\Omega'\xi) + B}\right)\right)e^{i\psi}, \tag{5.260}$$

$$U_{2,19} = \left(b_0 + \left(\frac{b_1}{2m}\right)\left(-l - \frac{\pm\sqrt{(-A^2 + B^2)}\Omega' + A\Omega' cos(\Omega'\xi)}{Asin(\Omega'\xi) + B}\right)\right)e^{i\psi}, \tag{5.261}$$

where two non-zero real constants A and B satisfies $A^2 - B^2 > 0$.

$$U_{2,20} = \left(b_0 - \frac{2b_1 kcos\left(\frac{\Omega'\xi}{2}\right)}{\Omega' sin\left(\frac{\Omega'\xi}{2}\right) + lcos\left(\frac{\Omega'\xi}{2}\right)}\right)e^{i\psi}, \tag{5.262}$$

$$U_{2,21} = \left(b_0 + \frac{2b_1 ksin\left(\frac{\Omega'\xi}{2}\right)}{\Omega'\left(cos\left(\frac{\Omega'\xi}{2}\right) - lsin\left(\frac{\Omega'\xi}{2}\right)\right)}\right)e^{i\psi}, \tag{5.263}$$

$$U_{2,22} = \left(b_0 - \frac{2b_1 kcos(\Omega'\xi)}{\Omega'\left(sin(\Omega'\xi) + lcos(\Omega'\xi)\right) \pm \Omega'}\right)e^{i\psi}, \tag{5.264}$$

$$U_{2,23} = \left(b_0 + \frac{\left(2b_1 ksin(\Omega'\xi)\right)}{\Omega'(cos(\Omega'\xi) - lsin(\Omega'\xi)) \pm \Omega'}\right)e^{i\psi}, \tag{5.265}$$

$$U_{2,24} = \left(b_0 + \frac{2b_1 ksin\left(\frac{\Omega'\xi}{2}\right)}{-lsin\left(\frac{\Omega'\xi}{2}\right) + 2\Omega' cosh\left(\frac{\Omega'\xi}{4}\right)^2 - \Omega'}\right)e^{i\psi}, \tag{5.266}$$

where,

$$\Omega=\sqrt{l^2-4mk},\Omega'=\sqrt{4mk-l^2}$$

$$\xi=\frac{\chi_1}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}+\frac{\chi_2}{\alpha}\left(y+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}-\frac{\lambda}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \tag{5.267}$$

$$\lambda=\chi_1\Upsilon_2\alpha_3+2\alpha_1\chi_1\Upsilon_1+2\alpha_2\chi_2\Upsilon_2+\chi_2\Upsilon_1\alpha_3,$$

$$\psi=\frac{\Upsilon_1}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}+\frac{\Upsilon_2}{\alpha}\left(y+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}+\frac{\omega}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha},$$

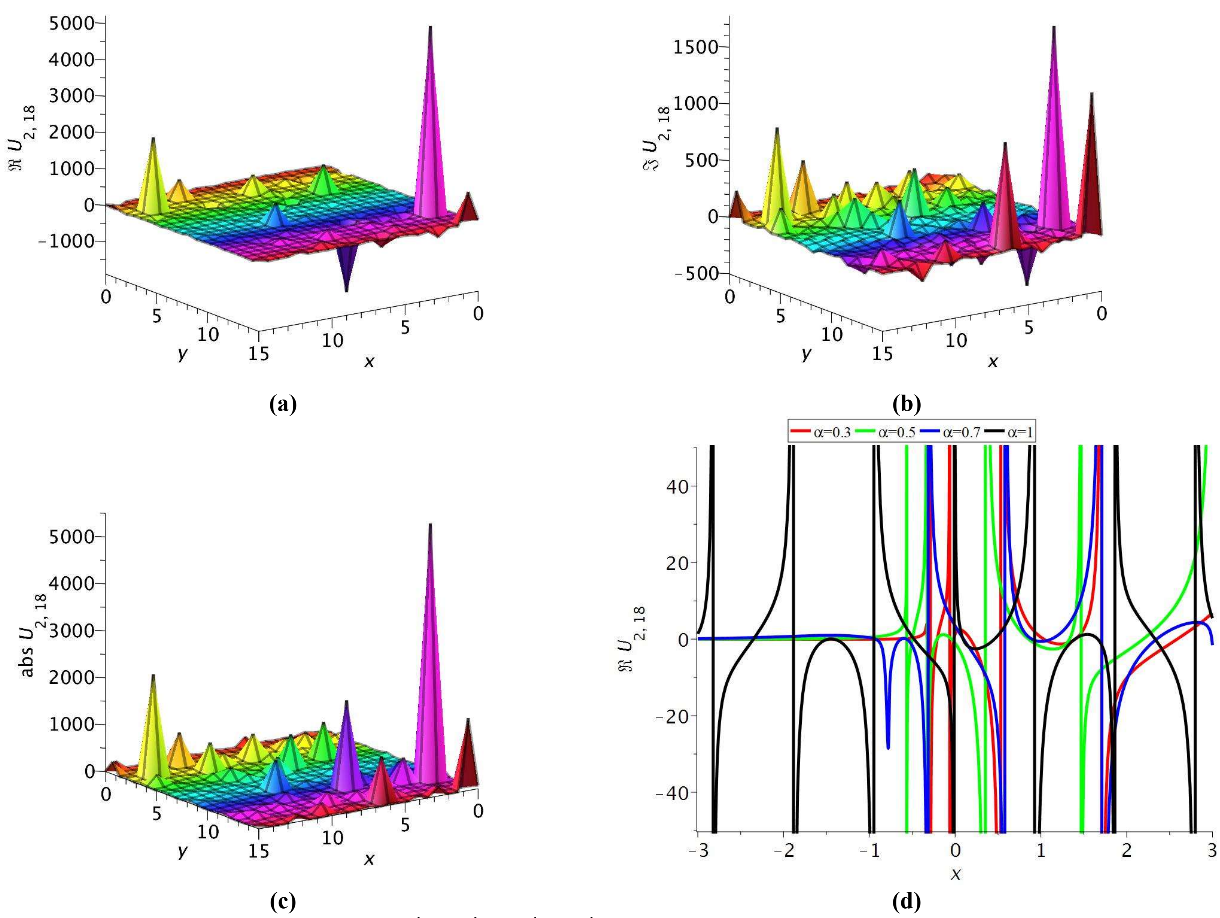


Figure 5.13: (a)-(c) 3D illustrarion of $Re(U_{2,18}), Im(U_{2,18}), abs(U_{2,18})$ by choosing arbitrary parameters $l=2, m=3, k=2, \alpha_1=1.5, \alpha_2=1.5, \alpha_3=1.5, \alpha_4=1.5, \Upsilon_1=2, \Upsilon_2=1.5, \chi_1=1.5, \chi_2=2, \alpha=0.3, A=3, B=2, t=1.5, x=0..15, y=0..15$, and (d) 2D illustrartion of $Re(U_{2,18})$ with , $\alpha=0.3, 0.5, 0.7, 1$ at $-3\le x\le 3$, $t=1.5, y=2$.

Family3:

When $k = 0$, and $lm \neq 0$ the solutions for Eq. (5.208) are as follows:

$$U_{2,25} = \left(b_0 - \frac{b_1 l d}{m\,(d + \cosh(l\xi) - \sinh(l\xi))}\right) e^{i\psi}, \tag{5.268}$$

where d is the arbitrary constant and

$$\omega = -\frac{\alpha_2(l^2){\chi_2}^2}{2} - \frac{\alpha_3\chi_1(l^2)\chi_2}{2} - \frac{\alpha_1(l^2){\chi_1}^2}{2} - \Upsilon_1\Upsilon_2\alpha_3 - {\Upsilon_2}^2\alpha_2 - {\Upsilon_1}^2\alpha_1.$$

Family4:

When $k = l = 0$, and $m \neq 0$ the rational solution of Eq. (5.208) is as follows:

$$U_{2,26} = \left(\frac{-b_1}{m\xi + c}\right) e^{i\psi}\,, \tag{5.269}$$

In this case,

$$\omega = -\Upsilon_1\Upsilon_2\alpha_3 - {\Upsilon_2}^2\alpha_2 - {\Upsilon_1}^2\alpha_1.$$

5.10 Graphical Explanation:

In this section, obtained results for conformable (2+1) dimensional Heisenberg ferromagnetic spin chain equation (HFM) is investigated. The graphs of some of the reported solutions that have been discussed here to have a good understanding of the physical properties of these types of solutions. We have constructed 3D graphs for the real, imaginary, and absolute values of some of obtained solutions such as dark, bright solitons, periodic wave solutions, singular periodic wave solutions, kink soliton solutions. Whereas 2D graphs have been plotted for real values of solutions to show pattern of wave propagation along $x - axis$ for choosing different values of α including classical and fractional order and we can see from these graphs that amplitude of wave increases with the increase in values of x for fractional values of α, and when $\alpha = 1$, we get complete wave with high amplitude for all values of x. Hence, amplitude of wave increases when x increases.

Figure 5.10 exhibits graphical representation of $U_{1,1}$, where 3D graphs (a), (b) represents $Re(U_{1,1})$ and $Im(U_{1,1})$ which are periodic in nature and figure (c) represents dark soliton for abs($U_{1,1}$) by taking parameters $l = 3, m = 1, k = 2, \alpha_1 = 1.5, \alpha_2 = 1.5, \alpha_3 = 1.5, \alpha_4 = 1.5, \Upsilon_1 = 1.3, \Upsilon_2 = 1.2, \chi_1 = 1.5, \chi_2 = 1, \alpha = 0.5, x = 0..15, y = 0..15$ and figure (d) represents 2D graphs of Re($U_{1,1}$) with different values of fractional order $\alpha = 0.3, 0.5, 0.7, 1$ at $-10 \le x \le 10, y = 3, t =$ *1.5*.

Figure 5.11 exhibits periodic solution of $U_{1,15}$, 3D graphs (a)-(c) represents Re($U_{1,15}$), Im($U_{1,15}$) and abs($U_{1,15}$) with $l = 3, m = 1, k = 2, \alpha_1 = 1.5, \alpha_2 = 1.5, \alpha_3 = 1.5, \alpha_4 = 1.5, \Upsilon_1 = 1.3, \Upsilon_2 = 1.2, \chi_1 = 1.5, \chi_2 = 1, \alpha = 0.8, x = 0..15, y = 0..15, t = 1.5$ and figure 2D- (d) represents Re($U_{1,15}$) with , $\alpha = 0.4, 0.6, 0.8, 1$ at $-10 \le x \le 10, t = 1.5, y = 2$

Figure 5.12 shows singular periodic wave solutions of $U_{2,2}$ where 3D graphs (a)-(c) exhibits Re($U_{2,2}$), Im($U_{2,2}$) and abs($U_{2,2}$) with $l = 2, m = 3, k = 2, \alpha_1 = 1.5, \alpha_2 = 1.5, \alpha_3 = 1.5, \alpha_4 = 1.5, \Upsilon_1 = 2, \Upsilon_2 = 1.2, \chi_1 = 1.5, \chi_2 = 2, \alpha = 0.5, x = 0..15, y = 0..15$, and 2D graph (d) represents Re($U_{2,2}$) with , $\alpha = 0.3, 0.5, 0.7, 1$ at $-3 \le x \le 3, y = 3, t = 0.5$.

Figure 5.13 exhibits the graph of singular periodic travelling wave solution of $U_{2,18}$, figures (a)-(c) exhibits 3D graphs of Re($U_{2,18}$), Im($U_{2,18}$) and abs($U_{2,18}$) with $l = 2, m = 3, k = 2, \alpha_1 = 1.5, \alpha_2 = 1.5, \alpha_3 = 1.5, \alpha_4 = 1.5, \Upsilon_1 = 2, \Upsilon_2 = 1.5, \chi_1 = 1.5, \chi_2 = 2, \alpha = 0.3, A = 3, B = 2, x = 0..15, y = 0..15$, and figure (d) shows 2D graphs of Re($U_{2,18}$) for various values of $\alpha = 0.3, 0.5, 0.7, 1$ at $-3 \le x \le 3, y = 2, t = 1.5$.

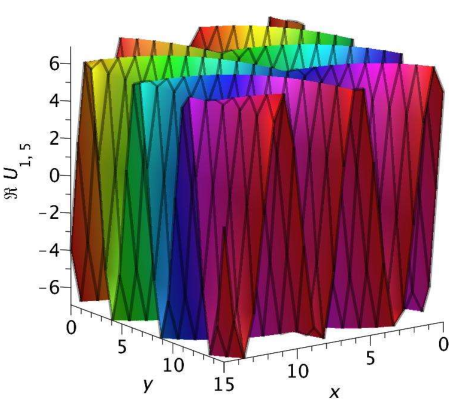
Figure 5.14 represents graphs of solution $U_{1,5}$ where, 3D:(a)-(b) exhibits periodic pattern of Re($U_{1,5}$), Im($U_{1,5}$) whereas figure (c) exhibits singular soliton for abs($U_{1,5}$) for $l = 4, m = 0.2, k = 4, \alpha_1 = 1.5, \alpha_2 = 1.5, \alpha_3 = 2, \alpha_4 = 1.5, \Upsilon_1 = 1.3, \Upsilon_2 = 1.2, \chi_1 = 1.5, \chi_2 = 1.5, \alpha = 0.2, x = 0..15, y = 0..15$, and figure 2D (d) exhibits Re($U_{1,5}$) by choosing $\alpha = 0.3, 0.5, 0.7, 1$ at $-10 \le x \le 10, y = 5, t = 2$

Figure 5.15 3D(a)-(c) exhibits graphs of periodic wave solution Re($U_{1,24}$), Im($U_{1,24}$) and abs($U_{1,24}$) with $l = 2, m = -1, k = 3, \alpha_1 = 2, \alpha_2 = 2, \alpha_3 = 2, \alpha_4 = 2, \Upsilon_1 = 1.3, \Upsilon_2 = 1.2, \chi_1 = 1.5, \chi_2 = 1.5, \alpha = 0.2, x = 0..15, y = 0..15, t = 1$, and figure (d) represents 2D graphs Re($U_{1,24}$) of with , $\alpha = 0.2, 0.6, 0.8, 1$ at $-10 \le x \le 10, y = 1, t = 1$.

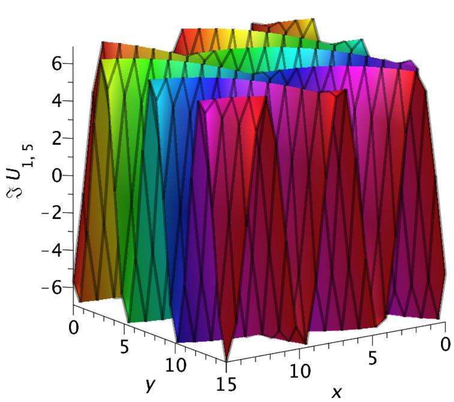
Figure 5.15 3D (a)-(b) shows graphs of solutions $\mathrm{Re}(\mathrm{U}_{2,24}), \mathrm{Im}(\mathrm{U}_{2,24})$ and are periodic in nature whereas figure (c) depicts graph of $\mathrm{abs}(\mathrm{U}_{2,24})$ which is kink soliton solution for parameters $l = 5, m = -1,\ k = 2,\ \alpha_1 = 1.5,\ \alpha_2 = 1.5,\ \alpha_3 = 1.5,\ \alpha_4 = 1.5,\ \Upsilon_1 = 2,\ \Upsilon_2 = 1.5,\ \chi_1 = 1.5,\ \chi_2 = 2, \alpha = 0.8, x = -15..15,\ y = -15..15, t = 3$,whereas figure 2D- (d) shows $\mathrm{Re}(\mathrm{U}_{2,24})$ with , $\alpha = 0.2, 0.6, 0.8, 1$ and $-10 \le x \le 10, y = 5, t = 3$.

Figure 5.16 3D (a)-(b) exhibits periodic wave solutions of $\mathrm{Re}(\mathrm{U}_{1,25}), \mathrm{Im}(\mathrm{U}_{1,25})$ while figure (c) exhibits kink soliton solution for $\mathrm{abs}(\mathrm{U}_{1,25})$ with l= $5, m = 1, k = 0, \alpha_1 = 1.5, \alpha_2 = 1.5, \alpha_3 = 1.5, \alpha_4 = 1.5,\ \Upsilon_1 = 2,\ \Upsilon_2 = 1.5,\ \chi_1 = 1.5,\ \chi_2 = 2, \alpha = 0.8,\ d = 1, x = -15..15,\ y = -15..15$, and figure 2D- (d) with $\mathrm{Re}(\mathrm{U}_{1,25})$ with $\alpha = 0.4, 0.6, 0.8, 1\quad -10 \le x \le 10,\ y = -1,\ t = 0.5$.

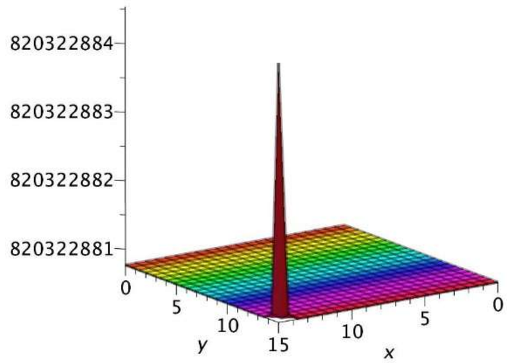
From these graphs, we can see that the shapes of the solutions change with by choosing different values of parameters and by slightly different values of the fractional derivative α behavior of wave changes.

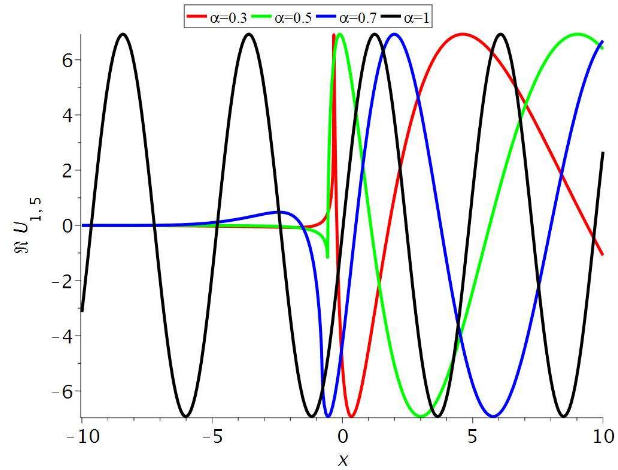


(a)



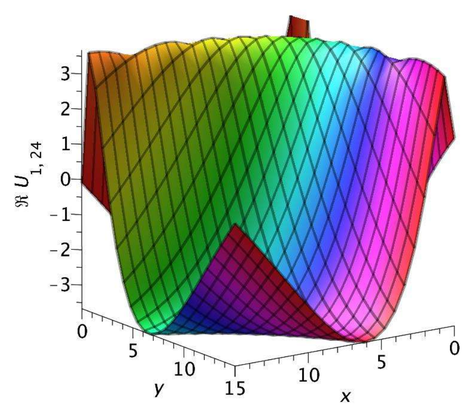
(b)

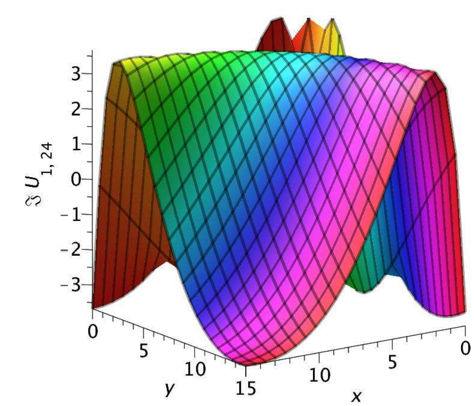




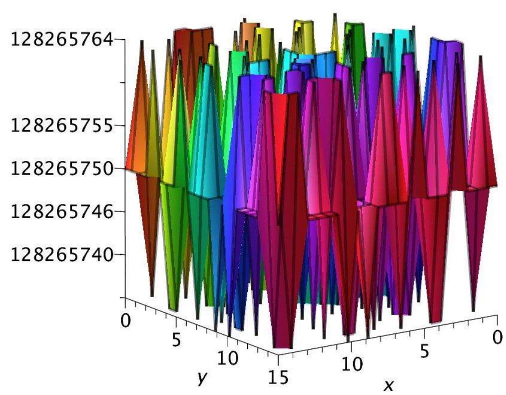
(c) **(d)**

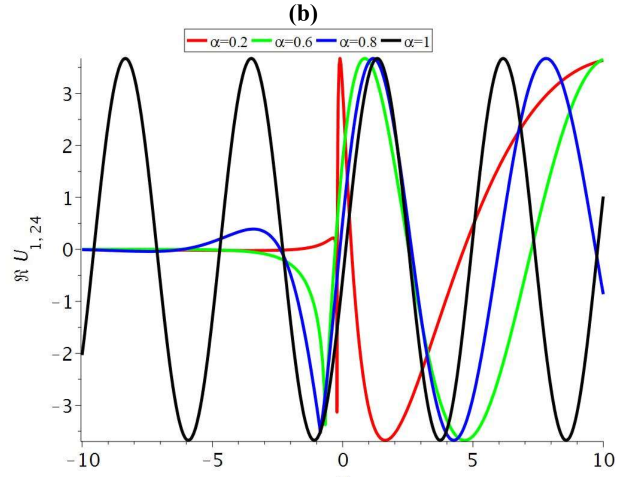
Figure 5.14: (a)-(c) 3D illustration of $Re(U_{1.5}), Im(U_{1.5}), abs(U_{1.5})$ by choosing parameters $l = 4, m = 0.2, k = 4, \alpha_1 = 1.5, \alpha_2 = 1.5, \alpha_3 = 2, \alpha_4 = 1.5, \rho = 1.3, \mu = 1.2, \beta = 1.5, \gamma = 1.5, \alpha = 0.2, t = 2, x = 0..15, y = 0..15$, and (d) 2D illustration of $Re(U_{1,5})$ with , $\alpha = 0.3, 0.5, 0.7, 1$ at $-10 \leq x \leq 10,\ t = 2, y = 5$.





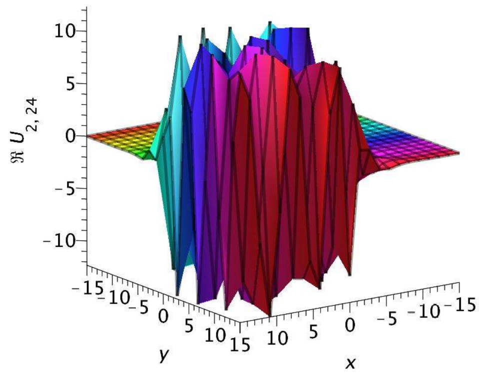
(a) **(b)**

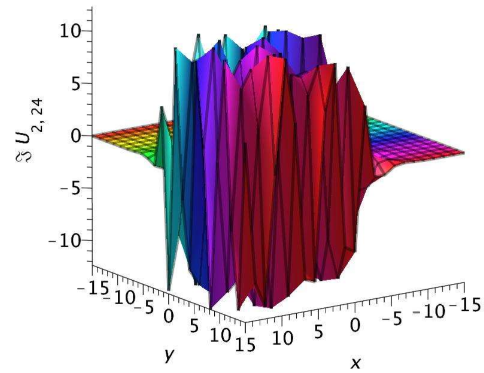




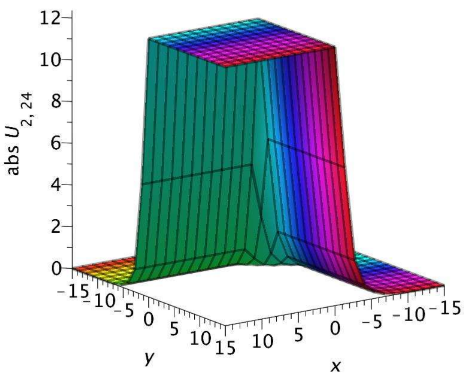
(c) **(d)**

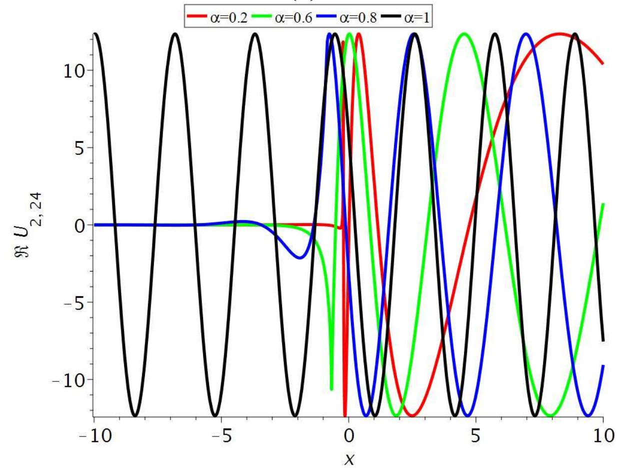
Figure 5.15: (a)-(c) 3D illustration $Re(U_{1,24}), Im(U_{1,24}), abs(U_{1,24})$with suitable parameters, $l = 2, m = -1, k = 3, \alpha_1 = 2, \alpha_2 = 2, \alpha_3 = 2, \alpha_4 = 2, \rho = 1.3, \mu = 1.2, \beta = 1.5, \gamma = 1.5, \alpha = 0.2, x = 0..15, y = 0..15, t = 1$, and (d) 2D illustration of $Re(U_{1.24})$ with , $\alpha = 0.2, 0.6, 0.8, 1$ at $x = -10..10,\ t = 1, y = 1$





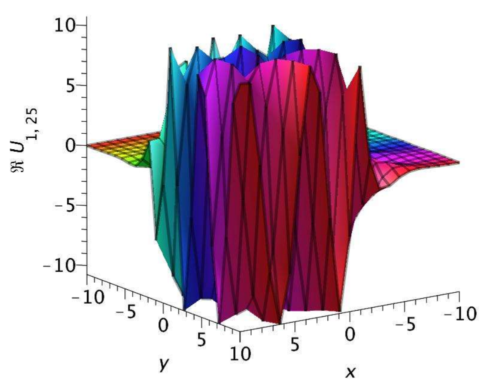
(a) **(b)**

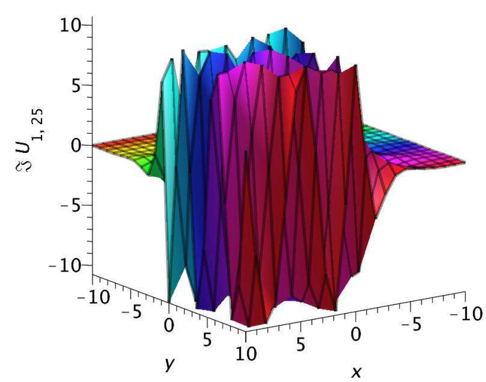




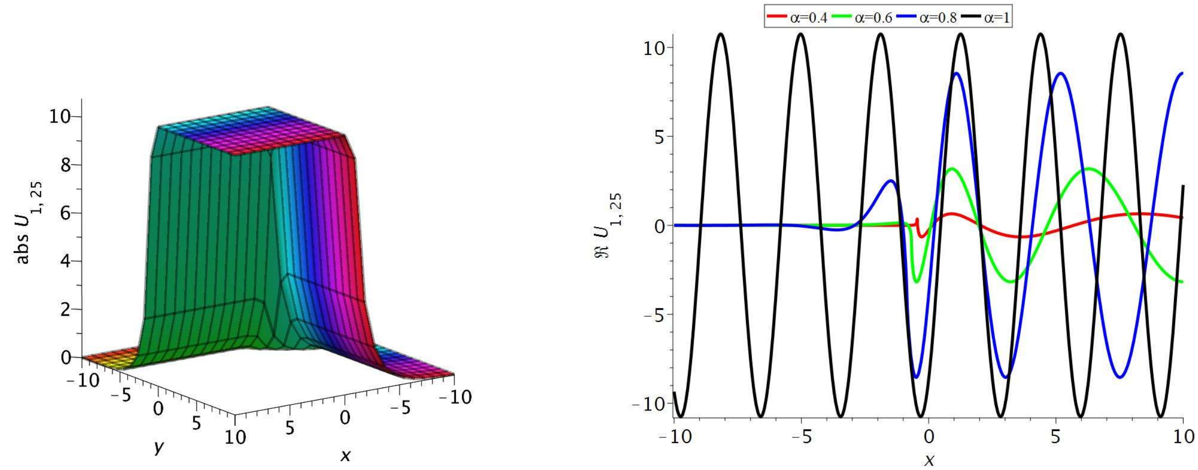
(c) **(d)**

Figure 5.16(a)-(c) 3D illustration of $Re(U_{2,24})$, $Im(U_{2,24})$, $abs(U_{2,24})$ with $l = 5, m = -1, k = 2, \alpha_1 = 1.5, \alpha_2 = 1.5, \alpha_3 = 1.5, \alpha_4 = 1.5, \rho = 2, \mu = 1.5, \beta = 1.5, \gamma = 2, \alpha = 0.8, x = -15..15,\ y = -15..15$, and (d) 2D illustration of $Re(U_{2,24})$ with , $\alpha = 0.2, 0.6, 0.8, 1\quad x = -10..10, y = 5,\ t = 3$





(a) **(b)**



(c) **(d)**

Figure 5.17 (a)-(c) 3D illustration of $Re(U_{1,25})$, $Im(U_{1,25})$, $abs(U_{1,25})$ by choosing $l = 5, m = 1, k = 0, \alpha_1 = 1.5, \alpha_2 = 1.5, \alpha_3 = 1.5, \alpha_4 = 1.5, \rho = 2, \mu = 1.5, \beta = 1.5, \gamma = 2, \alpha = 0.8, d = 1, x = -15..15,\ y = -15..15$, and (d) 2D illustration of $Re(U_{1,25})$ with , $\alpha = 0.4, 0.6, 0.8, 1\quad x = -10..10,\ y = -1,\ t = 0.5$

5.11 Conclusions:

We successfully derived exact solutions of conformable (2+1) dimensional Heisenberg ferromagnetic spin chain equation with the improved generalized Riccati mapping method. As a result, we established different solitary wave solution including dark and bright solitons, periodic wave solutions, singular solution, kink solitons and rational solution which have not been reported in literature previously. Moreover, this model has not been solved before using Antangan's fractional derivative. Computation software Maple has used to facilitate tedious algebraic calculations and all the results have been verified by backward substitution. We concluded that for different values of α including classical and fractional order, the graph represents wave solutions with high amplitude as $\alpha \rightarrow 1$. for fractional order the amplitude of the wave gradually increases with increase in values of x. Therefore this method is very effective technique in generating abundant solutions of various types. These results might be helpful in the study magnetic behavior in ferro-magnetic materials.

5.12 Summary:

This chapter incorporates with the well-known nonlinear PDEs in fractional order such as space-time fractional non-liner double dispersive equation (DDE), space-time fractional non-liner Telegraph equation, space-time fractional (2+1) dimensional Heisenberg ferromagnetic spin chain equation with the help of improved generalized Riccati equation mapping method. The efforts to extend the existing methods used to solve integer order NLPDEs to their fractional counterparts, and apply them to solve real life fractional models, have gained tremendous popularity. We succeed in generating many interesting types of solitary wave solutions that might be helpful in the study of these models. This chapter includes introduction of governing equations followed by main steps of methods used and derivation of solutions by proposed method. Finally graphical representation of some results followed by conclusion.

Chapter 6 includes the summary of previous chapters, significance of this research, contribution to the knowledge and conclusions. It also highlights limitations of our work and future recommendations to work in this field.

Chapter 6. Conclusions and Future recommendations

6.1 Conclusions:

This chapter discusses the overall conclusions of our work presented in this thesis.

The objective of this research work is to discover exact solitary wave solutions to nonlinear differential equations including integer order (NLPDEs) and non-integer order (NFPDEs) arising in various fields of science and technology for wave propagation. We have successfully found exact traveling wave solutions including solitons, periodic waves, kink wave solutions to several nonlinear partial differential equations representing real-life phenomena. These new solutions may be worthwhile in the field of ocean engineering, astrophysics, and aerodynamics, plasma physics and fluid mechanics to explain wave propagation of incompressible fluids. Each type of solitary wave has its importance in nonlinear media such as kink solitons which propagates in nonlinear physical phenomena having high order nonlinearity, high order nonlinear effects and self-steepening. These solitons have been studied extensively due to its perfect propagation through nonlinear media [106]. Singular solitons are also very important types of solitons that appear with singularity. These solitons likely provide information about formation of rouge waves, also another type of solitary waves are periodic wave solutions that plays notable role in the study of chemistry, physics, biology and many more [107]. The formation of solitary waves has been captured in the solution to NLPDEs corresponding to models of practical interest involving optic fiber signal transmission and wave propagation in different media.

Here we have used Tanh method, which was firstly presented by [48]. This method is straight forward, simple, and reliable that has ability to find solutions of variety of NPFDEs without reproducing many different forms of the same solution. We applied this method to a few well-known models, having applications in various fields such as Dodd-Bullough-Mikhailov equation, Sinh-Gordan equation, Liouville equation. The mentioned equation plays significant role in problems arising in fluid flows, solid state physics, nonlinear optics, quantum field theory and chemical kinetics [30]. We have also used this method on modified version of Benjamin-Bona-Mahony equation (BBM) called, $(3+1)$-Wazwaz-Benjamin-Bona-Mahony equation (WBBM) named by Wazwaz in 2017 [81]. BBM equation was derived by Benjamin, Bona and Mahony in 1972, which is also the improved version of Korteweg-de-Vries (KDV) equation for surfaced water waves in uniform channel and regularized version in shallow water waves [80]. A fair amount of work has been done on this equation due to its importance in surface wave water, in nonlinear dispersive system for long wave lengths, acoustic gravity waves in compressible liquids, hydromagnetic waves in plasma physics and many more.

Next, we have utilized innovative and efficient method called improved $\tanh(\frac{\varphi(\xi)}{2})$-expansion method (IThEM) for recently developed $(3+1)$-dimensional Boiti-Leon-Manna-Pempinelli equation. This model has applications in plasma physics, fluid dynamics, ocean engineering, astrophysics, and aerodynamics to explain wave propagation of incompressible fluids [31, 88, 92–96]. and, on fourth order Ablowitz-Kaup-Newell-Segur water wave (AKNS) equation. This equation is remarkable due to the fact that it can be reduce into some very prominent nonlinear equations such as KdV equation, mKdV equation which are used for the study of shallow water waves and wave propagation in plasma, $(2+1)$ dimensional Boussinesq wave equation which is used for the investigation of nonlinear wave effect on shallow water, sine-Gordan equation have application in different fields of physics and nonlinear Schrödinger equation has wide range of applications in optical physics, quantum mechanics and many more [32].

We have also used Auxiliary equation method (AEM) developed by Sirendaoreji [61] on Fokas system and $(2+1)$ Davey-Stewartson (DS) system which is the generalization of nonlinear Schrodinger equation used as governing equation to generate optical solitons that have showed significant effect in telecommunication field because of its key role in data transmission through optical fibers over large distances.

Moreover, we have utilized improved generalized Riccati equation mapping method on some fractional nonlinear models. The use of fractional calculus to model certain real-life phenomena is getting a great attention nowadays. NLFPDEs are generalizations of nonlinear partial differential equations (NPDEs) in which the orders of derivatives involved are fractional. We have studied space-time fractional nonlinear elastic inhomogeneous double dispersive equation for Murnaghan's rod. The doubly dispersive equation (DDE), which is an important nonlinear physical model describing the nonlinear wave propagation in the elastic inhomogeneous circular cylinder Murnaghan's rod. Space-time conformable telegraph equation commonly used to study electrical signals in transmission lines. And another important equation we have studied is the newly derived variant of Nonlinear Schrödinger Equation (NLSE) that describes time-space fractional (2+1)-dimensional Heisenberg ferromagnetic spin chains with bilinear and anisotropic interactions in the semi classical limit. Heisenberg ferromagnet model (HFM) is an interesting nonlinear model that exhibits magnetic solitons and, also very important to study magnetic behavior in magnetic materials [33]

The concluded wave structures can be helpful to understand the characteristics of nonlinear phenomena that develop in various realms of nonlinear sciences. Moreover, the outcome of this research can predict that this method is suitable to apply on various higher order nonlinear models to produce many interesting solutions involve in engineering, nonlinear optics, physics, and other life sciences.

6.2 Limitations:

Although analytical methods are powerful tool to generate exact solutions of numerous nonlinear PDEs and to understand the nonlinear behaviour of physical phenomena but still they have their weaknesses. These methods are applicable to many nonlinear systems but certain complex nonlinear PDEs are not solvable by these techniques alternatively these models have approximate or numerical solutions. These methods need clearly defined initial or boundary value problems. These types of techniques require a lot of computational work. Mostly computational software such as Maple/Mathematica used to intricate mathematical calculations. Which requires a lot of programming to extract solutions and for graphical representation of these results. Sometimes software gives up on solving long and complex system of linear systems. Finding coefficients of these linear systems are important step in finding the solutions of PDEs. Which is time consuming and tedious. Researchers need to derive methods that requires less computational work. Also, they can work on how to combine analytical and numerical methods to create a unified methods that can cater major portion of nonlinear systems.

6.3 Future Recommendations:

For future recommendations, we can modify some techniques used in this manuscript or in literature to improve their performance to get new types of solutions. There exist many NFPDEs in different fields of science and engineering which are still posed and unanswered in literature. We can increase the order of equations to make them integrable with higher order equations. Higher order nonlinear PDEs are considered very beneficial to describe physical mechanism. Multiple auxiliary equations methods are some other avenues for future endeavors. We can also use numerical methods along with analytical in our future work to check the accuracy of our results, as these solutions can help us to validate analytical solutions when complex partial differential equations are involved. Numerical solutions help us to analyze the behavior of solutions under certain parameters in a nice manner.

There is a recent growing trend to use artificial neural networks and machine learning to simulate certain real-life phenomena, the same can be used to simulate solitary motion of different traveling waves. It can be achieved both by data driven training networks or physics informed neural networks. We see a great potential in using deep learning to mimic solitary waves as well. Experimenting can be done with different learning and optimizing algorithms. We are hopeful these recommendations will be useful for anyone interested in working in this field in future.

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