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Norm and Numerical Radius Inequalities for Sums of Power Series of Operators in Hilbert Spaces

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Abstract: The main focus of this paper is on establishing inequalities for the norm and numerical radius of various operators applied to a power series with the complex coefficients $h(\lambda) = \sum_{k=0}^{\infty} a_k \lambda^k$ and its modified version $h_a(\lambda) = \sum_{k=0}^{\infty} |a_k| \lambda^k$. The convergence of $h(\lambda)$ is assumed on the open disk $\mathbb{D}(0, R)$, where R is the radius of convergence. Additionally, we explore some operator inequalities related to these concepts. The findings contribute to our understanding of operator behavior in bounded operator spaces and offer insights into norm and numerical radius inequalities.

Keywords: norm inequalities; numerical radius inequalities; power series; operators; Hilbert spaces

MSC: 47B65; 47A12; 47A13; 47A30

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1. Introduction and Preliminary

It is really important to understand and study operator inequalities, especially when they include norms and numerical radii. These concepts are used in different parts of mathematics. Previous research, including the studies referenced in [1–10], has extensively investigated mathematical inequalities and discovered significant findings. These studies provide a foundation for future research in this field. In particular, power series of operators have emerged as a key topic in functional analysis and operator theory. These series are essential for representing and studying operators systematically. In this paper, we focus on exploring norm and numerical radius inequalities that are specifically designed for sums of power series of operators within Hilbert spaces.

Power series are valuable tools for expressing complex mathematical ideas, and their application to operators helps us examine these ideas more effectively. By studying the relationship between power series and operator inequalities, we aim to uncover insights that contribute to a deeper understanding of mathematical structures. To fully grasp these concepts, we recommend readers explore the information available in references such as [11–16] and the sources mentioned therein. These resources not only complement our research but also provide valuable information for further exploration into the complexities of operator theory. As we delve into norm and numerical radius inequalities in the context of power series, we hope to shed light on new perspectives that could potentially influence broader discussions in mathematics.

We begin by considering power series with complex coefficients. A power series is an expression of the form $h(\lambda) = \sum_{k=0}^{\infty} a_k \lambda^k$, where a_k are complex numbers and λ is a complex variable. We assume that this power series converges within a certain region called the open disk $\mathbb{D}(0, R)$, which consists of all complex numbers λ with a distance less than R from the origin. If R is infinite, it means the power series converges for all complex numbers.

To understand the behavior of the coefficients in the power series, we define another series called $h_a(\lambda)$. This series is obtained by taking the absolute values of the coefficients in $h(\lambda)$, i.e., $h_a(\lambda) = \sum_{k=0}^{\infty} |a_k|\lambda^k$. It has the same convergence properties as $h(\lambda)$, but it focuses on the magnitudes of the coefficients. Natural examples include:

$$h(\lambda) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\lambda^n = \ln\left(\frac{1}{1+\lambda}\right),\ \lambda \in \mathbb{D}(0,1);$$
$$g(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\lambda^{2n} = \cos\lambda,\ \lambda \in \mathbb{C};$$
$$l(\lambda) = \sum_{n=0}^{\infty} (-1)^n\lambda^n = \frac{1}{1+\lambda},\ \lambda \in \mathbb{D}(0,1).$$

The corresponding functions with absolute values of coefficients are:

$$h_a(\lambda) = \sum_{n=1}^{\infty} \frac{1}{n}\lambda^n = \ln\left(\frac{1}{1-\lambda}\right),\ \lambda \in \mathbb{D}(0,1);$$
$$g_a(\lambda) = \sum_{n=0}^{\infty} \frac{1}{(2n)!}\lambda^{2n} = \cosh\lambda,\ \lambda \in \mathbb{C};$$
$$l_a(\lambda) = \sum_{n=0}^{\infty} \lambda^n = \frac{1}{1-\lambda},\ \lambda \in \mathbb{D}(0,1).$$

Other important examples of functions as power series with nonnegative coefficients are:

$$\exp(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!}\lambda^n, \quad \lambda \in \mathbb{C};$$
$$\frac{1}{2}\ln\left(\frac{1+\lambda}{1-\lambda}\right) = \sum_{n=1}^{\infty} \frac{1}{2n-1}\lambda^{2n-1}, \quad \lambda \in \mathbb{D}(0,1).$$

Before delving into our study, it is important to recall some definitions and terminologies. Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra consisting of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. We denote the identity operator as I. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be positive, denoted as $T \geq 0$, if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$. If $T \in \mathcal{B}(\mathcal{H})$ satisfies $T \geq 0$, there exists a unique positive operator $T^{\frac{1}{2}} \in \mathcal{B}(\mathcal{H})$ such that $T = (T^{\frac{1}{2}})^2$. The adjoint of an operator T is denoted as T^*. Furthermore, the absolute value of T, denoted by $|T|$, is given by $|T| = (T^*T)^{\frac{1}{2}}$.

Let $T \in \mathcal{B}(\mathcal{H})$. The operator norm of T, denoted by $\|T\|$, is defined as the supremum of $\|Tx\|$ over all unit vectors $\|x\| = 1$, i.e., $\|T\| = \sup_{\|x\|=1} \|Tx\|$. In this context, if x belongs to $\mathcal{H}$, the quantity $\|x\|$ is defined as the square root of the inner product $\langle x, x\rangle$, where $\langle\cdot,\cdot\rangle$ represents the inner product defined on $\mathcal{H}$. The operator norm $\|\cdot\|$ can be alternatively defined as $\|T\| = \sup_{\|x\|=\|y\|=1} |\langle Tx, y\rangle|$. In this definition, if we set $y = x$, we obtain a smaller quantity known as the numerical radius, denoted by $\omega(T)$. Therefore, for $T \in \mathcal{B}(\mathcal{H})$, the numerical radius of T is the scalar value $\omega(T) = \sup_{\|x\|=1} |\langle Tx, x\rangle|$. It can be easily verified that $\omega(\cdot)$ also defines a norm on $\mathcal{B}(\mathcal{H})$. However, there are significant differences between the norm properties of $\omega(\cdot)$ and $\|\cdot\|$. Specifically, the numerical radius is neither submultiplicative nor unitarily invariant, unlike the operator norm.

Although the definition of $\omega(\cdot)$ may appear simpler than $\|\cdot\|$, computing the numerical radius $\omega(\cdot)$ turns out to be more challenging. As a result, there has been significant interest within the research community in approximating the values of $\omega(\cdot)$ in terms of the operator norm $\|\cdot\|$. This is often accomplished by establishing sharp upper and lower bounds.

In this context, an important relation presented in ([17], Theorem 1.3-1) states that the for every $T \in \mathcal{B}(\mathcal{H})$, we have

$$\omega(T) \le \|T\| \le 2\omega(T). \tag{1}$$

This relation demonstrates the equivalence between the two norms, $\omega(\cdot)$ and $\|\cdot\|$. However, it is important to note that there can be a significant difference between the values on the left and right sides of (1). Consequently, researchers have devoted considerable efforts to finding tighter bounds for better approximations and deeper insights into these relationships. To provide information on norm and numerical radius inequalities, readers are encouraged to consult the following references [18–24] and the additional references cited therein.

The primary objective of this paper is to establish inequalities involving the norms and numerical radii of operators represented by power series. We aim to understand the relationship between the coefficients in the power series and the properties of the operators. By establishing norm and numerical radius inequalities for sums of power series of operators, we contribute to the field of functional analysis and operator theory.

In Section 2, our focus will be on proving different vector inequalities for operators. These inequalities involve the summation of power series of operators in Hilbert spaces along with their modified versions. Various generalizations of a Kato-type inequality for weighted sums of operators established in [25] are also provided. Among others, we showed that if the power series with complex coefficients $h(\lambda) = \sum_{k=0}^{\infty} a_k \lambda^k$ is convergent on $\mathbb{D}(0,R)$ and T_i, U_i, $V_i \in \mathcal{B}(\mathcal{H})$ with $\|T_i\| < R$, $i \in \{1,...,n\}$, then for non-negative constants $p_i \ge 0$ with $\sum_{i=1}^{n} p_i > 0$, it holds that

$$\begin{aligned}
&\sum_{i=1}^{n} p_i \left|\left\langle V_i^* T_i h(T_i) U_i x, y\right\rangle\right| \\
&\le \left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|) \left||T_i|^{\alpha} U_i\right|^2 x, x\right\rangle^{\frac{1}{2}} \left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|) \left||T_i^*|^{1-\alpha} V_i\right|^2 y, y\right\rangle^{\frac{1}{2}}
\end{aligned}$$

for all $x, y \in \mathcal{H}$ and $\alpha \in [0,1]$.

Moving on to Section 3, we will introduce a variety of inequalities related to the norm and numerical radius. As an excerpt, we mention the following result

$$\begin{aligned}
&\omega^2\left(\sum_{i=1}^{n} p_i V_i^* T_i h(T_i) U_i\right) \\
&\le \frac{1}{2}\left\|\sum_{i=1}^{n} p_i h_a(\|T_i\|) \left||T_i|^{\alpha} U_i\right|^2\right\| \left\|\sum_{i=1}^{n} p_i h_a(\|T_i\|) \left||T_i^*|^{1-\alpha} V_i\right|^2\right\| \\
&+ \frac{1}{2}\omega\left(\sum_{i=1}^{n} p_i h_a(\|T_i\|) \left||T_i^*|^{1-\alpha} V_i\right|^2 \sum_{i=1}^{n} p_i h_a(\|T_i\|) \left||T_i|^{\alpha} U_i\right|^2\right),
\end{aligned}$$

provided that the power series with complex coefficients $h(\lambda) = \sum_{k=0}^{\infty} a_k \lambda^k$ is convergent on $\mathbb{D}(0,R)$, T_i, U_i, $V_i \in \mathcal{B}(\mathcal{H})$ with $\|T_i\| < R$, $i \in \{1,...,n\}$, $\alpha \in [0,1]$ and $p_i \ge 0$ with $\sum_{i=1}^{n} p_i > 0$.

Various examples for fundamental operator functions such as the resolvent, the logarithm function, operator exponential, operator trigonometric and hyperbolic functions are given as well.

2. Power Series and Operator Vector Inequalities

In this section, we consider the power series with complex coefficients $h(\lambda) = \sum_{k=0}^{\infty} a_k \lambda^k$ with $a_k \in \mathbb{C}$ for $k \in \mathbb{N} = \{0, 1, \ldots\}$. We assume that this power series is convergent on the open disk $\mathbb{D}(0, R) = \{z \in \mathbb{C};\ |z| < R\}$. If $R = \infty$, then $\mathbb{D}(0, R) = \mathbb{C}$. We define $h_a(\lambda) = \sum_{k=0}^{\infty} |a_k| \lambda^k$, which has the same radius of convergence R. To prove our first result, we need to establish the following lemma.

Lemma 1. *Let $T, U, V \in \mathcal{B}(\mathcal{H})$ and $\alpha \in [0, 1]$. Then, for $n \geq 1$ we have*

$$\left|\left\langle V^* T^n U x, y\right\rangle\right|^2 \leq \|T\|^{2n-2} \left\langle \left||T|^{\alpha} U\right|^2 x, x\right\rangle \left\langle \left||T^*|^{1-\alpha} V\right|^2 y, y\right\rangle \tag{2}$$

for all $x, y \in \mathcal{H}$.

Proof. Firstly, observe that Kittaneh derived the following Schwarz-type inequality for powers of operators in ([26], Corollary 7). This inequality asserts that for every $T \in \mathcal{B}(\mathcal{H})$, and for all $x, y \in \mathcal{H}$, $\alpha \in [0, 1]$ and $n \geq 1$, the following holds:

$$\left|\left\langle T^n x, y\right\rangle\right|^2 \leq \|T\|^{2n-2} \left\langle |T|^{2\alpha} x, x\right\rangle \left\langle |T^*|^{2(1-\alpha)} y, y\right\rangle. \tag{3}$$

Now, let $x, y \in \mathcal{H}$; if we replace x by Ux and y by Vy in (3), then we obtain

$$\left|\left\langle V^* T^n U x, y\right\rangle\right|^2 \leq \|T\|^{2n-2} \left\langle U^* |T|^{2\alpha} U x, x\right\rangle \left\langle V^* |T^*|^{2(1-\alpha)} V y, y\right\rangle. \tag{4}$$

Observe that $U^* |T|^{2\alpha} U = \left||T|^{\alpha} U\right|^2$ and $V^* |T^*|^{2(1-\alpha)} V = \left||T^*|^{1-\alpha} V\right|^2$; then, from (4), we obtain (2). □

Now, we are able to establish the following result.

Proposition 1. *Assume that the power series with complex coefficients $h(\lambda) = \sum_{k=0}^{\infty} a_k \lambda^k$ is convergent on $\mathbb{D}(0, R)$ and $T, U, V \in \mathcal{B}(\mathcal{H})$ with $\|T\| < R$, then*

$$\left|\left\langle V^* T h(T) U x, y\right\rangle\right|^2 \leq h_a^2(\|T\|) \left\langle \left||T|^{\alpha} U\right|^2 x, x\right\rangle \left\langle \left||T^*|^{1-\alpha} V\right|^2 y, y\right\rangle \tag{5}$$

for $\alpha \in [0, 1]$ and $x, y \in \mathcal{H}$. In particular,

$$\left|\left\langle V^* T h(T) U x, y\right\rangle\right|^2 \leq h_a^2(\|T\|) \left\langle \left||T|^{\frac{1}{2}} U\right|^2 x, x\right\rangle \left\langle \left||T^*|^{\frac{1}{2}} V\right|^2 y, y\right\rangle \tag{6}$$

for $x, y \in \mathcal{H}$.

Proof. If we take $n = k + 1$, $k \in \mathbb{N}$ in (2) and take the square root, then we obtain

$$\left|\left\langle V^* T T^k U x, y\right\rangle\right| \leq \|T\|^k \left\langle \left||T|^{\alpha} U\right|^2 x, x\right\rangle^{\frac{1}{2}} \left\langle \left||T^*|^{1-\alpha} V\right|^2 y, y\right\rangle^{\frac{1}{2}}$$

for all $x, y \in \mathcal{H}$.

Further, if we multiply by $|a_k| \geq 0$, $k \in \{0, 1, \ldots\}$ and sum over k from 0 to m, then we obtain

$$\begin{aligned}
\left|\left\langle V^* T \sum_{k=0}^{m} a_k T^k U x, y\right\rangle\right| &= \left|\sum_{k=0}^{m} a_k \left\langle V^* T T^k U x, y\right\rangle\right| \\
&\leq \sum_{k=0}^{m} |a_k| \left|\left\langle V^* T T^k U x, y\right\rangle\right| \\
&\leq \sum_{k=0}^{m} |a_k| \|T\|^k \left\langle \left||T|^{\alpha} U\right|^2 x, x\right\rangle^{\frac{1}{2}} \left\langle \left||T^*|^{1-\alpha} V\right|^2 y, y\right\rangle^{\frac{1}{2}}
\end{aligned} \tag{7}$$

for all $x, y \in \mathcal{H}$.

Since $\|T\| < R$, then series $\sum_{k=0}^{\infty} a_k T^k$ and $\sum_{k=0}^{\infty} |a_k| \|T\|^k$ are convergent and

$$\sum_{k=0}^{\infty} a_k T^k = h(T) \text{ and } \sum_{k=0}^{\infty} |a_k| \|T\|^k = h_a(\|T\|).$$

By taking now the limit over $m \to \infty$ in (7), we deduce the desired result (5). □

The following remark is of great importance, as it reveals significant consequences derived from the preceding proposition.

Remark 1. *(1) If we take $h \equiv 1$, in (5) and (6), then we obtain the following Kato-type inequality [27]*

$$\left|\langle V^* T U x, y\rangle\right|^2 \leq \langle \left||T|^{\alpha} U\right|^2 x, x\rangle \langle \left||T^*|^{1-\alpha} V\right|^2 y, y\rangle$$

for $\alpha \in [0,1]$ and $x, y \in \mathcal{H}$. In particular,

$$\left|\langle V^* T U x, y\rangle\right|^2 \leq \langle \left||T|^{\frac{1}{2}} U\right|^2 x, x\rangle \langle \left||T^*|^{\frac{1}{2}} V\right|^2 y, y\rangle.$$

(2) If we take $U = V = I$ in (5) and (6), then we obtain for $\alpha \in [0,1]$ that

$$\left|\langle T h(T) x, y\rangle\right|^2 \leq h_a^2(\|T\|) \langle |T|^{2\alpha} x, x\rangle \langle |T^*|^{2(1-\alpha)} y, y\rangle \tag{8}$$

and

$$\left|\langle T h(T) x, y\rangle\right|^2 \leq h_a^2(\|T\|) \langle |T| x, x\rangle \langle |T^*| y, y\rangle$$

for $x, y \in \mathcal{H}$.

The case $h \equiv 1$ provides the original Kato's inequality [27], therefore (8) can be seen as a functional extension of Kato's celebrated result in the case when the function is given by a power series.

(3) If T is invertible and we take $V = I$, $U = T^{-1}$ in (5), then we obtain

$$\left|\langle h(T) x, y\rangle\right|^2 \leq h_a^2(\|T\|) \langle \left||T|^{\alpha} T^{-1}\right|^2 x, x\rangle \langle |T^*|^{2(1-\alpha)} y, y\rangle$$

for $\alpha \in [0,1]$ and $x, y \in \mathcal{H}$. In particular,

$$\left|\langle h(T) x, y\rangle\right|^2 \leq h_a^2(\|T\|) \langle \left||T|^{\frac{1}{2}} T^{-1}\right|^2 x, x\rangle \langle |T^*| y, y\rangle$$

for $x, y \in \mathcal{H}$.

(4) If $T > 0$ and we take $U = T^{-\beta}$, $V = T^{-1+\beta}$, $\beta \in [0,1]$, then we derive

$$\left|\langle h(T) x, y\rangle\right|^2 \leq h_a^2(\|T\|) \langle T^{2(\alpha-\beta)} x, x\rangle \langle T^{2(\beta-\alpha)} y, y\rangle$$

for $\alpha \in [0,1]$ and $x, y \in \mathcal{H}$.

To enhance our understanding of the previous result, we provide helpful examples in the following remark. This will aid in clarifying the concepts and implications presented earlier for some fundamental operator functions.

Remark 2. *(1) If $T, U, V \in \mathcal{B}(\mathcal{H})$ with $\|T\| < 1$, then for $\alpha \in [0,1]$ we have the following inequalities involving the resolvent functions $(I \pm T)^{-1}$*

$$\left|\langle V^* T (I \pm T)^{-1} U x, y\rangle\right|^2 \leq (1 - \|T\|)^{-2} \langle \left||T|^{\alpha} U\right|^2 x, x\rangle \langle \left||T^*|^{1-\alpha} V\right|^2 y, y\rangle \tag{9}$$

and inequalities involving the operator entropy functions $T\ln(I \pm T)$

$$\left|\langle V^*T\ln(I \pm T)Ux, y\rangle\right|^2 \le [\ln(1 - \|T\|)]^2 \langle \left||T|^\alpha U\right|^2 x, x\rangle \langle \left||T^*|^{1-\alpha} V\right|^2 y, y\rangle \tag{10}$$

for all $x, y \in \mathcal{H}$.

Remark 3. *For* $\alpha = \frac{1}{2}$ *in (9) and (10), we obtain*

$$\left|\langle V^*T(I \pm T)^{-1}Ux, y\rangle\right|^2 \le (1 - \|T\|)^{-2} \langle U^*|T|Ux, x\rangle \langle V^*|T^*|Vy, y\rangle$$

and

$$\left|\langle V^*T\ln(I \pm T)Ux, y\rangle\right|^2 \le [\ln(1 - \|T\|)]^2 \langle U^*|T|Ux, x\rangle \langle V^*|T^*|Vy, y\rangle$$

for all $x, y \in \mathcal{H}$.

Remark 4. *(2) If* $T, U, V \in \mathcal{B}(\mathcal{H})$ *and* $\alpha \in [0,1]$, *then we have the following results connecting the operator trigonometric and hyperbolic functions can be stated as well*

$$\left|\langle V^*T\sin(T)Ux, y\rangle\right|^2 \le [\sinh(\|T\|)]^2 \langle \left||T|^\alpha U\right|^2 x, x\rangle \langle \left||T^*|^{1-\alpha} V\right|^2 y, y\rangle \tag{11}$$

and

$$\left|\langle V^*T\cos(T)Ux, y\rangle\right|^2 \le [\cosh(\|T\|)]^2 \langle \left||T|^\alpha U\right|^2 x, x\rangle \langle \left||T^*|^{1-\alpha} V\right|^2 y, y\rangle \tag{12}$$

for all $x, y \in \mathcal{H}$.

Remark 5. *For* $\alpha = \frac{1}{2}$ *in (11) and (12) we obtain*

$$\left|\langle V^*T\sin(T)Ux, y\rangle\right|^2 \le [\sinh(\|T\|)]^2 \langle U^*|T|Ux, x\rangle \langle V^*|T^*|Vy, y\rangle$$

and

$$\left|\langle V^*T\cos(T)Ux, y\rangle\right|^2 \le [\cosh(\|T\|)]^2 \langle U^*|T|Ux, x\rangle \langle V^*|T^*|Vy, y\rangle$$

for all $x, y \in \mathcal{H}$.

Remark 6. *(3) Also, if* $T, U, V \in \mathcal{B}(\mathcal{H})$ *and* $\alpha \in [0,1]$, *then we have the following results involving the operator exponential and the hyperbolic functions*

$$\left|\langle V^*T\exp(T)Ux, y\rangle\right|^2 \le \exp(2\|T\|) \langle \left||T|^\alpha U\right|^2 x, x\rangle \langle \left||T^*|^{1-\alpha} V\right|^2 y, y\rangle,$$

$$\left|\langle V^*T\sinh(T)Ux, y\rangle\right|^2 \le [\sinh(\|T\|)]^2 \langle \left||T|^\alpha U\right|^2 x, x\rangle \langle \left||T^*|^{1-\alpha} V\right|^2 y, y\rangle$$

and

$$\left|\langle V^*T\cosh(T)Ux, y\rangle\right|^2 \le [\cosh(\|T\|)]^2 \langle \left||T|^\alpha U\right|^2 x, x\rangle \langle \left||T^*|^{1-\alpha} V\right|^2 y, y\rangle$$

for all $x, y \in \mathcal{H}$.

Remark 7. *For* $\alpha = \frac{1}{2}$ *in the last three equations, we obtain some simpler inequalities. However, we omit the details.*

Taking into account the above results, it is natural to extend them for finite sequences of operators as follows:

Theorem 1. *Assume that the power series with complex coefficients $h(\lambda) = \sum_{k=0}^{\infty} a_k \lambda^k$ is convergent on $\mathbb{D}(0,R)$ and $T_i, U_i, V_i \in \mathcal{B}(\mathcal{H})$ with $\|T_i\| < R$, $i \in \{1,\ldots,n\}$. Then, for non-negative constants $p_i \geq 0$ with $\sum_{i=1}^{n} p_i > 0$, it holds that*

$$\begin{aligned} &\sum_{i=1}^{n} p_i \left|\left\langle V_i^* T_i h(T_i) U_i x, y\right\rangle\right| \\ &\leq \left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|) \left| |T_i|^{\alpha} U_i \right|^2 x, x \right\rangle^{\frac{1}{2}} \left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|) \left| |T_i^*|^{1-\alpha} V_i \right|^2 y, y \right\rangle^{\frac{1}{2}} \end{aligned} \tag{13}$$

for all $x, y \in \mathcal{H}$ and $\alpha \in [0,1]$.

Proof. By taking the square root in (5), we obtain

$$\left|\left\langle V_i^* T_i h(T_i) U_i x, y\right\rangle\right| \leq \left\langle h_a(\|T_i\|) \left| |T_i|^{\alpha} U_i \right|^2 x, x \right\rangle^{\frac{1}{2}} \left\langle h_a(\|T_i\|) \left| |T_i^*|^{1-\alpha} V_i \right|^2 y, y \right\rangle^{\frac{1}{2}}$$

for all $x, y \in \mathcal{H}$ and $i \in \{1,\ldots,n\}$.

If we multiply by $p_i \geq 0$, $i \in \{1,\ldots,n\}$ and sum over i from 1 to n, then we obtain

$$\begin{aligned} &\sum_{i=1}^{n} p_i \left|\left\langle V_i^* T_i h(T_i) U_i x, y\right\rangle\right| \\ &\leq \sum_{i=1}^{n} p_i \left\langle h_a(\|T_i\|) \left| |T_i|^{\alpha} U_i \right|^2 x, x \right\rangle^{\frac{1}{2}} \left\langle h_a(\|T_i\|) \left| |T_i^*|^{1-\alpha} V_i \right|^2 y, y \right\rangle^{\frac{1}{2}} \end{aligned} \tag{14}$$

for all $x, y \in \mathcal{H}$.

By the Cauchy–Buniakowsky–Schwarz weighted inequality, we derive

$$\begin{aligned} &\sum_{i=1}^{n} p_i \left\langle h_a(\|T_i\|) \left| |T_i|^{\alpha} U_i \right|^2 x, x \right\rangle^{\frac{1}{2}} \left\langle h_a(\|T_i\|) \left| |T_i^*|^{1-\alpha} V_i \right|^2 y, y \right\rangle^{\frac{1}{2}} \\ &\leq \left[\sum_{i=1}^{n} p_i \left(\left\langle h_a(\|T_i\|) \left| |T_i|^{\alpha} U_i \right|^2 x, x \right\rangle^{\frac{1}{2}} \right)^2\right]^{\frac{1}{2}} \left[\sum_{i=1}^{n} p_i \left(\left\langle h_a(\|T_i\|) \left| |T_i^*|^{1-\alpha} V_i \right|^2 y, y \right\rangle^{\frac{1}{2}} \right)^2\right]^{\frac{1}{2}} \\ &= \left[\sum_{i=1}^{n} p_i \left\langle h_a(\|T_i\|) \left| |T_i|^{\alpha} U_i \right|^2 x, x \right\rangle\right]^{\frac{1}{2}} \left[\sum_{i=1}^{n} p_i \left\langle h_a(\|T_i\|) \left| |T_i^*|^{1-\alpha} V_i \right|^2 y, y \right\rangle\right]^{\frac{1}{2}} \\ &= \left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|) \left| |T_i|^{\alpha} U_i \right|^2 x, x \right\rangle^{\frac{1}{2}} \left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|) \left| |T_i^*|^{1-\alpha} V_i \right|^2 y, y \right\rangle^{\frac{1}{2}} \end{aligned} \tag{15}$$

for all $x, y \in \mathcal{H}$.

By making use of (14) and (15), we obtain the desired result (13). □

Remark 8. *We observe that if we take $h \equiv 1$ in (13) then we obtain*

$$\sum_{i=1}^{n} p_i \left|\left\langle V_i^* T_i U_i x, y\right\rangle\right| \leq \left\langle \sum_{i=1}^{n} p_i \left| |T_i|^{\alpha} U_i \right|^2 x, x \right\rangle^{\frac{1}{2}} \left\langle \sum_{i=1}^{n} p_i \left| |T_i^*|^{1-\alpha} V_i \right|^2 y, y \right\rangle^{\frac{1}{2}},$$

which is a generalization of the inequality

$$\sum_{i=1}^{n} p_i \left|\left\langle T_i x, y\right\rangle\right| \leq \left\langle \sum_{i=1}^{n} p_i |T_i|^{2\alpha} x, x \right\rangle^{\frac{1}{2}} \left\langle \sum_{i=1}^{n} p_i |T_i^*|^{2(1-\alpha)} y, y \right\rangle^{\frac{1}{2}},$$

obtained by the second author in ([25], Theorem 2). Therefore, the inequality (13) can be seen as a functional generalization of Dragomir's result [25] in the case when the function is given by a power series.

The next remark summarizes several useful consequences that arise from the above theorem. These consequences serve to further elucidate and expand upon the implications of the theorem.

Remark 9. *(1) It is clear that by the above theorem, we have*

$$\begin{aligned}&\sum_{i=1}^{n} p_i \left| \left\langle V_i^* T_i h(T_i) U_i x, y \right\rangle \right| \\ &\leq \left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|) \left| |T_i|^{\frac{1}{2}} U_i \right|^2 x, x \right\rangle^{\frac{1}{2}} \left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|) \left| |T_i^*|^{\frac{1}{2}} V_i \right|^2 y, y \right\rangle^{\frac{1}{2}}.\end{aligned}$$

Remark 10. *(2) Since $h_a(\cdot)$ is a nondecreasing function on $(0, R)$, then*

$$h_a(\|T_i\|) \leq h_a\left(\max_{k=1,\ldots,n} \|T_k\| \right) = \max_{k=1,\ldots,n} h_a(\|T_k\|),$$

then by (13) we derive for all $\alpha \in [0,1]$ that

$$\begin{aligned}&\sum_{i=1}^{n} p_i \left| \left\langle V_i^* T_i h(T_i) U_i x, y \right\rangle \right| \\ &\leq \max_{k=1,\ldots,n} h_a(\|T_k\|) \left\langle \sum_{i=1}^{n} p_i \left| |T_i|^{\alpha} U_i \right|^2 x, x \right\rangle^{\frac{1}{2}} \left\langle \sum_{i=1}^{n} p_i \left| |T_i^*|^{1-\alpha} V_i \right|^2 y, y \right\rangle^{\frac{1}{2}}\end{aligned}$$

for all $x, y \in \mathcal{H}$.

In particular, we have

$$\begin{aligned}&\sum_{i=1}^{n} p_i \left| \left\langle V_i^* T_i h(T_i) U_i x, y \right\rangle \right| \\ &\leq \max_{k=1,\ldots,n} h_a(\|T_k\|) \left\langle \sum_{i=1}^{n} p_i \left| |T_i|^{\frac{1}{2}} U_i \right|^2 x, x \right\rangle^{\frac{1}{2}} \left\langle \sum_{i=1}^{n} p_i \left| |T_i^*|^{\frac{1}{2}} V_i \right|^2 y, y \right\rangle^{\frac{1}{2}}\end{aligned}$$

for all $x, y \in \mathcal{H}$.

To facilitate a better understanding of our previous Theorem 1, we provide a set of useful examples in the next remark. These examples serve to illustrate and clarify the application and significance of the result discussed earlier.

Remark 11. *(1) If we take $V_i = U_i = I$, then for non-negative constants $p_i \geq 0$ with $\sum_{i=1}^{n} p_i > 0$, it holds from Theorem 1 that*

$$\begin{aligned}\sum_{i=1}^{n} p_i \left| \left\langle T_i h(T_i) x, y \right\rangle \right| &\leq \left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|) |T_i|^{2\alpha} x, x \right\rangle^{\frac{1}{2}} \left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|) |T_i^*|^{2(1-\alpha)} y, y \right\rangle^{\frac{1}{2}} \\ &\leq \max_{k=1,\ldots,n} h_a(\|T_k\|) \left\langle \sum_{i=1}^{n} p_i |T_i|^{2\alpha} x, x \right\rangle^{\frac{1}{2}} \left\langle \sum_{i=1}^{n} p_i |T_i^*|^{2(1-\alpha)} y, y \right\rangle^{\frac{1}{2}} \qquad (16)\end{aligned}$$

for all $x, y \in \mathcal{H}$ and $\alpha \in [0,1]$. In particular,

$$\begin{aligned}\sum_{i=1}^{n} p_i \left| \left\langle T_i h(T_i) x, y \right\rangle \right| &\leq \left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|) |T_i| x, x \right\rangle^{\frac{1}{2}} \left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|) |T_i^*| y, y \right\rangle^{\frac{1}{2}} \\ &\leq \max_{k=1,\ldots,n} h_a(\|T_k\|) \left\langle \sum_{i=1}^{n} p_i |T_i| x, x \right\rangle^{\frac{1}{2}} \left\langle \sum_{i=1}^{n} p_i |T_i^*| y, y \right\rangle^{\frac{1}{2}}\end{aligned}$$

for all $x, y \in \mathcal{H}$.

(2) If $T_i > 0$ and we take $U_i = T^{-\beta}$, $V_i = T^{-1+\beta}$, $\beta \in [0,1]$, then we derive from Theorem 1 that

$$\sum_{i=1}^{n} p_i \left| \left\langle T_i h(T_i) x, y \right\rangle \right| \le \left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|) T_i^{2(\alpha-\beta)} x, x \right\rangle^{\frac{1}{2}} \left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|) T_i^{2(\beta-\alpha)} y, y \right\rangle^{\frac{1}{2}}$$
$$\le \max_{k=1,\ldots,n} h_a(\|T_k\|) \left\langle \sum_{i=1}^{n} p_i T_i^{2(\alpha-\beta)} x, x \right\rangle^{\frac{1}{2}} \left\langle \sum_{i=1}^{n} p_i T_i^{2(\beta-\alpha)} y, y \right\rangle^{\frac{1}{2}}$$

for all $x, y \in \mathcal{H}$ and $\alpha \in [0,1]$.

(3) If we take $h(\lambda) = (1 \pm \lambda)^{-1}$ with $|\lambda| < 1$, then $h_a(\lambda) = (1-\lambda)^{-1}$ and by (16) we obtain for all $x, y \in \mathcal{H}$ and $\alpha \in [0,1]$ that

$$\sum_{i=1}^{n} p_i \left| \left\langle T_i (1 \pm T_i)^{-1} x, y \right\rangle \right|$$
$$\le \left\langle \sum_{i=1}^{n} p_i (1 - \|T_i\|)^{-1} |T_i|^{2\alpha} x, x \right\rangle^{\frac{1}{2}} \left\langle \sum_{i=1}^{n} p_i (1 - \|T_i\|)^{-1} |T_i^*|^{2(1-\alpha)} y, y \right\rangle^{\frac{1}{2}}$$
$$\le \left(1 - \max_{k=1,\ldots,n} \|T_k\| \right)^{-1} \left\langle \sum_{i=1}^{n} p_i |T_i|^{2\alpha} x, x \right\rangle^{\frac{1}{2}} \left\langle \sum_{i=1}^{n} p_i |T_i^*|^{2(1-\alpha)} y, y \right\rangle^{\frac{1}{2}},$$

where $\|T_i\| < 1$, and $p_i \ge 0$ with $\sum_{i=1}^{n} p_i > 0$.

(4) If we take $h(\lambda) = \exp(c\lambda)$ with $c, \lambda \in \mathbb{C}$, then $h_a(\lambda) = \exp(|c|\lambda)$ and by (16) we obtain for all $x, y \in \mathcal{H}$ and $\alpha \in [0,1]$ that

$$\sum_{i=1}^{n} p_i \left| \left\langle T_i \exp(cT_i) x, y \right\rangle \right|$$
$$\le \left\langle \sum_{i=1}^{n} p_i \exp(|c| \|T_i\|) |T_i|^{2\alpha} x, x \right\rangle^{\frac{1}{2}} \left\langle \sum_{i=1}^{n} p_i \exp(|c| \|T_i\|) |T_i^*|^{2(1-\alpha)} y, y \right\rangle^{\frac{1}{2}}$$
$$\le \exp\left(|c| \max_{k=1,\ldots,n} \|T_k\| \right) \left\langle \sum_{i=1}^{n} p_i |T_i|^{2\alpha} x, x \right\rangle^{\frac{1}{2}} \left\langle \sum_{i=1}^{n} p_i |T_i^*|^{2(1-\alpha)} y, y \right\rangle^{\frac{1}{2}},$$

where $T_i \in \mathcal{B}(\mathcal{H})$, $p_i \ge 0$ with $\sum_{i=1}^{n} p_i > 0$.

3. Norm and Numerical Radius Inequalities

In this section, we establish some norm and numerical radius inequalities for sums of power series of operators in Hilbert spaces. Our first result in this direction reads as follows.

Theorem 2. *Assume that the power series with complex coefficients $h(\lambda) = \sum_{k=0}^{\infty} a_k \lambda^k$ is convergent on $\mathbb{D}(0,R)$ and $T_i, U_i, V_i \in \mathcal{B}(\mathcal{H})$ with $\|T_i\| < R$, $i \in \{1,\ldots,n\}$. Then, for $\alpha \in [0,1]$, $p_i \ge 0$ with $\sum_{i=1}^{n} p_i > 0$, it holds that*

$$\left\| \sum_{i=1}^{n} p_i V_i^* T_i h(T_i) U_i \right\|^2 \le \left\| \sum_{i=1}^{n} p_i h_a(\|T_i\|) \left| |T_i|^{\alpha} U_i \right|^2 \right\| \left\| \sum_{i=1}^{n} p_i h_a(\|T_i\|) \left| |T_i^*|^{1-\alpha} V_i \right|^2 \right\|$$
$$\le h_a^2\left(\max_{k=1,\ldots,n} \|T_k\| \right) \left\| \sum_{i=1}^{n} p_i \left| |T_i|^{\alpha} U_i \right|^2 \right\| \left\| \sum_{i=1}^{n} p_i \left| |T_i^*|^{1-\alpha} V_i \right|^2 \right\|.$$

Proof. Because, from the generalized triangle inequality for the modulus, we have

$$\left| \left\langle \sum_{i=1}^{n} p_i V_i^* T_i h(T_i) U_i x, y \right\rangle \right| \le \sum_{i=1}^{n} p_i \left| \left\langle V_i^* T_i h(T_i) U_i x, y \right\rangle \right|,$$

then by (13), we obtain

$$\left|\left\langle \sum_{i=1}^{n} p_i V_i^* T_i h(T_i) U_i x, y \right\rangle\right| \tag{17}$$
$$\leq \left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|) \left||T_i|^{\alpha} U_i\right|^2 x, x \right\rangle^{\frac{1}{2}} \left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|) \left||T_i^*|^{1-\alpha} V_i\right|^2 y, y \right\rangle^{\frac{1}{2}}$$

for all $x, y \in \mathcal{H}$ and $\alpha \in [0,1]$.

By taking the supremum in (17), we obtain

$$\begin{aligned}
&\left\|\sum_{i=1}^{n} p_i V_i^* T_i h(T_i) U_i\right\| \\
&\leq \sup_{\|x\|=\|y\|=1} \left|\left\langle \sum_{i=1}^{n} p_i V_i^* T_i h(T_i) U_i x, y \right\rangle\right| \\
&\leq \sup_{\|x\|=1} \left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|) \left||T_i|^{\alpha} U_i\right|^2 x, x \right\rangle^{\frac{1}{2}} \sup_{\|y\|=1} \left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|) \left||T_i^*|^{1-\alpha} V_i\right|^2 y, y \right\rangle^{\frac{1}{2}} \\
&= \left\|\sum_{i=1}^{n} p_i h_a(\|T_i\|) \left||T_i|^{\alpha} U_i\right|^2\right\|^{\frac{1}{2}} \left\|\sum_{i=1}^{n} p_i h_a(\|T_i\|) \left||T_i^*|^{1-\alpha} V_i\right|^2\right\|^{\frac{1}{2}}
\end{aligned}$$

and thus the desired inequality is proved. □

Remark 12. *If we take Theorem 2 $h \equiv 1$, then we obtain the norm inequality*

$$\left\|\sum_{i=1}^{n} p_i V_i^* T_i U_i\right\|^2 \leq \left\|\sum_{i=1}^{n} p_i \left||T_i|^{\alpha} U_i\right|^2\right\| \left\|\sum_{i=1}^{n} p_i \left||T_i^*|^{1-\alpha} V_i\right|^2\right\|$$

that for the choice $U_i = V_i = I$, $i \in \{1, ..., n\}$, becomes

$$\left\|\sum_{i=1}^{n} p_i T_i\right\|^2 \leq \left\|\sum_{i=1}^{n} p_i |T_i|^{2\alpha}\right\| \left\|\sum_{i=1}^{n} p_i |T_i^*|^{2(1-\alpha)}\right\|,$$

which is a weighted version of an inequality of Kato-type from ([25], Criterion 1).

Moreover, if we only consider the resolvent function $h(z) = (1 \pm z)^{-1}$ for $|z| < 1$, the we obtain from Theorem 2 that

$$\begin{aligned}
&\left\|\sum_{i=1}^{n} p_i V_i^* T_i (I \pm T_i)^{-1} U_i\right\|^2 \\
&\leq \left\|\sum_{i=1}^{n} p_i (1 - \|T_i\|)^{-1} \left||T_i|^{\alpha} U_i\right|^2\right\| \left\|\sum_{i=1}^{n} p_i (1 - \|T_i\|)^{-1} \left||T_i^*|^{1-\alpha} V_i\right|^2\right\| \\
&\leq \left(1 - \max_{k=1,\ldots,n} \|T_k\|\right)^{-2} \left\|\sum_{i=1}^{n} p_i \left||T_i|^{\alpha} U_i\right|^2\right\| \left\|\sum_{i=1}^{n} p_i \left||T_i^*|^{1-\alpha} V_i\right|^2\right\|,
\end{aligned}$$

where $T_i, U_i, V_i \in \mathcal{B}(\mathcal{H})$ with $\|T_i\| < 1$, $i \in \{1, ..., n\}$.

We can state now some results that provide upper bounds for the numerical radius of a weighted sum of operators as follows:

Theorem 3. *Assume that the power series with complex coefficients $h(\lambda) = \sum_{k=0}^{\infty} a_k \lambda^k$ is convergent on $\mathbb{D}(0, R)$ and $T_i, U_i, V_i \in \mathcal{B}(\mathcal{H})$ with $\|T_i\| < R$, $i \in \{1, \ldots, n\}$. Then, for $\alpha \in [0, 1]$, $p_i \geq 0$ with $\sum_{i=1}^{n} p_i > 0$, it holds that:*

$$\omega\left(\sum_{i=1}^{n} p_i V_i^* T_i h(T_i) U_i\right) \leq \left\|\sum_{i=1}^{n} p_i h_a(\|T_i\|) \frac{\left||T_i|^\alpha U_i\right|^2 + \left||T_i^*|^{1-\alpha} V_i\right|^2}{2}\right\| \leq h_a\left(\max_{k=1,\ldots,n} \|T_k\|\right) \left\|\sum_{i=1}^{n} p_i \frac{\left||T_i|^\alpha U_i\right|^2 + \left||T_i^*|^{1-\alpha} V_i\right|^2}{2}\right\|.$$

Proof. From (17) we obtain for $y = x$ that

$$\left|\left\langle \sum_{i=1}^{n} p_i V_i^* T_i h(T_i) U_i x, x \right\rangle\right| \leq \left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|) \left||T_i|^\alpha U_i\right|^2 x, x \right\rangle^{\frac{1}{2}} \left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|) \left||T_i^*|^{1-\alpha} V_i\right|^2 x, x \right\rangle^{\frac{1}{2}} \tag{18}$$

for all $x \in \mathcal{H}$.

By the arithmetic–geometric mean inequality, we also have

$$\begin{aligned} &\left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|) \left||T_i|^\alpha U_i\right|^2 x, x \right\rangle^{\frac{1}{2}} \left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|) \left||T_i^*|^{1-\alpha} V_i\right|^2 x, x \right\rangle^{\frac{1}{2}} \\ &\leq \frac{1}{2}\left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|) \left||T_i|^\alpha U_i\right|^2 x, x \right\rangle + \frac{1}{2}\left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|) \left||T_i^*|^{1-\alpha} V_i\right|^2 x, x \right\rangle \\ &= \left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|) \frac{\left||T_i|^\alpha U_i\right|^2 + \left||T_i^*|^{1-\alpha} V_i\right|^2}{2} x, x \right\rangle \end{aligned} \tag{19}$$

for all $x \in \mathcal{H}$.

Therefore, by (18) and (19), we obtain

$$\begin{aligned} \omega\left(\sum_{i=1}^{n} p_i V_i^* T_i h(T_i) U_i\right) &= \sup_{\|x\|=1} \left|\left\langle \sum_{i=1}^{n} p_i V_i^* T_i h(T_i) U_i x, x \right\rangle\right| \\ &\leq \sup_{\|x\|=1} \left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|) \frac{\left||T_i|^\alpha U_i\right|^2 + \left||T_i^*|^{1-\alpha} V_i\right|^2}{2} x, x \right\rangle \\ &= \left\|\sum_{i=1}^{n} p_i h_a(\|T_i\|) \frac{\left||T_i|^\alpha U_i\right|^2 + \left||T_i^*|^{1-\alpha} V_i\right|^2}{2}\right\|, \end{aligned}$$

which proves the desired result. □

If we take $h \equiv 1$ in Theorem 3 we obtain

$$\omega\left(\sum_{i=1}^{n} p_i V_i^* T_i U_i\right) \leq \left\|\sum_{i=1}^{n} p_i \frac{\left||T_i|^\alpha U_i\right|^2 + \left||T_i^*|^{1-\alpha} V_i\right|^2}{2}\right\|,$$

while for the resolvent function, we obtain

$$\begin{aligned}
&\omega\left(\sum_{i=1}^{n} p_i V_i^* T_i (I \pm T_i)^{-1} U_i\right) \\
&\leq \left\| \sum_{i=1}^{n} p_i (1 - \|T_i\|)^{-1} \frac{\left||T_i|^{\alpha} U_i\right|^2 + \left||T_i^*|^{1-\alpha} V_i\right|^2}{2} \right\| \\
&\leq \left(1 - \max_{k=1,\ldots,n} \|T_k\|\right)^{-1} \left\| \sum_{i=1}^{n} p_i \frac{\left||T_i|^{\alpha} U_i\right|^2 + \left||T_i^*|^{1-\alpha} V_i\right|^2}{2} \right\|.
\end{aligned}$$

where $T_i, U_i, V_i \in \mathcal{B}(\mathcal{H})$ with $\|T_i\| < 1$, $i \in \{1, \ldots, n\}$.

Theorem 4. *Assume that the power series with complex coefficients $h(\lambda) = \sum_{k=0}^{\infty} a_k \lambda^k$ is convergent on $\mathbb{D}(0, R)$ and $T_i, U_i, V_i \in \mathcal{B}(\mathcal{H})$ with $\|T_i\| < R$, $i \in \{1, \ldots, n\}$. Then, for $\alpha \in [0, 1]$, $p_i \geq 0$ with $\sum_{i=1}^{n} p_i > 0$, it holds that*

$$\begin{aligned}
\omega^2\left(\sum_{i=1}^{n} p_i V_i^* T_i h(T_i) U_i\right) \leq{} & \frac{1}{2}\left\| \sum_{i=1}^{n} p_i h_a(\|T_i\|) \left||T_i|^{\alpha} U_i\right|^2 \right\| \left\| \sum_{i=1}^{n} p_i h_a(\|T_i\|) \left||T_i^*|^{1-\alpha} V_i\right|^2 \right\| \\
& + \frac{1}{2}\omega\left(\sum_{i=1}^{n} p_i h_a(\|T_i\|) \left||T_i^*|^{1-\alpha} V_i\right|^2 \sum_{i=1}^{n} p_i h_a(\|T_i\|) \left||T_i|^{\alpha} U_i\right|^2 \right).
\end{aligned}$$

Proof. Recall Buzano's inequality (see [28]), which states that

$$|\langle u, e\rangle\langle e, v\rangle| \leq \frac{1}{2}[\|u\|\|v\| + |\langle u, v\rangle|] \tag{20}$$

holds for any $u, v, e \in \mathcal{H}$ with $\|e\| = 1$.

Let $x \in \mathcal{H}$, $\|x\| = 1$; then, by (18) and (20), we have

$$\begin{aligned}
&\left|\left\langle \sum_{i=1}^{n} p_i V_i^* T_i h(T_i) U_i x, x \right\rangle\right|^2 \qquad (21) \\
&\leq \left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|) \left||T_i|^{\alpha} U_i\right|^2 x, x \right\rangle \left\langle x, \sum_{i=1}^{n} p_i h_a(\|T_i\|) \left||T_i^*|^{1-\alpha} V_i\right|^2 x \right\rangle \\
&\leq \frac{1}{2}\left\| \sum_{i=1}^{n} p_i h_a(\|T_i\|) \left||T_i|^{\alpha} U_i\right|^2 x \right\| \left\| \sum_{i=1}^{n} p_i h_a(\|T_i\|) \left||T_i^*|^{1-\alpha} V_i\right|^2 x \right\| \\
&+ \frac{1}{2}\left|\left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|) \left||T_i|^{\alpha} U_i\right|^2 x, \sum_{i=1}^{n} p_i h_a(\|T_i\|) \left||T_i^*|^{1-\alpha} V_i\right|^2 x \right\rangle\right| \\
&= \frac{1}{2}\left\| \sum_{i=1}^{n} p_i h_a(\|T_i\|) \left||T_i|^{\alpha} U_i\right|^2 x \right\| \left\| \sum_{i=1}^{n} p_i h_a(\|T_i\|) \left||T_i^*|^{1-\alpha} V_i\right|^2 x \right\| \\
&+ \frac{1}{2}\left|\left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|) \left||T_i^*|^{1-\alpha} V_i\right|^2 \sum_{i=1}^{n} p_i h_a(\|T_i\|) \left||T_i|^{\alpha} U_i\right|^2 x, x \right\rangle\right|.
\end{aligned}$$

By taking the supremum over $x \in \mathcal{H}$, $\|x\| = 1$, we obtain the desired result. □

If we take h to be as in Section 2, then we obtain various inequalities for several fundamental functions. We omit the details.

In order to establish our next result, we need to recall Young's inequality, which holds for $a, b \geq 0$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$:

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q. \tag{22}$$

We also require McCarthy's inequality (see [29]), which holds for $r \geq 1$ and a positive operator $P \geq 0$:

$$\langle Px, x\rangle^r \leq \langle P^r x, x\rangle \text{ for } x \in \mathcal{H} \text{ with } \|x\| = 1. \tag{23}$$

Theorem 5. *Assume that the power series with complex coefficients $h(\lambda) = \sum_{k=0}^{\infty} a_k \lambda^k$ is convergent on $\mathbb{D}(0, R)$ and $T_i, U_i, V_i \in \mathcal{B}(\mathcal{H})$ with $\|T_i\| < R$, $i \in \{1, ..., n\}$. Then, for $\alpha \in [0,1]$, $p_i \geq 0$ with $\sum_{i=1}^{n} p_i > 0$, it holds that*

$$\begin{aligned} &\omega^{2r}\left(\sum_{i=1}^{n} p_i V_i^* T_i h(T_i) U_i\right) \\ &\leq \left\| \frac{1}{p}\left(\sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i|^{\alpha} U_i\right|^2\right)^{rp} + \frac{1}{q}\left(\sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i^*|^{1-\alpha} V_i\right|^2\right)^{qr} \right\|, \end{aligned} \tag{24}$$

provided that $r > 0$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $pr, qr \geq 1$.

If $r \geq 1$, then

$$\begin{aligned} &\omega^{2r}\left(\sum_{i=1}^{n} p_i V_i^* T_i h(T_i) U_i\right) \\ &\leq \frac{1}{2}\left\|\sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i|^{\alpha} U_i\right|^2\right\|^r \left\|\sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i^*|^{1-\alpha} V_i\right|^2\right\|^r \\ &+ \frac{1}{2}\omega^r\left(\sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i^*|^{1-\alpha} V_i\right|^2 \sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i|^{\alpha} U_i\right|^2\right). \end{aligned} \tag{25}$$

If $r \geq 1$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $pr, qr \geq 2$, then also

$$\begin{aligned} &\omega^{2r}\left(\sum_{i=1}^{n} p_i V_i^* T_i h(T_i) U_i\right) \\ &\leq \frac{1}{2}\left\| \frac{1}{p}\left(\sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i|^{\alpha} U_i\right|^2\right)^{pr} + \frac{1}{q}\left(\sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i^*|^{1-\alpha} V_i\right|^2\right)^{qr} \right\| \\ &+ \frac{1}{2}\omega^r\left(\sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i^*|^{1-\alpha} V_i\right|^2 \sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i|^{\alpha} U_i\right|^2\right). \end{aligned} \tag{26}$$

Proof. If we raise both sides of Equation (18) to the power of $2r > 0$, we can utilize the inequalities (22) and (23) to deduce the following:

$$\begin{aligned} &\left|\left\langle \sum_{i=1}^{n} p_i V_i^* T_i h(T_i) U_i x, x\right\rangle\right|^{2r} \\ &\leq \left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i|^{\alpha} U_i\right|^2 x, x\right\rangle^r \left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i^*|^{1-\alpha} V_i\right|^2 x, x\right\rangle^r \\ &\leq \frac{1}{p}\left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i|^{\alpha} U_i\right|^2 x, x\right\rangle^{rp} + \frac{1}{q}\left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i^*|^{1-\alpha} V_i\right|^2 x, x\right\rangle^{qr} \\ &\leq \frac{1}{p}\left\langle \left(\sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i|^{\alpha} U_i\right|^2\right)^{rp} x, x\right\rangle + \frac{1}{q}\left\langle \left(\sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i^*|^{1-\alpha} V_i\right|^2\right)^{qr} x, x\right\rangle \\ &= \left\langle \frac{1}{p}\left(\sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i|^{\alpha} U_i\right|^2\right)^{rp} + \frac{1}{q}\left(\sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i^*|^{1-\alpha} V_i\right|^2\right)^{qr} x, x\right\rangle \end{aligned} \tag{27}$$

for $x \in \mathcal{H}$ with $\|x\| = 1$. By taking the supremum over $\|x\| = 1$, then we obtain the desired result (24).

By taking the power $r \geq 1$ in (21) and using the convexity of the power function, we obtain

$$\begin{aligned}
&\left|\left\langle \sum_{i=1}^{n} p_i V_i^* T_i h(T_i) U_i x, x\right\rangle\right|^{2r} \\
&= \left(\frac{1}{2}\left\|\sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i|^{\alpha} U_i\right|^2 x\right\|\left\|\sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i^*|^{1-\alpha} V_i\right|^2 x\right\| \right.\\
&\left. + \frac{1}{2}\left|\left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i^*|^{1-\alpha} V_i\right|^2 \sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i|^{\alpha} U_i\right|^2 x, x\right\rangle\right|\right)^r \\
&\leq \frac{1}{2}\left\|\sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i|^{\alpha} U_i\right|^2 x\right\|^r\left\|\sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i^*|^{1-\alpha} V_i\right|^2 x\right\|^r \\
&+ \frac{1}{2}\left|\left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i^*|^{1-\alpha} V_i\right|^2 \sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i|^{\alpha} U_i\right|^2 x, x\right\rangle\right|^r.
\end{aligned}$$

By taking the supremum over $\|x\| = 1$, then we obtain (25).

Also,

$$\begin{aligned}
&\left\|\sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i|^{\alpha} U_i\right|^2 x\right\|^r\left\|\sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i^*|^{1-\alpha} V_i\right|^2 x\right\|^r \\
&\leq \frac{1}{p}\left\|\sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i|^{\alpha} U_i\right|^2 x\right\|^{pr} + \frac{1}{q}\left\|\sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i^*|^{1-\alpha} V_i\right|^2 x\right\|^{qr} \\
&= \frac{1}{p}\left\|\sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i|^{\alpha} U_i\right|^2 x\right\|^{2\frac{pr}{2}} + \frac{1}{q}\left\|\sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i^*|^{1-\alpha} V_i\right|^2 x\right\|^{2\frac{qr}{2}} \\
&= \frac{1}{p}\left\langle \left(\sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i|^{\alpha} U_i\right|^2\right)^2 x, x\right\rangle^{\frac{pr}{2}} + \frac{1}{q}\left\langle \left(\sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i^*|^{1-\alpha} V_i\right|^2\right)^2 x, x\right\rangle^{\frac{qr}{2}} \\
&\leq \frac{1}{p}\left\langle \left(\sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i|^{\alpha} U_i\right|^2\right)^{pr} x, x\right\rangle + \frac{1}{q}\left\langle \left(\sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i^*|^{1-\alpha} V_i\right|^2\right)^{qr} x, x\right\rangle \\
&= \left\langle \left[\frac{1}{p}\left(\sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i|^{\alpha} U_i\right|^2\right)^{pr} + \frac{1}{q}\left(\sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i^*|^{1-\alpha} V_i\right|^2\right)^{qr}\right] x, x\right\rangle
\end{aligned}$$

for $\|x\| = 1$, then

$$\begin{aligned}
&\frac{1}{2}\left\|\sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i|^{\alpha} U_i\right|^2 x\right\|^r\left\|\sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i^*|^{1-\alpha} V_i\right|^2 x\right\|^r \qquad (28)\\
&+ \frac{1}{2}\left|\left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i^*|^{1-\alpha} V_i\right|^2 \sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i|^{\alpha} U_i\right|^2 x, x\right\rangle\right|^r \\
&\leq \frac{1}{2}\left\langle \left[\frac{1}{p}\left(\sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i|^{\alpha} U_i\right|^2\right)^{pr} + \frac{1}{q}\left(\sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i^*|^{1-\alpha} V_i\right|^2\right)^{qr}\right] x, x\right\rangle \\
&+ \frac{1}{2}\left|\left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i^*|^{1-\alpha} V_i\right|^2 \sum_{i=1}^{n} p_i h_a(\|T_i\|)\left||T_i|^{\alpha} U_i\right|^2 x, x\right\rangle\right|^r
\end{aligned}$$

for $\|x\| = 1$.

By utilizing (27) and (28), we then deduce the desired result (26). □

Finally, we can also state the following result.

Theorem 6. *Assume that the power series with complex coefficients $h(\lambda) = \sum_{k=0}^{\infty} a_k \lambda^k$ is convergent on $\mathbb{D}(0,R)$ and $T_i, U_i, V_i \in \mathcal{B}(\mathcal{H})$ with $\|T_i\| < R$, $i \in \{1,...,n\}$. Then, for $\alpha, \lambda \in [0,1]$, $p_i \geq 0$ with $\sum_{i=1}^{n} p_i > 0$, it holds that*

$$\begin{aligned} &\omega^2\left(\sum_{i=1}^{n} p_i V_i^* T_i h(T_i) U_i\right) \\ &\leq \left\| \sum_{i=1}^{n} p_i h_a(\|T_i\|)\left[(1-\lambda)\big||T_i|^{\alpha} U_i\big|^2 + \lambda\Big||T_i^*|^{1-\alpha} V_i\Big|^2\right]\right\| \\ &\times \left\|\sum_{i=1}^{n} p_i h_a(\|T_i\|)\big||T_i|^{\alpha} U_i\big|^2\right\|^{\lambda} \left\|\sum_{i=1}^{n} p_i h_a(\|T_i\|)\Big||T_i^*|^{1-\alpha} V_i\Big|^2\right\|^{1-\lambda} \end{aligned} \tag{29}$$

and

$$\begin{aligned} &\omega^2\left(\sum_{i=1}^{n} p_i V_i^* T_i h(T_i) U_i\right) \\ &\leq \left\| \sum_{i=1}^{n} p_i h_a(\|T_i\|)\left[(1-\lambda)\big||T_i|^{\alpha} U_i\big|^2 + \lambda\Big||T_i^*|^{1-\alpha} V_i\Big|^2\right]\right\| \\ &\times \left\| \sum_{i=1}^{n} p_i h_a(\|T_i\|)\left[\lambda\big||T_i|^{\alpha} U_i\big|^2 + (1-\lambda)\Big||T_i^*|^{1-\alpha} V_i\Big|^2\right]\right\|. \end{aligned} \tag{30}$$

Proof. From (17), we obtain

$$\begin{aligned} &\left|\left\langle \sum_{i=1}^{n} p_i V_i^* T_i h(T_i) U_i x, x\right\rangle\right|^2 \\ &\leq \left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|)\big||T_i|^{\alpha} U_i\big|^2 x, x\right\rangle \left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|)\Big||T_i^*|^{1-\alpha} V_i\Big|^2 x, x\right\rangle \\ &= \left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|)\big||T_i|^{\alpha} U_i\big|^2 x, x\right\rangle^{1-\lambda} \left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|)\Big||T_i^*|^{1-\alpha} V_i\Big|^2 x, x\right\rangle^{\lambda} \\ &\times \left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|)\big||T_i|^{\alpha} U_i\big|^2 x, x\right\rangle^{\lambda} \left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|)\Big||T_i^*|^{1-\alpha} V_i\Big|^2 x, x\right\rangle^{1-\lambda} \end{aligned}$$

for all $x \in \mathcal{H}$, $\|x\| = 1$.

By the weighted arithmetic-geometric mean inequality, we also have

$$\begin{aligned} &\left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|)\big||T_i|^{\alpha} U_i\big|^2 x, x\right\rangle^{1-\lambda} \left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|)\Big||T_i^*|^{1-\alpha} V_i\Big|^2 x, x\right\rangle^{\lambda} \\ &\leq (1-\lambda)\left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|)\big||T_i|^{\alpha} U_i\big|^2 x, x\right\rangle + \lambda\left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|)\Big||T_i^*|^{1-\alpha} V_i\Big|^2 x, x\right\rangle \\ &= \left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|)\left[(1-\lambda)\big||T_i|^{\alpha} U_i\big|^2 + \lambda\Big||T_i^*|^{1-\alpha} V_i\Big|^2\right] x, x\right\rangle \end{aligned}$$

for all $x \in \mathcal{H}$, $\|x\| = 1$.

Then, we obtain

$$\begin{aligned}&\left|\left\langle \sum_{i=1}^{n} p_i V_i^* T_i h(T_i) U_i x, x \right\rangle\right|^2\\&\leq \left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|) \left[(1-\lambda)\left||T_i|^{\alpha} U_i\right|^2 + \lambda \left||T_i^*|^{1-\alpha} V_i\right|^2 \right] x, x \right\rangle\\&\times \left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|) \left||T_i|^{\alpha} U_i\right|^2 x, x \right\rangle^{\lambda} \left\langle \sum_{i=1}^{n} p_i h_a(\|T_i\|) \left||T_i^*|^{1-\alpha} V_i\right|^2 x, x \right\rangle^{1-\lambda}\end{aligned}$$

for all $x \in \mathcal{H}$, $\|x\| = 1$.

If we take the supremum over $\|x\| = 1$, then we obtain (29).

The inequality (30) follows in a similar way. □

If we take h to be as in Section 2, then we obtain various inequalities for several fundamental functions. We omit the details.

4. Conclusions

In this paper, we have focused on establishing inequalities for the norm and numerical radius of various operators applied to power series of operators in Hilbert spaces. Specifically, we have considered the power series with complex coefficients $h(\lambda) = \sum_{k=0}^{\infty} a_k \lambda^k$ and its modified version $h_a(\lambda) = \sum_{k=0}^{\infty} |a_k| \lambda^k$, assuming the convergence of $h(\lambda)$ on the open disk $\mathbb{D}(0, R)$, where R is the radius of convergence. Additionally, we have explored several operator inequalities associated with these concepts.

The findings of this study significantly contribute to our understanding of operator behavior in bounded operator spaces. By establishing these inequalities, we have gained insights into the relationships between power series of operators and various operator properties.

Moreover, this study serves as a starting point for future investigations in this field. It provides a foundation for exploring other topics, such as Hölder-type inequalities for power series of operators in Hilbert spaces. By extending our work to consider different types of inequalities, we can deepen our understanding of the behavior of power series of operators in Hilbert spaces.

Furthermore, we speculate that our paper, particularly when $n = 1$, can be connected with recent results by Bhunia [30]. This connection suggests potential avenues for further research and opens up the possibility of establishing connections between different lines of inquiry.

In conclusion, this paper contributes to the field of operator theory by establishing norm and numerical radius inequalities for sums of power series of operators in Hilbert spaces. The findings presented here provide valuable insights into operator behavior and lay the groundwork for future research in this area.

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