VICTORIA UNIVERSITY
MELBOURNE AUSTRALIA

Hermite-Hadamard-type inequalities for generalized trigonometrically and hyperbolic ρ-convex functions in two dimension

This is the Published version of the following publication

Dragomir, Sever S, Jleli, Mohamed and Samet, Bessem (2024) Hermite-Hadamard-type inequalities for generalized trigonometrically and hyperbolic ρ-convex functions in two dimension. Open Mathematics, 22 (1). ISSN 2391-5455

The publisher's official version can be found at
https://www.degruyter.com/document/doi/10.1515/math-2024-0028/html
Note that access to this version may require subscription.

Downloaded from VU Research Repository https://vuir.vu.edu.au/49149/

Research Article

Silvestru Sever Dragomir, Mohamed Jleli, and Bessem Samet*

Hermite-Hadamard-type inequalities for generalized trigonometrically and hyperbolic ρ-convex functions in two dimension

https://doi.org/10.1515/math-2024-0028
received February 10, 2024; accepted May 16, 2024

Abstract: In this article, we establish Hermite-Hadamard-type inequalities for the two classes of functions $X_{\pm\lambda}(\Omega) = \{f \in C^2(\Omega) : \Delta f \pm \lambda f \geq 0\}$, where $\lambda > 0$ and Ω is an open subset of $\mathbb{R}^2$. We also obtain a characterization of the set $X_{-\lambda}(\Omega)$. Notice that in the one-dimensional case, if $\Omega = I$ (an open interval of $\mathbb{R}$) and $\lambda = \rho^2$, $\rho > 0$, then $X_\lambda(\Omega)$ (resp. $X_{-\lambda}(\Omega)$) reduces to the class of functions $f \in C^2(I)$ such that f is trigonometrically ρ-convex (resp. hyperbolic ρ-convex) on I.

Keywords: Hermite-Hadamard-type inequalities, trigonometrically ρ-convex functions, hyperbolic ρ-convex functions, Bessel functions

MSC 2020: 26B25, 26A51, 26D15, 35A23

1 Introduction

Convex functions are widely used in various branches of mathematics, such as functional analysis, optimization theory, and numerical analysis. One of the important inequalities related to convex functions is the (right-side) Hermite-Hadamard inequality [1,2] that can be stated as follows: Let $\sigma_1 < \sigma_2$ and $f : [\sigma_1, \sigma_2] \to \mathbb{R}$ be a convex function. Then

$$\frac{1}{\sigma_2 - \sigma_1}\int_{\sigma_1}^{\sigma_2} f(x)\mathrm{d}x \leq \frac{\sum_{j=1}^{2} f(\sigma_j)}{2}. \tag{1.1}$$

Inequality (1.1) has been refined and generalized in various directions. For more details, we refer to the monograph [3] collecting many results in that direction.

In [4], among other results, the first author extended inequality (1.1) to the class of convex functions on a disk. Namely, he proved that, if $f : \overline{B(C,R)} \to \mathbb{R}$ is convex, then

$$\frac{1}{\pi R^2}\int_{B(C,R)} f(z)\mathrm{d}z \leq \frac{1}{2\pi R}\int_{\partial B(C,R)} f(y)\mathrm{d}l(y), \tag{1.2}$$

*** Corresponding author: Bessem Samet,** Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh, 11451, Saudi Arabia, e-mail: bsamet@ksu.edu.sa

Silvestru Sever Dragomir: Mathematics College of Engineering & Science, Victoria University, P.O. Box 14428, Melbourne City, VIC 8001, Australia; School of Computer Science & Applied Mathematics, University of the Witwatersrand Private Bag 3, Johannesburg 2050, South Africa, e-mail: sever.dragomir@vu.edu.au

Mohamed Jleli: Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh, 11451, Saudi Arabia, e-mail: jleli@ksu.edu.sa

Open Access. © 2024 the author(s), published by De Gruyter. This work is licensed under the Creative Commons Attribution 4.0 International License.

where $B(C,R)$ is the open ball of $\mathbb{R}^2$ with center $C \in \mathbb{R}^2$ and radius $R > 0$, $\partial B(C,R)$ is the boundary of $B(C,R)$ and $\overline{B(C,R)}$ is the closure of $B(C,R)$. For further results related to Hermite-Hadamard-type inequalities in higher dimensions, see [5–11] and the references therein.

Let $\rho > 0$ and $f : I \to \mathbb{R}$, where I is an interval of $\mathbb{R}$. We say that f is a trigonometrically ρ-convex function on I [12], if for all $a, b \in I$ with $0 < b - a < \frac{\pi}{\rho}$, we have

$$f(x) \le \frac{\sin[\rho(b-x)]}{\sin[\rho(b-a)]} f(a) + \frac{\sin[\rho(x-a)]}{\sin[\rho(b-a)]} f(b), \quad a \le x \le b.$$

In [13], the first author proved (among many other results) that, if $f : I \to \mathbb{R}$ is a trigonometrically ρ-convex function on I, then for all $a, b \in I$ with $0 < b - a < \frac{\pi}{\rho}$, we have

$$\int_a^b f(x)\mathrm{d}x \le \frac{f(a) + f(b)}{\rho} \tan\left[\frac{\rho(b-a)}{2}\right].$$

For other results related to trigonometrically ρ-convex functions, see [14–17] and the references therein.

Let $\rho > 0$ and $f : I \to \mathbb{R}$. Assume that for all $a, b \in I$ with $a < b$, we have

$$f(x) \le \frac{\sinh[\rho(b-x)]}{\sinh[\rho(b-a)]} f(a) + \frac{\sinh[\rho(x-a)]}{\sinh[\rho(b-a)]} f(b), \quad a \le x \le b.$$

In this case, the function f is said to be a hyperbolic ρ-convex function on I [18]. The first author in [19] established that if $f : I \to \mathbb{R}$ is a hyperbolic ρ-convex function on I, then for all $a, b \in I$ with $a < b$, we have

$$\int_a^b f(x)\mathrm{d}x \le \frac{f(a) + f(b)}{\rho} \tanh\left[\frac{\rho(b-a)}{2}\right].$$

We also refer to [20,21] for further results related to hyperbolic ρ-convex functions.

On the other hand, it was shown in [12] that $f \in C^2(I)$ is trigonometrically ρ-convex on I if and only if f satisfies the second-order differential inequality:

$$f'' + \rho^2 f \ge 0 \quad \text{in } I. \tag{1.3}$$

In [18], it was proven that $f \in C^2(I)$ is hyperbolic ρ-convex on I if and only if f satisfies the second-order differential inequality:

$$f'' - \rho^2 f \ge 0 \quad \text{in } I. \tag{1.4}$$

In this article, motivated by the aforementioned cited works, we establish Hermite-Hadamard-type inequalities for the two classes of functions

$$X_{\pm\lambda}(\Omega) = \{f \in C^2(\Omega) : \Delta f \pm \lambda f \ge 0\},$$

where $\lambda > 0$, Ω is an open subset of $\mathbb{R}^2$, and Δ denotes, as usual, the Laplacian operator. Moreover, we prove a characterization of the set $X_{-\lambda}(\Omega)$. Remark that in the one-dimensional case, by (1.3), if $\Omega = I$ and $\lambda = \rho^2$, $\rho > 0$, then $X_\lambda(\Omega)$ reduces to the class of functions $f \in C^2(I)$ such that f is trigonometrically ρ-convex on I. Similarly, in the one-dimensional case, by (1.4), if $\Omega = I$ and $\lambda = \rho^2$, $\rho > 0$, then $X_{-\lambda}(\Omega)$ reduces to the class of functions $f \in C^2(I)$, such that f is hyperbolic ρ-convex on I.

The structure of the rest of the article is as follows. In Section 2, we study some properties of the sets $X_{\pm\lambda}(\Omega)$. In Section 3, we derive Hermite-Hadamard-type inequalities for both classes of functions $X_{\pm\lambda}(\Omega)$. Finally, in Section 5, we establish a characterization of the set $X_{-\lambda}(\Omega)$.

2 The classes of functions $X_{\pm\lambda}(\Omega)$

Throughout this article, by Ω, we mean an open subset of $\mathbb{R}^2$. For $\lambda > 0$, let

$$X_{\pm\lambda}(\Omega) = \{f \in C^2(\Omega) : \Delta f \pm \lambda f \geq 0\}.$$

We also consider the class of functions

$$X_0(\Omega) = \{f \in C^2(\Omega) : \Delta f \geq 0\}.$$

In this section, we provide some properties of the above classes of functions.

Proposition 2.1. *Let $\lambda > 0$ and $0 \leq \theta \leq 1$. The following properties hold:*

(*i*) $\bigcap_{\mu>0} X_{+\mu}(\Omega) \subset X_0(\Omega)$.

(*ii*) $f \in X_0(\Omega), f \geq 0 \;\Rightarrow\; f \in \bigcap_{\mu>0} X_{+\mu}(\Omega)$.

(*iii*) $f \in X_{+\lambda}(\Omega), f \geq 0 \;\Rightarrow\; f \in \bigcap_{\mu\geq\lambda} X_{+\mu}(\Omega)$.

(*iv*) $f \in X_{+\lambda}(\Omega), \alpha \geq 0\ (\alpha \in \mathbb{R}) \;\Rightarrow\; \alpha f \in X_{+\lambda}(\Omega)$.

(*v*) $f, g \in X_{+\lambda}(\Omega) \;\Rightarrow\; f + g \in X_{+\lambda}(\Omega)$.

(*vi*) $f \in X_{\theta\lambda}(\Omega), g \in X_{(1-\theta)\lambda}(\Omega), f \geq 0, g \geq 0, \nabla f \cdot \nabla g \geq 0 \;\Rightarrow\; fg \in X_{+\lambda}(\Omega)$, *where ∇ is the gradient operator and $\cdot$ is the inner product in $\mathbb{R}^2$.*

Proof. We only prove (i) and (vi). The other properties are immediate, so we omit the details.

(i) Let $f \in \bigcap_{\mu>0} X_{+\mu}(\Omega)$. If $z \in \Omega$ is fixed, then for all $\mu > 0$, we have

$$\Delta f(z) + \mu f(z) \geq 0.$$

Passing to the limit as $\mu \to 0^+$, we obtain $\Delta f(z) \geq 0$, which shows that $f \in X_0(\Omega)$.

(vi) Let $f \in X_{\theta\lambda}(\Omega)$ and $g \in X_{(1-\theta)\lambda}(\Omega)$ be such that

$$f \geq 0, g \geq 0, \nabla f \cdot \nabla g \geq 0.$$

By the property

$$\Delta(fg) = f\Delta g + g\Delta f + 2\nabla f \cdot \nabla g,$$

we obtain

$$\Delta(fg) + \lambda fg = \Delta(fg) + \lambda\theta fg + (1-\theta)\lambda fg = f(\Delta g + (1-\theta)\lambda g) + g(\Delta f + \lambda\theta f) + 2\nabla f \cdot \nabla g \geq 0,$$

which shows that $fg \in X_{+\lambda}(\Omega)$. □

Similarly, we have the following properties.

Proposition 2.2. *Let $\lambda > 0$ and $0 \leq \theta \leq 1$. The following properties hold:*

(*i*) $\bigcap_{\mu>0} X_{-\mu}(\Omega) \subset X_0(\Omega)$.

(*ii*) $f \in X_0(\Omega), f \leq 0 \;\Rightarrow\; f \in \bigcap_{\mu>0} X_{-\mu}(\Omega)$.

(*iii*) $f \in X_{-\lambda}(\Omega), f \geq 0 \;\Rightarrow\; f \in \bigcap_{0\leq\mu\leq\lambda} X_{+\mu}(\Omega)$.

(*iv*) $f \in X_{-\lambda}(\Omega), \alpha \geq 0\ (\alpha \in \mathbb{R}) \;\Rightarrow\; \alpha f \in X_{-\lambda}(\Omega)$.

(*v*) $f, g \in X_{-\lambda}(\Omega) \;\Rightarrow\; f + g \in X_{-\lambda}(\Omega)$.

(*vi*) $f \in X_{-\theta\lambda}(\Omega), g \in X_{-(1-\theta)\lambda}(\Omega), f \geq 0, g \geq 0, \nabla f \cdot \nabla g \geq 0 \;\Rightarrow\; fg \in X_{-\lambda}(\Omega)$.

3 Hermite-Hadamard-type inequalities on a disk

In this section, we are concerned with the extension of inequality (1.2) to the classes of functions $X_{\pm\lambda}(\Omega)$. We need to recall some properties related to Bessel functions. For more details about such functions, we refer to [22].

3.1 Some preliminaries

For all $n \in \mathbb{N}$ (n is a natural number), let J_n be the Bessel function of the first kind of order n. This function can be written as an infinite power series as follows:

$$J_n(x) = \sum_{\iota=0}^{\infty} \frac{(-1)^\iota}{\iota!(n+\iota)!}\left(\frac{x}{2}\right)^{n+2\iota}.$$

Let $j_{n,1}$ be the first positive zero of J_n.

Lemma 3.1. *Let $n \in \mathbb{N}$. We have the following properties:*

(*i*) *The function J_n satisfies the Bessel differential equation:*

$$x^2 J_n''(x) + xJ_n'(x) + (x^2 - n^2)J_n(x) = 0.$$

(*ii*) $J_n'(x) = -J_{n+1}(x) + \frac{n}{x}J_n(x)$.

(*iii*) $J_n(x) \sim \frac{1}{n!}\left(\frac{x}{2}\right)^n$ *as* $x \to 0$.

(*iv*) *The function J_0 is decreasing in the interval* $[0, j_{0,1}]$.

Similarly, for all $n \in \mathbb{N}$, let I_n be the modified Bessel function of the first kind of order n. This function can be written as an infinite power series as follows:

$$I_n(x) = \left(\frac{x}{2}\right)^n \sum_{\iota=0}^{\infty} \frac{1}{\iota!(n+\iota)!}\left(\frac{x^2}{4}\right)^{\iota}.$$

Lemma 3.2. *Let $n \in \mathbb{N}$. We have the following properties:*

(*i*) *The function I_n satisfies the modified Bessel differential equation:*

$$x^2 I_n''(x) + xI_n'(x) - (x^2 + n^2)I_n(x) = 0.$$

(*ii*) $I_0'(x) = I_1(x)$.

(*iii*) $I_n(x) \sim \frac{1}{n!}\left(\frac{x}{2}\right)^n$ *as* $x \to 0$.

3.2 The class of functions $X_{+\lambda}(\Omega)$

The following result is an extension of inequality (1.2) to the class of functions $X_{+\lambda}(\Omega)$.

Theorem 3.3. *Let $\lambda > 0$ and $f \in X_{+\lambda}(\Omega)$. Assume that there exist $C \in \Omega$ and $R \in \left[0, \frac{j_{0,1}}{\sqrt{\lambda}}\right]$ such that $\overline{B(C,R)} \subset \Omega$. Then, the following inequality holds:*

$$\int_{B(C,R)} f(z)\mathrm{d}z \le \frac{J_1(\sqrt{\lambda}R)}{\sqrt{\lambda}J_0(\sqrt{\lambda}R)} \int_{\partial B(C,R)} f(y)\mathrm{d}l(y). \tag{3.1}$$

Proof. Let $C \in \Omega$ and $R \in \left[0, \frac{j_{0,1}}{\sqrt{\lambda}}\right]$ be such that $\overline{B(C,R)} \subset \Omega$. We introduce the function

$$\xi(z) = \frac{1}{\lambda}\left(\frac{J_0(\sqrt{\lambda}\|z-C\|)}{J_0(\sqrt{\lambda}R)} - 1\right), \quad z \in \overline{B(C,R)}. \tag{3.2}$$

We first establish some properties of the above function. Since $0 < R \leq \frac{j_{0,1}}{\sqrt{\lambda}}$, then $J_0(\sqrt{\lambda}R) \neq 0$, and the function ξ is well defined. Taking $\|z-C\| = R$ in (3.2), we obtain

$$\xi(z) = 0, \quad z \in \partial B(C,R). \tag{3.3}$$

On the other hand, from the property (iv) in Lemma 3.1, we deduce that for all $z \in \overline{B(C,R)}$,

$$J_0(\sqrt{\lambda}\|z-C\|) \leq J_0(\sqrt{\lambda}R),$$

which shows that

$$\xi(z) \geq 0, \quad z \in \overline{B(C,R)}. \tag{3.4}$$

For all $0 \leq r \leq R$, let

$$\tilde{\xi}(r) = \frac{1}{\lambda}\left(\frac{J_0(\sqrt{\lambda}r)}{J_0(\sqrt{\lambda}R)} - 1\right).$$

Since ξ is a radial function, that is,

$$\xi(z) = \tilde{\xi}(r), r = \|z-C\|,$$

then for all $z \in B(C,R)$, we have

$$\Delta\xi(z) = \tilde{\zeta}''(r) + \frac{1}{r}\tilde{\xi}'(r). \tag{3.5}$$

Furthermore, we have

$$\tilde{\xi}'(r) = \frac{1}{\sqrt{\lambda}}\frac{J_0'(\sqrt{\lambda}r)}{J_0(\sqrt{\lambda}R)} \tag{3.6}$$

and

$$\tilde{\xi}''(r) = \frac{J_0''(\sqrt{\lambda}r)}{J_0(\sqrt{\lambda}R)},$$

which imply by (3.5) that

$$\Delta\xi(z) = \frac{J_0''(\sqrt{\lambda}r)}{J_0(\sqrt{\lambda}R)} + \frac{1}{\sqrt{\lambda}r}\frac{J_0'(\sqrt{\lambda}r)}{J_0(\sqrt{\lambda}R)} = \frac{1}{J_0(\sqrt{\lambda}R)(\sqrt{\lambda}r)^2}((\sqrt{\lambda}r)^2 J_0''(\sqrt{\lambda}r) + \sqrt{\lambda}rJ_0'(\sqrt{\lambda}r)).$$

Then, from the property (i) in Lemma 3.1, we deduce that

$$\begin{aligned}\Delta\xi(z) &= \frac{1}{J_0(\sqrt{\lambda}R)(\sqrt{\lambda}r)^2}(-(\sqrt{\lambda}r)^2 J_0(\sqrt{\lambda}r)) \\ &= -\frac{J_0(\sqrt{\lambda}r)}{J_0(\sqrt{\lambda}R)} \\ &= -1 - \left(\frac{J_0(\sqrt{\lambda}r)}{J_0(\sqrt{\lambda}R)} - 1\right) \\ &= -1 - \lambda\tilde{\xi}(r) \\ &= -1 - \lambda\xi(z),\end{aligned}$$

that is,

$$\Delta\xi(z) + \lambda\xi(z) = -1, \quad z \in B(C,R). \tag{3.7}$$

We now use (3.7) to obtain

$$\int_{B(C,R)} f(z)\mathrm{d}z = -\int_{B(C,R)} \Delta\xi(z) f(z)\mathrm{d}z - \lambda \int_{B(C,R)} f(z)\xi(z)\mathrm{d}z. \tag{3.8}$$

Moreover, thanks to Green's formula, we have

$$-\int_{B(C,R)} \Delta\xi(z) f(z)\mathrm{d}z = \int_{B(C,R)} \nabla f(z)\cdot\nabla\xi(z)\mathrm{d}z - \tilde{\xi}'(R) \int_{\partial B(C,R)} f(\gamma)\mathrm{d}l(\gamma). \tag{3.9}$$

By using again Green's formula together with (3.3), we obtain

$$-\int_{B(C,R)} \Delta f(z)\xi(z)\mathrm{d}z = \int_{B(C,R)} \nabla f(z)\cdot\nabla\xi(z)\mathrm{d}z. \tag{3.10}$$

Then, it follows from (3.8), (3.9), and (3.10) that

$$\int_{B(C,R)} f(z)\mathrm{d}z = -\tilde{\xi}'(R) \int_{\partial B(C,R)} f(\gamma)\mathrm{d}l(\gamma) - \int_{B(C,R)} (\Delta f(z) + \lambda f(z))\xi(z)\mathrm{d}z.$$

We also have by (3.6) and the property (ii) of Lemma 3.1 that

$$\tilde{\xi}'(R) = \frac{-1}{\sqrt{\lambda}} \frac{J_1(\sqrt{\lambda}R)}{J_0(\sqrt{\lambda}R)}.$$

Therefore, it holds that

$$\int_{B(C,R)} f(z)\mathrm{d}z = \frac{1}{\sqrt{\lambda}} \frac{J_1(\sqrt{\lambda}R)}{J_0(\sqrt{\lambda}R)} \int_{\partial B(C,R)} f(\gamma)\mathrm{d}l(\gamma) - \int_{B(C,R)} (\Delta f(z) + \lambda f(z))\xi(z)\mathrm{d}z. \tag{3.11}$$

On the other hand, due to (3.4) and since $f \in X_{+\lambda}(\Omega)$, we have

$$\int_{B(C,R)} (\Delta f(z) + \lambda f(z))\xi(z)\mathrm{d}z \geq 0. \tag{3.12}$$

Finally, (3.1) follows from (3.11) and (3.12). □

Remark 3.4. By the property (iii) in Lemma 3.1, one has

$$\lim_{\lambda \to 0^+} \frac{J_1(\sqrt{\lambda}R)}{\sqrt{\lambda} J_0(\sqrt{\lambda}R)} = \frac{R}{2}.$$

Hence, if $\lambda \to 0^+$, (3.1) reduces to (1.2).

3.3 The class of functions $X_{-\lambda}(\Omega)$

The following result is an extension of inequality (1.2) to the class of functions $X_{-\lambda}(\Omega)$

Theorem 3.5. *Let $\lambda > 0$ and $f \in X_{-\lambda}(\Omega)$. Then, for all $C \in \Omega$ and $R > 0$ with $\overline{B(C,R)} \subset \Omega$, we have*

$$\int_{B(C,R)} f(z)\mathrm{d}z \leq \frac{I_1(\sqrt{\lambda}R)}{\sqrt{\lambda} I_0(\sqrt{\lambda}R)} \int_{\partial B(C,R)} f(\gamma)\mathrm{d}l(\gamma). \tag{3.13}$$

Proof. Let $C \in \Omega$ and $R > 0$ be such that $\overline{B(C,R)} \subset \Omega$. We consider the function

$$\zeta(z) = \frac{1}{\lambda}\left(1 - \frac{I_0(\sqrt{\lambda}\|z - C\|)}{I_0(\sqrt{\lambda}R)}\right), \quad z \in \overline{B(C,R)}. \tag{3.14}$$

Taking $\|z - C\| = R$ in (3.14), we obtain

$$\zeta(z) = 0, \quad z \in \partial B(C,R). \tag{3.15}$$

On the other hand, from the property (ii) in Lemma 3.2, the function I_0 is nondecreasing in the interval $[0, +\infty]$. Hence, for all $z \in \overline{B(C,R)}$, we have

$$I_0(\sqrt{\lambda}\|z - C\|) \le I_0(\sqrt{\lambda}R),$$

which shows that

$$\zeta(z) \ge 0, \quad z \in \overline{B(C,R)}. \tag{3.16}$$

For all $0 \le r \le R$, let

$$\tilde{\zeta}(r) = \frac{1}{\lambda}\left(1 - \frac{I_0(\sqrt{\lambda}r)}{I_0(\sqrt{\lambda}R)}\right).$$

Since ζ is a radial function, that is,

$$\zeta(z) = \tilde{\zeta}(r), \quad r = \|z - C\|,$$

then for all $z \in B(C,R)$, we have

$$\Delta\zeta(z) = \tilde{\zeta}''(r) + \frac{1}{r}\tilde{\zeta}'(r). \tag{3.17}$$

Furthermore, we have

$$\tilde{\zeta}'(r) = -\frac{1}{\sqrt{\lambda}}\frac{I_0'(\sqrt{\lambda}r)}{I_0(\sqrt{\lambda}R)} \tag{3.18}$$

and

$$\tilde{\zeta}''(r) = -\frac{I_0''(\sqrt{\lambda}r)}{I_0(\sqrt{\lambda}R)},$$

which imply by (3.17) that

$$\begin{aligned}\Delta\xi(z) &= -\frac{I_0''(\sqrt{\lambda}r)}{I_0(\sqrt{\lambda}R)} - \frac{1}{\sqrt{\lambda}r}\frac{I_0'(\sqrt{\lambda}r)}{I_0(\sqrt{\lambda}R)}\\ &= -\frac{1}{I_0(\sqrt{\lambda}R)(\sqrt{\lambda}r)^2}((\sqrt{\lambda}r)^2 I_0''(\sqrt{\lambda}r) + \sqrt{\lambda}rI_0'(\sqrt{\lambda}r)).\end{aligned}$$

Then, from the property (i) in Lemma 3.2, we deduce that

$$\begin{aligned}\Delta\zeta(z) &= -\frac{1}{J_0(\sqrt{\lambda}R)(\sqrt{\lambda}r)^2}((\sqrt{\lambda}r)^2 I_0(\sqrt{\lambda}r))\\ &= -\frac{I_0(\sqrt{\lambda}r)}{I_0(\sqrt{\lambda}R)}\\ &= -1 + \left(1 - \frac{I_0(\sqrt{\lambda}r)}{I_0(\sqrt{\lambda}R)}\right)\\ &= -1 + \lambda\tilde{\zeta}(r)\\ &= -1 + \lambda\zeta(z),\end{aligned}$$

that is,

$$\Delta\zeta(z) - \lambda\zeta(z) = -1, \quad z \in B(C,R). \tag{3.19}$$

Then, in view of (3.19), we have

$$\int_{B(C,R)} f(z)\mathrm{d}z = -\int_{B(C,R)} \Delta\zeta(z)f(z)\mathrm{d}z + \lambda\int_{B(C,R)} f(z)\zeta(z)\mathrm{d}z. \tag{3.20}$$

On the other hand, by Green's formula and (3.15), we have

$$-\int_{B(C,R)} \Delta\zeta(z)f(z)\mathrm{d}z = -\int_{B(C,R)} \Delta f(z)\zeta(z)\mathrm{d}z - \tilde{\zeta}'(R)\int_{\partial B(C,R)} f(y)\mathrm{d}l(y). \tag{3.21}$$

Combining (3.20) with (3.21), we obtain

$$\int_{B(C,R)} f(z)\mathrm{d}z = -\tilde{\zeta}'(R)\int_{\partial B(C,R)} f(y)\mathrm{d}l(y) - \int_{B(C,R)} (\Delta f(z) - \lambda f(z))\zeta(z)\mathrm{d}z. \tag{3.22}$$

Making use of (3.18) and the property (ii) in Lemma 3.2, we obtain

$$\tilde{\zeta}'(R) = -\frac{1}{\sqrt{\lambda}}\frac{I_1(\sqrt{\lambda}R)}{I_0(\sqrt{\lambda}R)},$$

which implies together with (3.22) that

$$\int_{B(C,R)} f(z)\mathrm{d}z = \frac{I_1(\sqrt{\lambda}R)}{\sqrt{\lambda}I_0(\sqrt{\lambda}R)}\int_{\partial B(C,R)} f(y)\mathrm{d}l(y) - \int_{B(C,R)} (\Delta f(z) - \lambda f(z))\zeta(z)\mathrm{d}z.$$

Finally, since $f \in X_{-\lambda}(\Omega)$ and $\zeta \geq 0$ by (3.16), the aforementioned inequality yields (3.13). □

Remark 3.6. By the property (iii) in Lemma 3.2, one has

$$\lim_{\lambda\to 0^+} \frac{I_1(\sqrt{\lambda}R)}{\sqrt{\lambda}I_0(\sqrt{\lambda}R)} = \frac{R}{2}.$$

Hence, if $\lambda \to 0^+$, (3.13) reduces to (1.2).

Let Ω_i, $i = 1, 2$, be two open subsets of $\mathbb{R}^2$. For $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$ with $\lambda_i > 0$, $i = 1, 2$, we consider the class of functions

$$X_{-(\lambda_1,\lambda_2)}(\Omega_1 \times \Omega_2) = \{f = f(x,y) \in C^2(\Omega_1 \times \Omega_2) : \Delta_x f - \lambda_1 f \geq 0, \Delta_y f - \lambda_2 f \geq 0\},$$

where for all $x = (x_1, x_2) \in \Omega_1$ and $y = (y_1, y_2) \in \Omega_2$,

$$\Delta_x f(x,y) = \frac{\partial^2 f}{\partial x_1^2}(x,y) + \frac{\partial^2 f}{\partial x_2^2}(x,y)$$

and

$$\Delta_y f(x,y) = \frac{\partial^2 f}{\partial y_1^2}(x,y) + \frac{\partial^2 f}{\partial y_2^2}(x,y).$$

Remark that $f = f(x,y) \in X_{-(\lambda_1,\lambda_2)}(\Omega_1 \times \Omega_2)$ means that for all $(x,y) \in \Omega_1 \times \Omega_2$,

$$f(\cdot, y) \in X_{-\lambda_1}(\Omega_1) \quad \text{and} \quad f(x,\cdot) \in X_{-\lambda_2}(\Omega_2),$$

where

$$f(\cdot,y) : \Omega_1 \ni x \mapsto f(x,y) \in \mathbb{R} \quad \text{and} \quad f(x,\cdot) : \Omega_2 \ni y \mapsto f(x,y) \in \mathbb{R}.$$

From Theorem 3.5, we deduce the following Hermite-Hadamard-type inequalities for the class of functions $X_{-(\lambda_1,\lambda_2)}(\Omega_1 \times \Omega_2)$.

Corollary 3.7. *Let Ω_i, $i = 1, 2$, be two open subsets of $\mathbb{R}^2$ and $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$ with $\lambda_i > 0$, $i = 1, 2$. If $f \in X_{-(\lambda_1,\lambda_2)}(\Omega_1 \times \Omega_2)$, then for all $(C_1, C_2) \in \Omega_1 \times \Omega_2$ and $R_1, R_2 > 0$ with $\overline{B(C_i, R_i)} \subset \Omega_i$, $i = 1, 2$, we have*

$$\begin{aligned}\int_{B(C_1,R_1)\times B(C_2,R_2)} f(x,y)\,\mathrm{d}x\mathrm{d}y &\le \frac{I_1(\sqrt{\lambda_1}R_1)}{2\sqrt{\lambda_1}I_0(\sqrt{\lambda_1}R_1)} \int_{\partial B(C_1,R_1)} \left(\int_{B(C_2,R_2)} f(\gamma,y)\,\mathrm{d}y \right) \mathrm{d}l(\gamma) \\ &\quad + \frac{I_1(\sqrt{\lambda_2}R_2)}{2\sqrt{\lambda_2}I_0(\sqrt{\lambda_2}R_2)} \int_{\partial B(C_2,R_2)} \left(\int_{B(C_1,R_1)} f(x,\sigma)\,\mathrm{d}x \right) \mathrm{d}l(\sigma) \\ &\le \frac{I_1(\sqrt{\lambda_1}R_1)I_1(\sqrt{\lambda_2}R_2)}{\sqrt{\lambda_1\lambda_2}\,I_0(\sqrt{\lambda_1}R_1)I_0(\sqrt{\lambda_2}R_2)} \int_{\partial B(C_1,R_1)\times\partial B(C_2,R_2)} f(\gamma,\sigma)\,\mathrm{d}l(\gamma)\mathrm{d}l(\sigma).\end{aligned} \tag{3.23}$$

Proof. Let $R_i > 0$ be such that $\overline{B(C_i, R_i)} \subset \Omega_i$, $i = 1, 2$. Let $y \in \Omega_2$ be fixed. Since $f(\cdot, y) \in X_{-\lambda_1}(\Omega_1)$, then by Theorem 3.5 we have

$$\int_{B(C_1,R_1)} f(x,y)\,\mathrm{d}x \le \frac{I_1(\sqrt{\lambda_1}R_1)}{\sqrt{\lambda_1}I_0(\sqrt{\lambda_1}R_1)} \int_{\partial B(C_1,R_1)} f(\gamma,y)\,\mathrm{d}l(\gamma).$$

Integrating the aforementioned inequality over $y \in B(C_2, R_2)$, we obtain

$$\int_{B(C_1,R_1)\times B(C_2,R_2)} f(x,y)\,\mathrm{d}x\mathrm{d}y \le \frac{I_1(\sqrt{\lambda_1}R_1)}{\sqrt{\lambda_1}I_0(\sqrt{\lambda_1}R_1)} \int_{\partial B(C_1,R_1)} \left(\int_{B(C_2,R_2)} f(\gamma,y)\,\mathrm{d}y \right) \mathrm{d}l(\gamma). \tag{3.24}$$

Similarly, for a fixed $x \in \Omega_1$, since $f(x, \cdot) \in X_{-\lambda_2}(\Omega_2)$, then by Theorem 3.5, we have

$$\int_{B(C_2,R_2)} f(x,y)\,\mathrm{d}y \le \frac{I_1(\sqrt{\lambda_2}R_2)}{\sqrt{\lambda_2}I_0(\sqrt{\lambda_2}R_2)} \int_{\partial B(C_2,R_2)} f(x,\sigma)\,\mathrm{d}l(\sigma).$$

Integrating the aforementioned inequality over $x \in B(C_1, R_1)$, we obtain

$$\int_{B(C_1,R_1)\times B(C_2,R_2)} f(x,y)\,\mathrm{d}x\mathrm{d}y \le \frac{I_1(\sqrt{\lambda_2}R_2)}{\sqrt{\lambda_2}I_0(\sqrt{\lambda_2}R_2)} \int_{\partial B(C_2,R_2)} \left(\int_{B(C_1,R_1)} f(x,\sigma)\,\mathrm{d}x \right) \mathrm{d}l(\sigma). \tag{3.25}$$

By summing (3.24) and (3.25), we obtain

$$\begin{aligned}\int_{B(C_1,R_1)\times B(C_2,R_2)} f(x,y)\,\mathrm{d}x\mathrm{d}y &\le \frac{I_1(\sqrt{\lambda_1}R_1)}{2\sqrt{\lambda_1}I_0(\sqrt{\lambda_1}R_1)} \int_{\partial B(C_1,R_1)} \left(\int_{B(C_2,R_2)} f(\gamma,y)\,\mathrm{d}y \right) \mathrm{d}l(\gamma) \\ &\quad + \frac{I_1(\sqrt{\lambda_2}R_2)}{2\sqrt{\lambda_2}I_0(\sqrt{\lambda_2}R_2)} \int_{\partial B(C_2,R_2)} \left(\int_{B(C_1,R_1)} f(x,\sigma)\,\mathrm{d}x \right) \mathrm{d}l(\sigma),\end{aligned}$$

which proves the first inequality in (3.23). On the other hand, since for all $\gamma \in \partial B(C_1, R_1)$, the function $f(\gamma, \cdot) \in X_{-\lambda_2}(\Omega_2)$, then by Theorem 3.5, we have

$$\int_{B(C_2,R_2)} f(\gamma,y)\,\mathrm{d}y \le \frac{I_1(\sqrt{\lambda_2}R_2)}{\sqrt{\lambda_2}I_0(\sqrt{\lambda_2}R_2)} \int_{\partial B(C_2,R_2)} f(\gamma,\sigma)\,\mathrm{d}l(\sigma).$$

Integrating the aforementioned inequality over $y \in \partial B(C_1, R_1)$, we obtain

$$\int_{\partial B(C_1,R_1)} \left(\int_{B(C_2,R_2)} f(y,y)\mathrm{d}y \right) \mathrm{d}l(y) \leq \frac{I_1(\sqrt{\lambda}_2 R_2)}{\sqrt{\lambda}_2 I_0(\sqrt{\lambda}_2 R_2)} \int_{\partial B(C_1,R_1)\times \partial B(C_2,R_2)} f(y,\sigma)\mathrm{d}l(y)\mathrm{d}l(\sigma). \tag{3.26}$$

Similarly, since for all $\sigma \in \partial B(C_2, R_2)$, the function $f(\cdot, \sigma) \in X_{-\lambda_1}(\Omega_1)$, then by Theorem 3.5, we have

$$\int_{B(C_1,R_1)} f(x,\sigma)\mathrm{d}x \leq \frac{I_1(\sqrt{\lambda}_1 R_1)}{\sqrt{\lambda}_1 I_0(\sqrt{\lambda}_1 R_1)} \int_{\partial B(C_1,R_1)} f(y,\sigma)\mathrm{d}l(y),$$

which implies after integration over $\sigma \in \partial B(C_2, R_2)$ that

$$\int_{\partial B(C_2,R_2)} \left(\int_{B(C_1,R_1)} f(x,\sigma)\mathrm{d}x \right) \mathrm{d}l(\sigma) \leq \frac{I_1(\sqrt{\lambda}_1 R_1)}{\sqrt{\lambda}_1 I_0(\sqrt{\lambda}_1 R_1)} \int_{\partial B(C_1,R_1)\times \partial B(C_2,R_2)} f(y,\sigma)\mathrm{d}l(y)\mathrm{d}l(\sigma). \tag{3.27}$$

Then, it follows from (3.26) and (3.27) that

$$\begin{aligned} &\frac{I_1(\sqrt{\lambda}_1 R_1)}{2\sqrt{\lambda}_1 I_0(\sqrt{\lambda}_1 R_1)} \int_{\partial B(C_1,R_1)} \left(\int_{B(C_2,R_2)} f(y,y)\mathrm{d}y \right) \mathrm{d}l(y) + \frac{I_1(\sqrt{\lambda}_2 R_2)}{2\sqrt{\lambda}_2 I_0(\sqrt{\lambda}_2 R_2)} \int_{\partial B(C_2,R_2)} \left(\int_{B(C_1,R_1)} f(x,\sigma)\mathrm{d}x \right) \mathrm{d}l(\sigma) \\ &\leq \frac{I_1(\sqrt{\lambda}_1 R_1) I_1(\sqrt{\lambda}_2 R_2)}{\sqrt{\lambda_1\lambda_2}\, I_0(\sqrt{\lambda}_1 R_1) I_0(\sqrt{\lambda}_2 R_2)} \int_{\partial B(C_1,R_1)\times \partial B(C_2,R_2)} f(y,\sigma)\mathrm{d}l(y)\mathrm{d}l(\sigma), \end{aligned}$$

which proves the second inequality in (3.23). □

4 Characterization of the class of functions $X_{-\lambda}(\Omega)$

In this section, we show that inequality (3.13) provides a characterization of the class of functions $X_{-\lambda}(\Omega)$. Namely, we have the following result.

Theorem 4.1. *Let $f \in C^2(\Omega)$ and $\lambda > 0$. Assume that for all $C \in \Omega$ and $R > 0$ with $\overline{B(C,R)} \subset \Omega$, (3.13) holds. Then $f \in X_{-\lambda}(\Omega)$.*

Proof. Let $C \in \Omega$ and $\varepsilon > 0$ small enough so that $\overline{B(C,\varepsilon)} \subset \Omega$. Let

$$\zeta_\varepsilon(z) = \widetilde{\zeta_\varepsilon}(r) = \frac{1}{\lambda}\left(1 - \frac{I_0(\sqrt{\lambda}\, r)}{I_0(\sqrt{\lambda}\, \varepsilon)}\right), \quad z \in \overline{B(C,\varepsilon)},\ r = \|z - C\|.$$

From the proof of Theorem 3.5, we have

$$\zeta_\varepsilon(z) = 0, \quad z \in \partial B(C,\varepsilon), \tag{4.1}$$

$$\zeta_\varepsilon(z) \geq 0, \quad z \in \overline{B(C,\varepsilon)}, \tag{4.2}$$

$$\Delta\zeta_\varepsilon(z) - \lambda\zeta_\varepsilon(z) = -1, \quad z \in B(C,\varepsilon), \tag{4.3}$$

$$\widetilde{\zeta_\varepsilon}'(\varepsilon) = -\frac{1}{\sqrt{\lambda}} \frac{I_1(\sqrt{\lambda}\,\varepsilon)}{I_0(\sqrt{\lambda}\,\varepsilon)}. \tag{4.4}$$

Making use of (5.1), (5.3), (5.4), and using two times Green's formula, we obtain

$$\begin{aligned}\int_{B(C,\varepsilon)} f(z)\mathrm{d}z &= -\int_{B(C,\varepsilon)} \Delta\zeta_\varepsilon(z)f(z)\mathrm{d}z + \lambda\int_{B(C,\varepsilon)} f(z)\zeta_\varepsilon(z)\mathrm{d}z\\ &= -\int_{B(C,\varepsilon)} (\Delta f(z) - \lambda f(z))\zeta_\varepsilon(z)\mathrm{d}z - \widetilde{\zeta_\varepsilon}'(\varepsilon)\int_{\partial B(C,\varepsilon)} f(y)\mathrm{d}l(y)\\ &= -\int_{B(C,\varepsilon)} (\Delta f(z) - \lambda f(z))\zeta_\varepsilon(z)\mathrm{d}z + \frac{I_1(\sqrt{\lambda}\varepsilon)}{\sqrt{\lambda}I_0(\sqrt{\lambda}\varepsilon)}\int_{\partial B(C,\varepsilon)} f(y)\mathrm{d}l(y),\end{aligned}$$

that is,

$$\int_{B(C,\varepsilon)} f(z)\mathrm{d}z - \frac{I_1(\sqrt{\lambda}\varepsilon)}{\sqrt{\lambda}I_0(\sqrt{\lambda}\varepsilon)}\int_{\partial B(C,\varepsilon)} f(y)\mathrm{d}l(y) = -\int_{B(C,\varepsilon)} (\Delta f(z) - \lambda f(z))\zeta_\varepsilon(z)\mathrm{d}z.$$

Then, from (3.13) with $R = \varepsilon$, we deduce that

$$-\int_{B(C,\varepsilon)} (\Delta f(z) - \lambda f(z))\zeta_\varepsilon(z)\mathrm{d}z \leq 0.$$

Since $\zeta_\varepsilon \geq 0$ by (5.2), we deduce the existence of $z_\varepsilon \in B(C,\varepsilon)$ so that

$$\Delta f(z_\varepsilon) - \lambda f(z_\varepsilon) \geq 0.$$

Furthermore, since $f \in C^2(\Omega)$, passing to the limit as $\varepsilon \to 0^+$ in the aforementioned inequality, we obtain

$$\Delta f(C) - \lambda f(C) \geq 0.$$

Since C is an arbitrary point of Ω, it holds that $\Delta f - \lambda f \geq 0$ in Ω, which shows that $f \in X_{-\lambda}(\Omega)$. □

5 Characterization of the class of functions $X_{-\lambda}(\Omega)$

In this section, we show that inequality (3.13) provides a characterization of the class of functions $X_{-\lambda}(\Omega)$. Namely, we have the following result.

Theorem 5.1. *Let $f \in C^2(\Omega)$ and $\lambda > 0$. Assume that for all $C \in \Omega$ and $R > 0$ with $\overline{B(C,R)} \subset \Omega$,* (3.13) *holds. Then $f \in X_{-\lambda}(\Omega)$.*

Proof. Let $C \in \Omega$ and $\varepsilon > 0$ small enough so that $\overline{B(C,\varepsilon)} \subset \Omega$. Let

$$\zeta_\varepsilon(z) = \widetilde{\zeta_\varepsilon}(r) = \frac{1}{\lambda}\left(1 - \frac{I_0(\sqrt{\lambda}r)}{I_0(\sqrt{\lambda}\varepsilon)}\right), \quad z \in \overline{B(C,\varepsilon)}, r = \|z - C\|.$$

From the proof of Theorem 3.5, we have

$$\zeta_\varepsilon(z) = 0, \quad z \in \partial B(C,\varepsilon), \tag{5.1}$$

$$\zeta_\varepsilon(z) \geq 0, \quad z \in \overline{B(C,\varepsilon)}, \tag{5.2}$$

$$\Delta\zeta_\varepsilon(z) - \lambda\zeta_\varepsilon(z) = -1, \quad z \in B(C,\varepsilon), \tag{5.3}$$

$$\widetilde{\zeta_\varepsilon}'(\varepsilon) = -\frac{1}{\sqrt{\lambda}}\frac{I_1(\sqrt{\lambda}\varepsilon)}{I_0(\sqrt{\lambda}\varepsilon)}. \tag{5.4}$$

Making use of (5.1), (5.3), (5.4), and using two times Green's formula, we obtain

$$\begin{aligned}\int\limits_{B(C,\varepsilon)} f(z)\mathrm{d}z &= -\int\limits_{B(C,\varepsilon)} \Delta\zeta_\varepsilon(z)f(z)\mathrm{d}z + \lambda\int\limits_{B(C,\varepsilon)} f(z)\zeta_\varepsilon(z)\mathrm{d}z\\ &= -\int\limits_{B(C,\varepsilon)} (\Delta f(z) - \lambda f(z))\zeta_\varepsilon(z)\mathrm{d}z - \widetilde{\zeta_\varepsilon}'(\varepsilon)\int\limits_{\partial B(C,\varepsilon)} f(y)\mathrm{d}l(y)\\ &= -\int\limits_{B(C,\varepsilon)} (\Delta f(z) - \lambda f(z))\zeta_\varepsilon(z)\mathrm{d}z + \frac{I_1(\sqrt{\lambda}\varepsilon)}{\sqrt{\lambda}I_0(\sqrt{\lambda}\varepsilon)}\int\limits_{\partial B(C,\varepsilon)} f(y)\mathrm{d}l(y),\end{aligned}$$

that is,

$$\int\limits_{B(C,\varepsilon)} f(z)\mathrm{d}z - \frac{I_1(\sqrt{\lambda}\varepsilon)}{\sqrt{\lambda}I_0(\sqrt{\lambda}\varepsilon)}\int\limits_{\partial B(C,\varepsilon)} f(y)\mathrm{d}l(y) = -\int\limits_{B(C,\varepsilon)} (\Delta f(z) - \lambda f(z))\zeta_\varepsilon(z)\mathrm{d}z.$$

Then, from (3.13) with $R = \varepsilon$, we deduce that

$$-\int\limits_{B(C,\varepsilon)} (\Delta f(z) - \lambda f(z))\zeta_\varepsilon(z)\mathrm{d}z \le 0.$$

Since $\zeta_\varepsilon \ge 0$ by (5.2), we deduce the existence of $z_\varepsilon \in B(C,\varepsilon)$ so that

$$\Delta f(z_\varepsilon) - \lambda f(z_\varepsilon) \ge 0.$$

Furthermore, since $f \in C^2(\Omega)$, passing to the limit as $\varepsilon \to 0^+$ in the above inequality, we obtain

$$\Delta f(C) - \lambda f(C) \ge 0.$$

Since C is an arbitrary point of Ω, it holds that $\Delta f - \lambda f \ge 0$ in Ω, which shows that $f \in X_{-\lambda}(\Omega)$. □

From Theorems 3.5 and 5.1, we deduce the following characterization.

Corollary 5.2. *Let* $f \in C^2(\Omega)$ *and* $\lambda > 0$. *The following statements are equivalent*:
(*i*) $f \in X_{-\lambda}(\Omega)$.
(*ii*) *For all* $C \in \Omega$ *and* $R > 0$ *with* $\overline{B(C,R)} \subset \Omega$, (3.13) *holds*.

6 Conclusion

We considered the two classes of functions $X_{\pm\lambda}(\Omega)$, where Ω is an open subset of $\mathbb{R}^2$ and $\lambda > 0$. We extended inequality (1.2) to functions $f \in X_{\pm\lambda}(\Omega)$ (Theorems 3.3 and 3.5). We also obtained a characterization of the class of functions $X_{-\lambda}(\Omega)$ (Corollary 5.2). In this article, only the two-dimensional case is considered. It will be interesting to extend the obtained results to the N-dimensional case (i.e., Ω is an open subset of $\mathbb{R}^N$). It will be also interesting to extend the obtained results to the more general classes of functions

$$X_{\pm\lambda,\alpha}(\Omega) = \{f \in C^2(\Omega) : \operatorname{div}(\alpha\nabla f) \pm \lambda f \ge 0\},$$

where $\alpha \in C^1(\Omega)$. Here, div is the divergence operator. Notice that, if $\alpha \equiv 1$, then $X_{\pm\lambda,\alpha}(\Omega) = X_{\pm\lambda}(\Omega)$.

Acknowledgement: The third author was supported by Researchers Supporting Project number (RSP2024R4), King Saud University, Riyadh, Saudi Arabia.

Author contributions: All authors contributed equally to this work and have read and agreed to the published version of the manuscript.

Conflict of interest: The authors state no conflicts of interest. Bessem Samet was a Guest Editor of the Open Mathematics journal and was not involved in the review and decision-making process of this article.

Data availability statement: Data sharing is not applicable to this article as no datasets were generated or analyzed during this study.

References

[1] J. Hadamard, *Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann*, J. Math. Pures Appl. **58** (1893), 171–215.
[2] C. Hermite, *Sur deux limites daune intégrale défine*, Mathesis **3** (1883), 1–82.
[3] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, Melbourne, 2000.
[4] S. S. Dragomir, *On Hadamard's inequality on a disk*, J. Inequal. Pure Appl. Math. **1** (2000), no. 1, 2.
[5] F. Chen, *A note on the Hermite-Hadamard inequality for convex functions on the co-ordinates*, J. Math. Inequal. **8** (2014), 915–923.
[6] M. Feng, J. Ruan, and X. Ma, *Hermite-Hadamard type inequalities for multidimensional strongly h-convex functions*, Math. Inequal. Appl. **24** (2021), 897–911.
[7] M. Jleli and B. Samet, *On Hermite-Hadamard-type inequalities for subharmonic functions over circular ring domains*, Numer. Funct. Anal. Optim. **44** (2023), 1395–1408.
[8] M. Matloka, *On Hadamard's inequality for h-convex function on a disk*, Appl. Math. Comput. **235** (2014), 118–123.
[9] Z. Pavić, *The Jensen and Hermite-Hadamard inequality on the triangle*, J. Math. Inequal. **11** (2017), 1099–1112.
[10] M. Z. Sarikaya, *Some inequalities for differentiable co-ordinated convex mappings*, Asian-Eur. J. Math. **8** (2015), 1550058.
[11] X. Wang, J. Ruan, and X. Ma, *On the Hermite-Hadamard inequalities for h-convex functions on balls and ellipsoids*, Filomat **33** (2019), 5817–5886.
[12] M. S. S. Ali, *On certain properties of trigonometrically ρ-convex functions*, Adv. Pure Math. **2** (2012), 337–340.
[13] S. S. Dragomir, *Some inequalities of Hermite-Hadamard type for trigonometrically ρ-convex functions*, Preprint RGMIA Res. Rep. Coll. **21** (2018), 9.
[14] H. Budak, *Refinements of Hermite-Hadamard inequality for trigonometrically ρ-convex functions*, Math. Morav. **23** (2019), 87–96.
[15] H. Budak, H. Kara, and M. E. Kiri, *On Hermite-Hadamard type inequalities for co-ordinated trigonometrically ρ-convex functions*, Tbilisi Math. J. **13** (2020), 1–26.
[16] H. Budak, H. Kara, and M. E. Kiris, *On Hermite-Hadamard-Fejér-type inequalities for co-ordinated trigonometrically ρ-convex functions*, Asian-Eur. J. Math. **16** (2023), 2350043.
[17] S. S. Dragomir, *Some inequalities of Fejér type for trigonometrically ρ-convex functions*, Preprint RGMIA Res. Rep. Coll. **21** (2018), 10.
[18] M. S. S. Ali, *On certain properties for two classes of generalized convex functions*, Abstr. Appl. Anal. **2016** (2016), 4652038.
[19] S. S. Dragomir, *Some inequalities of Hermite-Hadamard type for hyperbolic p-convex functions*, Preprint RGMIA Res. Rep. Coll. **21** (2018), 13.
[20] S. S. Dragomir, *Some inequalities of Ostrowski and trapezoid type for hyperbolic ρ-convex functions*, Preprint RGMIA Res. Rep. Coll. **21** (2018), 15.
[21] H. Kara, H. Budak, and M. E. Kiris, *On Fejér type inequalities for co-ordinated hyperbolic p-onvex functions*, AIMS Math. **5** (2020), 4681–4701.
[22] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions: With Formulas, Graphs, and Mathematical Tables*, Dover Publications, New York, NY, USA, 1992.