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Improved Bounds for the Euclidean Numerical Radius of Operator Pairs in Hilbert Spaces

Najla Altwaijry [1,†], **Silvestru Sever Dragomir** [2,†] and **Kais Feki** [3,*,†]

[1] Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia; najla@ksu.edu.sa
[2] Applied Mathematics Research Group, ISILC, Victoria University, P.O. Box 14428, Melbourne, VIC 8001, Australia; sever.dragomir@ajmaa.org or sever.dragomir@vu.edu.au
[3] Laboratory Physics-Mathematics and Applications (LR/13/ES-22), Faculty of Sciences of Sfax, University of Sfax, Sfax 3018, Tunisia
* Correspondence: kais.feki@hotmail.com
† These authors contributed equally to this work.

Abstract: This paper presents new lower and upper bounds for the Euclidean numerical radius of operator pairs in Hilbert spaces, demonstrating improvements over recent results by other authors. Additionally, we derive new inequalities for the numerical radius and the Davis–Wielandt radius as natural consequences of our findings.

Keywords: Euclidean numerical radius; operator pairs; Hilbert spaces; Davis–Wielandt radius; numerical radius inequalities

MSC: 47A30; 46C05; 47A63; 47A99

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1. Introduction

Let H be a complex Hilbert space equipped with an inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|$. The C^*-algebra of bounded linear operators on H is denoted by $\mathcal{B}(H)$. For any operator $A\in\mathcal{B}(H)$, the adjoint of A is denoted by A^*, and $|A|=(A^*A)^{\frac{1}{2}}$ represents the positive square root of A^*A. The numerical range of A, denoted by $W(A)$, is the set of values $\{\langle Ax,x\rangle : x\in H, \|x\|=1\}$.

The operator norm and numerical radius of A are denoted by $\|A\|$ and $w(A)$, respectively. The operator norm is defined as

$$\|A\|=\sup\{|\langle Ax,y\rangle| : x,y\in H, \|x\|=\|y\|=1\},$$

while the numerical radius is given by

$$w(A)=\sup\{|\langle Ax,x\rangle| : x\in H, \|x\|=1\}.$$

It is known that the numerical radius $w(\cdot)$ defines a norm on $\mathcal{B}(H)$ that is equivalent to the operator norm $\|\cdot\|$. In particular, the following double inequality is valid:

$$\frac{1}{2}\|A\|\le w(A)\le\|A\| \tag{1}$$

for any operator $A\in\mathcal{B}(H)$. These inequalities are sharp. The first inequality becomes an equality if $A^2=0$, while the second inequality becomes an equality if and only if A is a normal operator. An improvement to these inequalities was established by Kittaneh [1], who proved that

$$\frac{1}{4}\||A|^2+|A^*|^2\|\le w^2(A)\le\frac{1}{2}\||A|^2+|A^*|^2\|. \tag{2}$$

For further advancements related to inequalities (1) and (2), interested readers can refer to sources [2–8] and the references therein.

Let $(C,D) \in \mathcal{B}(H)^2 := \mathcal{B}(H) \times \mathcal{B}(H)$. The Euclidean operator radius is defined by

$$w_e(C,D) := \sup_{\substack{x \in H \\ \|x\|=1}} \left(|\langle Cx, x\rangle|^2 + |\langle Dx, x\rangle|^2 \right)^{1/2}.$$

As pointed out in [9], $w_e : \mathcal{B}(H)^2 \to [0,\infty)$ is a norm and the following inequality holds:

$$\frac{\sqrt{2}}{4} \left\| |C|^2 + |D|^2 \right\|^{1/2} \leq w_e(C,D) \leq \left\| |C|^2 + |D|^2 \right\|^{1/2} \tag{3}$$

for $(C,D) \in \mathcal{B}(H)^2$, where the constants $\frac{\sqrt{2}}{4}$ and 1 are best possible in (3).

In [10], the second author obtained the following lower bound:

$$\frac{\sqrt{2}}{2} \left[w\left(B^2 + C^2\right) \right]^{\frac{1}{2}} \leq w_e(B,C).$$

The constant $\frac{\sqrt{2}}{2}$ is best possible in the sense that it cannot be replaced by a larger constant. In the same paper, the following results were obtained as well:

$$\frac{\sqrt{2}}{2} \max\{w(B+C), w(B-C)\} \leq w_e(B,C) \leq \frac{\sqrt{2}}{2} \left[w^2(B+C) + w^2(B-C) \right]^{\frac{1}{2}},$$

with constant $\frac{\sqrt{2}}{2}$ sharp in both inequalities,

$$w_e^2(B,C) \leq \max\left\{ \|B\|^2, \|C\|^2 \right\} + w(C^*B),$$

with the inequality being sharp, and also

$$w_e^2(B,C) \leq \frac{1}{2}\left[\left\| |B|^2 + |C|^2 \right\| + \left\| |B|^2 - |C|^2 \right\| \right] + w(C^*B),$$

which is sharp as well.

By taking $(B,C) = (A, A^*)$ or $(B,C) = (\Re(A), \Im(A))$ for $A \in B(H)$, where we recall that

$$\Re(A) := \frac{A + A^*}{2} \text{ and } \Im(A) := \frac{A - A^*}{2i},$$

the second author obtained in [10] several norm and numerical radius inequalities of interest for one operator A. Note that $\Re(A)$ and $\Im(A)$ refer to the real and imaginary parts of A, respectively.

For some recent results involving the estimation of the Euclidean numerical radius in terms of similar upper and lower bounds, see [11–13]. These estimations appear to be simpler to calculate when the involved operators are self-adjoint, since in that case, for instance, $|B|^2 = B^2$, $|C|^2 = C^2$, and $C^*B = CB$. Moreover, if B and C are projections orthogonal to each other, then $B^2 = B$, $C^2 = C$, and $CB = 0$, which make the bounds even simpler to calculate.

The Davis–Wielandt radius of an operator $T \in \mathcal{B}(H)$, denoted by $dw(T)$ is defined as follows [14,15]:

$$dw(T) = \sup_{\substack{x \in H \\ \|x\|=1}} \left(|\langle Tx, x\rangle|^2 + \|Tx\|^4 \right)^{1/2}.$$

It is evident that $dw(T) \geq 0$, and $dw(T) = 0$ if and only if $T = 0$. For any $\lambda \in \mathbb{C}$, the following inequalities hold: $dw(\lambda T) > |\lambda| dw(T)$ if $|\lambda| > 1$, $dw(\lambda T) < |\lambda| dw(T)$ if $|\lambda| < 1$, and $dw(\lambda T) = |\lambda| dw(T)$ when $|\lambda| = 1$. Note that the triangle inequality

$dw(T+S) \le dw(T) + dw(S)$ does not always hold for arbitrary operators $T, S \in \mathcal{B}(H)$. However, this inequality holds when $\Re(T^*S) = 0$, as shown in ([16], Corollary 2.2). It is also straightforward to verify that

$$\max\left\{w(T), \|T\|^2\right\} \le dw(T) \le \left(w^2(T) + \|T\|^4\right)^{1/2},$$

and these inequalities are sharp, as noted in ([16], Corollary 2.2).

Additionally, observe that for $C = T$ and $D = |T|^2$, we have

$$\begin{aligned} w_e(T, |T|^2) &:= \sup_{\|x\|=1} \left(|\langle Tx, x\rangle|^2 + |\langle |T|^2 x, x\rangle|^2\right)^{1/2} \\ &= \sup_{\|x\|=1} \left(|\langle Tx, x\rangle|^2 + \|Tx\|^4\right)^{1/2} = dw(T). \end{aligned}$$

If we set $C = T$ and $D = |T|^2$ in (3), then we obtain

$$\frac{1}{2}\left\||T|^2 + |T|^4\right\| \le dw^2(T) \le \left\||T|^2 + |T|^4\right\|, \tag{4}$$

which provides the upper bound from (5) and a corresponding lower bound.

Zamani and Shebrawi ([17], Theorem 2.1) proved that

$$dw(T) \le \left[w^2(T - |T|^2) + 2\|T\|^2 w(T)\right]^{1/2}.$$

Furthermore, in ([17], Theorems 2.13, 2.14, and 2.17), they also established the following inequalities:

$$dw^2(T) \le \|T\|^2 \max\left\{1, \|T\|^2\right\} + \sqrt{2}w\left(|T|^2 T\right),$$

$$dw^2(T) \le \frac{1}{2}\left(\left\||T|^4 + |T|^2\right\| + \left\||T|^4 - |T|^2\right\| + \sqrt{2}w\left(|T|^2 T\right)\right),$$

and

$$dw^2(T) \le \|T\| \max\left\{w(T), \|T\|^2\right\}\left[1 + \|T\|^2 + 2w(T)\right]^{1/2}$$

for any operator $T \in \mathcal{B}(H)$.

Recently, Bhunia et al. in ([16], Theorem 2.4) obtained the following upper bound:

$$dw(T) \le \left\||T|^4 + |T|^2\right\|^{1/2} \tag{5}$$

for $T \in \mathcal{B}(H)$.

Additionally, in [18], the authors derived inequalities for the sum of operators. Specifically, we have

$$\begin{aligned} dw(T+S) &\le \left[2\left(dw^2(T) + dw^2(S)\right) + 6\left\||T|^4 + |S|^4\right\|\right]^{1/2} \\ &\le 2\sqrt{2}\left[dw^2(T) + dw^2(S)\right]^{1/2} \end{aligned}$$

for $T, S \in \mathcal{B}(H)$.

Motivated by these results, we present new lower and upper bounds for the Euclidean numerical radius of operator pairs in this paper, demonstrating that some of these bounds improve upon those recently established by other authors.

2. Lower Bounds for the Euclidean Numerical Radius of Operator Pairs

In this section, we derive and establish several lower bounds for the Euclidean numerical radius of pairs of operators. Specifically, we present and prove the following lower bounds for the Euclidean numerical radius.

Theorem 1. *For any $B, C \in \mathcal{B}(H)$, we have*

$$w_e^2(B,C) \geq \frac{1}{2}\max\left\{w\left(B^2+C^2\right), w(BC+CB)\right\} + \frac{1}{2}\max\{w(B), w(C)\}|w(B+C)-w(B-C)| \tag{6}$$

and

$$w_e^2(B,C) \geq \frac{1}{2}\max\left\{w\left(B^2+C^2\right), w\left(B^2-C^2\right)\right\} + \frac{1}{2}\max\{w(B+C), w(B-C)\}|w(B)-w(C)|. \tag{7}$$

Proof. As in the proof of Theorem 1 of [10] (see also [12], Theorem 2.2), we have

$$\begin{aligned} w_e^2(B,C) &= \sup_{\|x\|=1}\left[|\langle Bx,x\rangle|^2 + |\langle Cx,x\rangle|^2\right] \\ &\geq \frac{1}{2}\sup_{\|x\|=1}\left(|\langle Bx,x\rangle| + |\langle Cx,x\rangle|\right)^2 \\ &\geq \frac{1}{2}\sup_{\|x\|=1}|\langle Bx,x\rangle \pm \langle Cx,x\rangle|^2 \\ &= \frac{1}{2}\sup_{\|x\|=1}|\langle (B\pm C)x,x\rangle|^2 = \frac{1}{2}w^2(B\pm C). \end{aligned}$$

This implies that

$$\begin{aligned} w_e^2(B,C) &\geq \frac{1}{2}\max\left\{w^2(B+C), w^2(B-C)\right\} \\ &\geq \frac{1}{2}\max\left\{w\left((B+C)^2\right), w\left((B-C)^2\right)\right\} \\ &= \frac{1}{4}\left[w\left((B+C)^2\right) + w\left((B-C)^2\right)\right] \\ &\quad + \frac{1}{4}\left|w\left((B+C)^2\right) - w\left((B-C)^2\right)\right| \\ &= \frac{1}{4}\left[w\left((B+C)^2\right) + w\left((B-C)^2\right)\right] \\ &\quad + \frac{1}{4}[w(B+C)+w(B-C)]|w(B+C)-w(B-C)| \end{aligned} \tag{8}$$

By the triangle inequality, we have

$$w\left((B+C)^2\right) + w\left((B-C)^2\right) \geq w\left[(B+C)^2 \pm (B-C)^2\right],$$

which gives

$$w\left((B+C)^2\right) + w\left((B-C)^2\right) \geq 2\max\left\{w\left(B^2+C^2\right), w(BC+CB)\right\}.$$

Also,

$$w(B+C) + w(B-C) \geq w[(B+C) \pm (B-C)],$$

which gives

$$w(B+C)+w(B-C)\geq 2\max\{w(B),w(C)\}.$$

By (8), we obtain (6).

If we replace B by $B+C$ and C by $B-C$ in (6), then we obtain

$$\begin{aligned}&w_e^2(B+C,B-C)\\&\geq \frac{1}{2}\max\Big\{w\Big((B+C)^2+(B-C)^2\Big),w((B+C)(B-C)+(B-C)(B+C))\Big\}\\&+\max\{w(B+C),w(B-C)\}|w(B)-w(C)|.\end{aligned}\tag{9}$$

Since

$$(B+C)^2+(B-C)^2=2\Big(B^2+C^2\Big)$$

and

$$(B+C)(B-C)+(B-C)(B+C)=2\Big(B^2-C^2\Big),$$

we can apply (9) to obtain

$$\begin{aligned}w_e^2(B+C,B-C)\geq \max\Big\{w\Big(B^2+C^2\Big),w\Big(B^2-C^2\Big)\Big\}\\+\max\{w(B+C),w(B-C)\}|w(B)-w(C)|,\end{aligned}$$

which, by noting that

$$w_e^2(B+C,B-C)=2w_e^2(B,C),$$

yields the desired result (7). □

Remark 1. *Inequality (6) is better than the following result obtained recently in ([12], Theorem 2.2). Indeed, for any $B,C\in\mathcal{B}(H)$, we have*

$$\begin{aligned}w_e^2(B,C)&\geq \frac{1}{2}\Big[w\Big(B^2+C^2\Big)+\max\{w(B),w(C)\}|w(B+C)-w(B-C)|\Big]\\&\geq \frac{1}{2}w\Big(B^2+C^2\Big).\end{aligned}\tag{10}$$

Notice that the second lower bound in (10) was obtained in 2006 by the second author (see [10]).

Several consequences of Theorem 1 can be drawn. We begin with the following corollary.

Corollary 1. *For any $B,C\in\mathcal{B}(H)$ and $\alpha,\beta\in\mathbb{C}$, we have*

$$\begin{aligned}&\max\Big\{|\alpha|^2,|\beta|^2\Big\}w_e^2(B,C)\\&\geq \frac{1}{2}\max\Big\{w\Big(\alpha^2B^2+\beta^2C^2\Big),|\alpha\beta|w(BC+CB)\Big\}\\&+\frac{1}{2}\max\{|\alpha|w(B),|\beta|w(C)\}|w(\alpha B+\beta C)-w(\alpha B-\beta C)|\end{aligned}\tag{11}$$

and

$$\begin{aligned}&\max\Big\{|\alpha|^2,|\beta|^2\Big\}w_e^2(B,C)\\&\geq \frac{1}{2}\max\Big\{w\Big(\alpha^2B^2+\beta^2C^2\Big),w\Big(\alpha^2B^2-\beta^2C^2\Big)\Big\}\\&+\frac{1}{2}\max\{w(\alpha B+\beta C),w(\alpha B-\beta C)\}||\alpha|w(B)-|\beta|w(C)|.\end{aligned}\tag{12}$$

Proof. By replacing B with αB and C with βC in Theorem 1, we obtain

$$\begin{aligned}&w_e^2(\alpha B, \beta C)\\ &\geq \frac{1}{2}\max\left\{w\left(\alpha^2 B^2+\beta^2 C^2\right), w(\alpha\beta BC+\alpha\beta CB)\right\}\\ &+\frac{1}{2}\max\{w(\alpha B), w(\beta C)\}|w(\alpha B+\beta C)-w(\alpha B-\beta C)|\end{aligned}$$

and

$$\begin{aligned}&w_e^2(B, C)\\ &\geq \frac{1}{2}\max\left\{w\left(\alpha^2 B^2+\beta^2 C^2\right), w\left(\alpha^2 B^2-\beta^2 C^2\right)\right\}\\ &+\frac{1}{2}\max\{w(\alpha B+\beta C), w(\alpha B-\beta C)\}|w(\alpha B)-w(\beta C)|.\end{aligned}$$

Since

$$\max\left\{|\alpha|^2, |\beta|^2\right\}w_e^2(B, C) \geq w_e^2(\alpha B, \beta C),$$
$$w(\alpha\beta BC+\alpha\beta CB) = |\alpha\beta| w(BC+CB)$$

and

$$w(\alpha B) = |\alpha| w(B),\ w(\beta C) = |\beta| w(C);$$

hence, the inequalities (11) and (12) are obtained. □

Remark 2. *If $\alpha, \beta \in \mathbb{C}$ with $|\alpha| = |\beta| = 1$, then from Corollary 1, we derive the simpler inequalities*

$$\begin{aligned}w_e^2(B, C) \geq{}& \frac{1}{2}\max\left\{w\left(\alpha^2 B^2+\beta^2 C^2\right), w(BC+CB)\right\}\\ &+\frac{1}{2}\max\{w(B), w(C)\}|w(\alpha B+\beta C)-w(\alpha B-\beta C)|\end{aligned} \tag{13}$$

and

$$\begin{aligned}w_e^2(B, C) \geq{}& \frac{1}{2}\max\left\{w\left(\alpha^2 B^2+\beta^2 C^2\right), w\left(\alpha^2 B^2-\beta^2 C^2\right)\right\}\\ &+\frac{1}{2}\max\{w(\alpha B+\beta C), w(\alpha B-\beta C)\}|w(B)-w(C)|.\end{aligned} \tag{14}$$

Moreover for $\alpha = 1$ and $\beta = i$, we obtain from (13) and (14) that

$$\begin{aligned}w_e^2(B, C) \geq{}& \frac{1}{2}\max\left\{w\left(B^2-C^2\right), w(BC+CB)\right\}\\ &+\frac{1}{2}\max\{w(B), w(C)\}|w(B+iC)-w(B-iC)|\end{aligned}$$

and

$$\begin{aligned}w_e^2(B, C) \geq{}& \frac{1}{2}\max\left\{w\left(B^2-C^2\right), w\left(B^2+C^2\right)\right\}\\ &+\frac{1}{2}\max\{w(B+iC), w(B-iC)\}|w(B)-w(C)|.\end{aligned}$$

We also present the following corollary, which offers inequalities that improve upon the lower bound established in Corollary 2.4 of the recent paper [12].

Corollary 2. *For any self-adjoint operators B, C we have the inequalities*

$$\begin{aligned} w_e^2(B,C) &\geq \frac{1}{2}\max\left\{\left\|B^2+C^2\right\|, \|BC+CB\|\right\} \\ &\quad + \frac{1}{2}\max\{\|B\|,\|C\|\}|\|B+C\|-\|B-C\|| \\ &\geq \frac{1}{2}\left\|B^2+C^2\right\| \end{aligned} \tag{15}$$

and

$$\begin{aligned} w_e^2(B,C) &\geq \frac{1}{2}\max\left\{\left\|B^2+C^2\right\|, \left\|B^2-C^2\right\|\right\} \\ &\quad + \frac{1}{2}\max\{\|B+C\|,\|B-C\|\}|\|B\|-\|C\||. \end{aligned}$$

Proof. The proof follows directly from Theorem 1, noting that the numerical radius of a self-adjoint operator is equal to its norm. □

The following numerical example demonstrates that the first lower bound in (15) is significantly better than the second one, which was initially obtained in [10].

Example 1. *Consider the diagonal* 2×2 *matrices*

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2 \text{ and } C = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \text{ with } a > 0.$$

Then,

$$B^2 = I_2, C^2 = \begin{pmatrix} 1 & 0 \\ 0 & a^2 \end{pmatrix}, BC+CB = \begin{pmatrix} 2 & 0 \\ 0 & 2a \end{pmatrix},$$

$$B+C = \begin{pmatrix} 2 & 0 \\ 0 & a+1 \end{pmatrix}, B-C = \begin{pmatrix} 0 & 0 \\ 0 & 1-a \end{pmatrix}$$

and

$$B^2+C^2 = \begin{pmatrix} 2 & 0 \\ 0 & a^2+1 \end{pmatrix}.$$

It is well known that if $A = diag(\lambda_1, \cdots, \lambda_n)$*, then* $\|A\| = \max_{i=1,\cdots,n}|\lambda_i|$*. Therefore,*

$$\|B\| = 1,\ \|C\| = \max\{1,a\},\ \|B+C\| = \max\{2,1+a\} = 1+\max\{1,a\},$$

$$\|B-C\| = |1-a|,\ \|BC+CB\| = 2\max\{1,a\}$$

and

$$\left\|B^2+C^2\right\| = \max\left\{2, a^2+1\right\} = 1 + (\max\{1,a\})^2$$

for $a > 0$.

Observe also that for $\bar{z} = (x,y) \in \mathbb{C}^2$*, we have that*

$$\langle B\bar{z},\bar{z}\rangle = |x|^2+|y|^2,\ \langle C\bar{z},\bar{z}\rangle = |x|^2+a|y|^2.$$

Therefore,

$$\begin{aligned} w_e^2(B,C) &= \sup_{|x|^2+|y|^2=1}\left(\left(|x|^2+|y|^2\right)^2+\left(|x|^2+a|y|^2\right)^2\right) \\ &= \sup_{|x|^2+|y|^2=1}\left(1+\left(1+(a-1)y^2\right)^2\right) \\ &= \sup_{|x|^2+|y|^2=1}\left(1+1+2(a-1)y^2+(a-1)^2y^4\right) \\ &= 2+\sup_{|x|^2+|y|^2=1}\left[(a-1)y^2\left(2+(a-1)y^2\right)\right] \end{aligned}$$

for $a>0$.

If we take $a=2$, *then*

$$w_e^2(B,C)=5,$$

the middle term in (15)

$$\frac{1}{2}\left[\max\left\{\left\|B^2+C^2\right\|,\|BC+CB\|\right\}+\max\{\|B\|,\|C\|\}|\|B+C\|-\|B-C\||\right] \\ =4.5$$

and the right term

$$\frac{1}{2}\left\|B^2+C^2\right\|=2.5.$$

This numerically shows that the middle term in (15) provides a significantly better lower bound for the Euclidean numerical radius than the previous one from [10].

We also have the following corollary.

Corollary 3. *For any* $A\in\mathcal{B}(H)$ *and* $\alpha,\beta\in\mathbb{C}$ *we have*

$$\begin{aligned} &\left(|\alpha|^2+|\beta|^2\right)w^2(A) \\ &\geq\frac{1}{2}\max\left\{w\left(\alpha^2A^2+\beta^2\left(A^2\right)^*\right),|\alpha\beta|\left\||A|^2+|A^*|^2\right\|\right\} \\ &+\frac{1}{2}\max\{|\alpha|,|\beta|\}w(A)|w(\alpha A+\beta A^*)-w(\alpha A-\beta A^*)| \end{aligned} \tag{16}$$

and

$$\begin{aligned} &\left(|\alpha|^2+|\beta|^2\right)w^2(A) \\ &\geq\frac{1}{2}\max\left\{w\left(\alpha^2A^2+\beta^2\left(A^2\right)^*\right),w\left(\alpha^2A^2-\beta^2\left(A^2\right)^*\right)\right\} \\ &+\frac{1}{2}||\alpha|-|\beta||w(A)\max\{w(\alpha A+\beta A^*),w(\alpha A-\beta A^*)\}. \end{aligned} \tag{17}$$

Proof. We take $B=\alpha A$ and $C=\beta A^*$ in Theorem 1 to obtain

$$\begin{aligned} w_e^2(\alpha A,\beta A^*)\geq{}&\frac{1}{2}\max\left\{w\left(\alpha^2A^2+\beta^2\left(A^2\right)^*\right),w(\alpha\beta AA^*+\alpha\beta A^*A)\right\} \\ &+\frac{1}{2}\max\{w(\alpha A),w(\beta A^*)\}|w(\alpha A+\beta A^*)-w(\alpha A-\beta A^*)| \end{aligned}$$

and

$$w_e^2(\alpha A, \beta A^*) \geq \frac{1}{2}\max\left\{w\left(\alpha^2 A^2 + \beta^2\left(A^2\right)^*\right), w\left(\alpha^2 A^2 - \beta^2\left(A^2\right)^*\right)\right\}$$
$$+ \frac{1}{2}\max\{w(\alpha A + \beta A^*), w(\alpha A - \beta A^*)\}|w(\alpha A) - w(\beta A^*)|.$$

Observe that

$$w_e^2(\alpha A, \beta A^*) = \left(|\alpha|^2 + |\beta|^2\right)w^2(A),$$

$$w(\alpha\beta AA^* + \alpha\beta A^*A) = |\alpha\beta|w(AA^* + A^*A) = |\alpha\beta|\left\||A|^2 + |A^*|^2\right\|$$

and

$$w(\beta A^*) = |\beta|w(A),$$

which proves the desired results (16) and (17). □

Remark 3. *If we take* $\alpha = \cos\theta$ *and* $\beta = \sin\theta$, $\theta \in \mathbb{R}$ *in (17), we obtain*

$$w^2(A) \geq \frac{1}{2}\max\left\{w\left(\cos^2\theta A^2 + \sin^2\theta\left(A^2\right)^*\right), |\sin(2\theta)|\left\|\frac{|A|^2 + |A^*|^2}{2}\right\|\right\}$$
$$+ \frac{1}{2}\max\{|\cos\theta|, |\sin\theta|\}w(A)|w(\cos\theta A + \sin\theta A^*) - w(\cos\theta A - \sin\theta A^*)| \tag{18}$$

and

$$w^2(A) \geq \frac{1}{2}\max\left\{w\left(\cos^2\theta A^2 + \sin^2\theta\left(A^2\right)^*\right), w\left(\cos^2\theta A^2 - \sin^2\theta\left(A^2\right)^*\right)\right\}$$
$$+ \frac{1}{2}||\cos\theta| - |\sin\theta||w(A)\max\{w(\cos\theta A + \sin\theta A^*), w(\cos\theta A - \sin\theta A^*)\}.$$

If we take $\theta = \pi/4$ *in (18), we obtain*

$$w^2(A) \geq \frac{1}{2}\max\left\{\left\|\frac{A^2 + (A^2)^*}{2}\right\|, \left\|\frac{|A|^2 + |A^*|^2}{2}\right\|\right\}$$
$$+ \frac{1}{2}w(A)\left|\left\|\frac{A + A^*}{2}\right\| - \left\|\frac{A - A^*}{2}\right\|\right|.$$

We now present the following corollary, which offers an improvement of the lower bound from ([12], Corollary 2.5).

Corollary 4. *For any* $A \in \mathcal{B}(H)$, *we have*

$$w^2(A) \geq \frac{1}{2}\max\left\{\left\|\frac{|A|^2 + |A^*|^2}{2}\right\|, \left\|\frac{(A^*)^2 - A^2}{2}\right\|\right\}$$
$$+ \frac{1}{2}\max\left\{\left\|\frac{A + A^*}{2}\right\|, \left\|\frac{A^* - A}{2}\right\|\right\}\left|\left(\left\|\frac{(1-i)A + (1+i)A^*}{2}\right\|\right) - \left\|\frac{(1+i)A + (1-i)A^*}{2}\right\|\right| \tag{19}$$

and

$$w^2(A) \geq \frac{1}{2}\max\left\{\left\|\frac{|A|^2 + |A^*|^2}{2}\right\|, \left\|\frac{A^2 + (A^*)^2}{2}\right\|\right\}$$
$$+ \frac{1}{2}\max\left\{\left\|\frac{(1-i)A + (1+i)A^*}{2}\right\|, \left\|\frac{(1+i)A + (1-i)A^*}{2}\right\|\right\}\left|\left\|\frac{A + A^*}{2}\right\| - \left\|\frac{A^* - A}{2}\right\|\right|. \tag{20}$$

Proof. Let $A = B + iC$ be the Cartesian decomposition of A, where

$$B = \mathfrak{R}(A) = \frac{A + A^*}{2} \text{ and } C = \mathfrak{I}(A) = \frac{A - A^*}{2i} = \frac{i}{2}(A^* - A).$$

Observe that

$$w_e^2(B, C) = w^2(A), \; B^2 + C^2 = \frac{|A|^2 + |A^*|^2}{2}$$

$$\begin{aligned} BC + CB &= \frac{i}{2}\left[\frac{(A + A^*)(A^* - A)}{2} + \frac{(A^* - A)(A + A^*)}{2}\right] \\ &= \frac{i}{4}\left[AA^* + (A^*)^2 - A^2 - A^*A + A^*A - A^2 + (A^*)^2 - AA^*\right] \\ &= \frac{i}{2}\left((A^*)^2 - A^2\right), \end{aligned}$$

$$\begin{aligned} B^2 - C^2 &= \left(\frac{A + A^*}{2}\right)^2 - \left(\frac{i}{2}(A^* - A)\right)^2 \\ &= \left(\frac{A + A^*}{2}\right)^2 + \left(\frac{A^* - A}{2}\right)^2 \\ &= \frac{A^2 + AA^* + A^*A + (A^*)^2 + A^2 - AA^* - A^*A + (A^*)^2}{4} \\ &= \frac{1}{2}\left(A^2 + (A^*)^2\right) \end{aligned}$$

$$B + C = \frac{A + A^*}{2} + \frac{i(A^* - A)}{2} = \frac{(1 - i)A + (1 + i)A^*}{2}$$

and

$$B - C = \frac{A + A^*}{2} - \frac{i(A^* - A)}{2} = \frac{(1 + i)A + (1 - i)A^*}{2}.$$

By making use of (6) and (7), we deduce the desired results (19) and (20). □

We have the following lower bounds for the Davis–Wielandt radius. These lower bounds are better than the one offered by the first inequality in (4).

Corollary 5. *For any $T \in \mathcal{B}(H)$, we have*

$$\begin{aligned} dw^2(T) \geq{}& \frac{1}{2}\max\left\{w\left(T^2 + |T|^4\right), w\left(T|T|^2 + |T|^2T\right)\right\} \\ &+ \frac{1}{2}\max\left\{w(T), \|T\|^2\right\}\left|w\left(T + |T|^2\right) - w\left(T - |T|^2\right)\right| \end{aligned}$$

and

$$\begin{aligned} dw^2(T) \geq{}& \frac{1}{2}\max\left\{w\left(T^2 + |T|^4\right), w\left(T^2 - |T|^4\right)\right\} \\ &+ \frac{1}{2}\max\left\{w\left(T + |T|^2\right), w\left(T - |T|^2\right)\right\}\left|w(T) - \|T\|^2\right|. \end{aligned}$$

Proof. The proof follows by taking $(B, C) = \left(T, |T|^2\right)$ in Theorem 1. □

Further, we recall the following representation result obtained in ([11], Theorem 2.2).

Lemma 1. *Let* $X, Y \in \mathcal{B}(H)$*. Then,*

$$w_e^2(X, Y) = \sup_{\substack{\mu^2+\nu^2=1, \\ \mu,\nu\geq 0}} \sup_{\theta\in\mathbb{R}} w^2\left(\mu e^{i\theta} X + \nu e^{-i\theta} Y\right).$$

By utilizing this result, we can prove the following lower bound for the Euclidean numerical radius.

Proposition 1. *For* $B, C \in \mathcal{B}(H)$*, we have*

$$w_e^2(B, C) \geq \frac{1}{2} w_e\left(B^2, C^2\right). \tag{21}$$

Proof. Let $\mu, \nu \geq 0$ be such that $\mu^2 + \nu^2 = 1$. Let also $\theta \in \mathbb{R}$. From (11), we obtain for $\alpha = \sqrt{\mu} e^{i\frac{\theta}{2}}$ and $\beta = \sqrt{\nu} e^{-i\frac{\theta}{2}}$ that

$$\max\{\mu, \nu\} w_e^2(B, C) \geq \frac{1}{2} w\left(\mu e^{i\theta} B^2 + \nu e^{-i\theta} C^2\right).$$

So, we deduce that

$$\begin{aligned} \sup_{\substack{\mu^2+\nu^2=1, \\ \mu,\nu\geq 0}} \max\{\mu, \nu\} w_e^2(B, C) &\geq \frac{1}{2} \sup_{\substack{\mu^2+\nu^2=1, \\ \mu,\nu\geq 0}} \sup_{\theta\in\mathbb{R}} w\left(\mu e^{i\theta} B^2 + \nu e^{-i\theta} C^2\right) \\ &= \frac{1}{2} w_e\left(B^2, C^2\right) \end{aligned}$$

and since $\sup_{\substack{\mu^2+\nu^2=1, \\ \mu,\nu\geq 0}} \max\{\mu, \nu\} = 1$, we deduce inequality (21). □

Remark 4. *If we take* $B = A$ *and* $C = A^*$ *in (21), we obtain*

$$w^2(A) \geq \frac{1}{4} w_e\left(A^2, (A^*)^2\right).$$

If we take $B = \mathfrak{R}(A)$ *and* $C = \mathfrak{I}(A)$*, then by (21) we obtain*

$$w^2(A) \geq \frac{1}{2} \left\| \mathfrak{R}^2(A) + \mathfrak{I}^2(A) \right\|^{1/2}.$$

Indeed, we have

$$\begin{aligned} w_e^2\left(\mathfrak{R}^2(A), \mathfrak{I}^2(A)\right) &= \sup_{\|x\|=1} \left[\langle \mathfrak{R}^2(A)x, x\rangle + \langle \mathfrak{I}^2(A)x, x\rangle\right] \\ &= \sup_{\|x\|=1} \left[\left\langle \left(\mathfrak{R}^2(A) + \mathfrak{I}^2(A)\right)x, x\right\rangle\right] \\ &= \omega\left(\mathfrak{R}^2(A) + \mathfrak{I}^2(A)\right) = \left\|\mathfrak{R}^2(A) + \mathfrak{I}^2(A)\right\|. \end{aligned}$$

If we take $B = T$ *and* $C = |T|^2$*, then by (21), we obtain*

$$dw^2(T) \geq \frac{1}{2} w_e\left(T^2, |T|^4\right).$$

The following representation is also known (see, for instance, [19]).

Lemma 2. *Let $X, Y \in \mathcal{B}(H)$. Then,*

$$w\begin{pmatrix} 0 & X \\ Y^* & 0 \end{pmatrix} = \frac{1}{2}\sup_{\theta\in\mathbb{R}}\left\| e^{i\theta}X + e^{-i\theta}Y \right\|.$$

We are able now to prove the following proposition.

Proposition 2. *For $B, C \in \mathcal{B}(H)$, we have*

$$w_e^2(B,C) \geq \frac{1}{2} w\begin{pmatrix} 0 & B^2 \\ (C^*)^2 & 0 \end{pmatrix}. \tag{22}$$

Proof. By using (12) for $\alpha = e^{i\frac{\theta}{2}}$ and $\beta = e^{-i\frac{\theta}{2}}$, we obtain

$$\begin{aligned} w_e^2(B,C) &\geq \frac{1}{2}\max\left\{ w\left(e^{i\theta}B^2 + e^{-i\theta}C^2\right), w\left(e^{-i\theta}B^2 - e^{-i\theta}C^2\right)\right\} \\ &\geq \frac{1}{4}\max\left\{ \left\| e^{i\theta}B^2 + e^{-i\theta}C^2 \right\|, \left\| e^{-i\theta}B^2 - e^{-i\theta}C^2 \right\| \right\} \end{aligned}$$

for all $\theta \in \mathbb{R}$.

If we take the supremum over $\theta \in \mathbb{R}$, we obtain

$$w_e^2(B,C) \geq \frac{1}{4}\max\left\{ \sup_{\theta\in\mathbb{R}}\left\| e^{i\theta}B^2 + e^{-i\theta}C^2 \right\|, \sup_{\theta\in\mathbb{R}}\left\| e^{-i\theta}B^2 - e^{-i\theta}C^2 \right\| \right\}. \tag{23}$$

Since, by Lemma 2,

$$\frac{1}{2}\sup_{\theta\in\mathbb{R}}\left\| e^{i\theta}B^2 + e^{-i\theta}C^2 \right\| = w\begin{pmatrix} 0 & B^2 \\ (C^2)^* & 0 \end{pmatrix} = w\begin{pmatrix} 0 & B^2 \\ (C^*)^2 & 0 \end{pmatrix}$$

and

$$\frac{1}{2}\sup_{\theta\in\mathbb{R}}\left\| e^{-i\theta}B^2 - e^{-i\theta}C^2 \right\| = w\begin{pmatrix} 0 & B^2 \\ -(C^*)^2 & 0 \end{pmatrix} = w\begin{pmatrix} 0 & B^2 \\ (C^*)^2 & 0 \end{pmatrix}.$$

Then, by (23), we deduce (22) as requested. □

Remark 5. *If we take $B = A$ and $C = A^*$ in (22), we obtain*

$$w^2(A) \geq \frac{1}{4} w\begin{pmatrix} 0 & A^2 \\ A^2 & 0 \end{pmatrix} = \frac{1}{4} w\left(A^2\right).$$

If we take $B = \mathfrak{R}(A)$ and $C = \mathfrak{I}(A)$, then by (22), we obtain

$$w^2(A) \geq \frac{1}{2} w\begin{pmatrix} 0 & \mathfrak{R}^2(A) \\ \mathfrak{I}^2(A) & 0 \end{pmatrix}.$$

Moreover, if we take $B = T$ and $C = |T|^2$, then by (21), we obtain

$$dw^2(T) \geq \frac{1}{2} w\begin{pmatrix} 0 & T^2 \\ |T|^4 & 0 \end{pmatrix}.$$

3. Upper Bounds for the Euclidean Numerical Radius of Operator Pairs

In this section, we aim to establish several upper bounds for the Euclidean numerical radius of operator pairs in Hilbert spaces. To derive our first result, we recall two well-known inequalities. The first is the *Kato inequality* (see [20]), which is given by

$$\left|\langle Tx, y\rangle\right|^2 \leq \langle |T|^{2\lambda}x, x\rangle\langle |T^*|^{2(1-\lambda)}y, y\rangle \tag{24}$$

for any $T \in \mathcal{B}(H)$, $\lambda \in [0,1]$, and any $x, y \in H$.

The second is the well-known McCarthy inequality (see [21]), which asserts that for any positive operator P and for $s \geq 1$, we have

$$\langle Px, x\rangle^s \leq \langle P^s x, x\rangle, \quad x \in H, \ \|x\| = 1. \tag{25}$$

With these preliminaries, we can now present the following result.

Theorem 2. *For any $B, C \in \mathcal{B}(H)$, $\lambda \in [0,1]$ and $\alpha, \beta, \gamma, \delta \in \{-1,1\}$, we have*

$$\begin{aligned} w_e^2(B,C) &\leq w\Big(\alpha|B|^{2\lambda} + i\beta|C|^{2\lambda}\Big) w\Big(\gamma|B^*|^{2(1-\lambda)} + i\delta|C^*|^{2(1-\lambda)}\Big) \\ &\leq \Big\||B|^{4\lambda} + |C|^{4\lambda}\Big\|^{1/2} \Big\||B^*|^{4(1-\lambda)} + |C^*|^{4(1-\lambda)}\Big\|^{1/2} \end{aligned} \tag{26}$$

and

$$\begin{aligned} w_e^2(B,C) &\leq w\Big(\alpha|B|^{2\lambda} + i\delta|C^*|^{2(1-\lambda)}\Big) w\Big(\gamma|B^*|^{2(1-\lambda)} + i\beta|C|^{2\lambda}\Big) \\ &\leq \Big\||B|^{4\lambda} + |C^*|^{4(1-\lambda)}\Big\|^{1/2} \Big\||B^*|^{4(1-\lambda)} + |C|^{4\lambda}\Big\|^{1/2}. \end{aligned} \tag{27}$$

Proof. By (24), we have

$$|\langle Bx, x\rangle|^2 + |\langle Cx, x\rangle|^2 \leq \langle |B|^{2\lambda}x, x\rangle\langle |B^*|^{2(1-\lambda)}x, x\rangle + \langle |C|^{2\lambda}x, x\rangle\langle |C^*|^{2(1-\lambda)}x, x\rangle \tag{28}$$

for any $B, C \in \mathcal{B}(H)$, $x \in H$ and $\lambda \in [0,1]$.

By the Cauchy–Schwarz inequality, we have

$$\begin{aligned} &\langle |B|^{2\lambda}x, x\rangle\langle |B^*|^{2(1-\lambda)}x, x\rangle + \langle |C|^{2\lambda}x, x\rangle\langle |C^*|^{2(1-\lambda)}x, x\rangle \\ &\leq \Big(\langle |B|^{2\lambda}x, x\rangle^2 + \langle |C|^{2\lambda}x, x\rangle^2\Big)^{1/2} \Big(\langle |B^*|^{2(1-\lambda)}x, x\rangle^2 + \langle |C^*|^{2(1-\lambda)}x, x\rangle^2\Big)^{1/2} \\ &= \Big(\langle \alpha|B|^{2\lambda}x, x\rangle^2 + \langle \beta|C|^{2\lambda}x, x\rangle^2\Big)^{1/2} \Big(\langle \gamma|B^*|^{2(1-\lambda)}x, x\rangle^2 + \langle \delta|C^*|^{2(1-\lambda)}x, x\rangle^2\Big)^{1/2}, \end{aligned} \tag{29}$$

for $x \in H$, where $\alpha, \beta, \gamma, \delta \in \{-1,1\}$.

Since

$$\begin{aligned} \langle \alpha|B|^{2\lambda}x, x\rangle^2 + \langle \beta|C|^{2\lambda}x, x\rangle^2 &= \Big|\langle \alpha|B|^{2\lambda}x, x\rangle + i\langle \beta|C|^{2\lambda}x, x\rangle\Big|^2 \\ &= \Big|\Big\langle \Big(\alpha|B|^{2\lambda} + i\beta|C|^{2\lambda}\Big)x, x\Big\rangle\Big|^2 \end{aligned}$$

and

$$\langle \gamma|B^*|^{2(1-\lambda)}x, x\rangle^2 + \langle \delta|C^*|^{2(1-\lambda)}x, x\rangle^2 = \Big|\Big\langle \Big(\gamma|B^*|^{2(1-\lambda)} + i\delta|C^*|^{2(1-\lambda)}\Big)x, x\Big\rangle\Big|^2,$$

then by taking (29) into account, we obtain

$$\begin{aligned} &\langle |B|^{2\lambda}x, x\rangle\langle |B^*|^{2(1-\lambda)}x, x\rangle + \langle |C|^{2\lambda}x, x\rangle\langle |C^*|^{2(1-\lambda)}x, x\rangle \\ &\leq \Big|\Big\langle \Big(\alpha|B|^{2\lambda} + i\beta|C|^{2\lambda}\Big)x, x\Big\rangle\Big| \Big|\Big\langle \Big(\gamma|B^*|^{2(1-\lambda)} + i\delta|C^*|^{2(1-\lambda)}\Big)x, x\Big\rangle\Big| \end{aligned} \tag{30}$$

for $x \in H$.

By (28) and (30), we derive

$$\begin{aligned} &|\langle Bx, x\rangle|^2 + |\langle Cx, x\rangle|^2 \\ &\leq \Big|\Big\langle \Big(\alpha|B|^{2\lambda} + i\beta|C|^{2\lambda}\Big)x, x\Big\rangle\Big| \Big|\Big\langle \Big(\gamma|B^*|^{2(1-\lambda)} + i\delta|C^*|^{2(1-\lambda)}\Big)x, x\Big\rangle\Big| \end{aligned}$$

for $x \in H$.

By taking the supremum over $x \in H$, $\|x\| = 1$, we obtain

$$\begin{aligned} w_e^2(B,C) &= \sup_{\|x\|=1} \left(\left|\langle Bx, x\rangle\right|^2 + \left|\langle Cx, x\rangle\right|^2 \right) \\ &\le \sup_{\|x\|=1} \left[\left| \left\langle \left(\alpha|B|^{2\lambda} + i\beta|C|^{2\lambda} \right) x, x \right\rangle \right| \left| \left\langle \left(\gamma|B^*|^{2(1-\lambda)} + i\delta|C^*|^{2(1-\lambda)} \right) x, x \right\rangle \right| \right] \\ &\le \sup_{\|x\|=1} \left| \left\langle \left(\alpha|B|^{2\lambda} + i\beta|C|^{2\lambda} \right) x, x \right\rangle \right| \sup_{\|x\|=1} \left| \left\langle \left(\gamma|B^*|^{2(1-\lambda)} + i\delta|C^*|^{2(1-\lambda)} \right) x, x \right\rangle \right| \\ &= w\left(\alpha|B|^{2\lambda} + i\beta|C|^{2\lambda} \right) w\left(\gamma|B^*|^{2(1-\lambda)} + i\delta|C^*|^{2(1-\lambda)} \right), \end{aligned}$$

which proves (26).

By the Cauchy–Schwarz inequality, we also have

$$\begin{aligned} &\langle |B|^{2\lambda} x, x\rangle \langle |B^*|^{2(1-\lambda)} x, x\rangle + \langle |C|^{2\lambda} x, x\rangle \langle |C^*|^{2(1-\lambda)} x, x\rangle \\ &\le \left(\langle |B|^{2\lambda} x, x\rangle^2 + \langle |C^*|^{2(1-\lambda)} x, x\rangle^2 \right)^{1/2} \left(\langle |B^*|^{2(1-\lambda)} x, x\rangle^2 + \langle |C|^{2\lambda} x, x\rangle^2 \right)^{1/2} \\ &= \left(\langle \alpha|B|^{2\lambda} x, x\rangle^2 + \langle \delta|C^*|^{2(1-\lambda)} x, x\rangle^2 \right)^{1/2} \left(\langle \gamma|B^*|^{2(1-\lambda)} x, x\rangle^2 + \langle \beta|C|^{2\lambda} x, x\rangle^2 \right)^{1/2} \end{aligned}$$

for $x \in H$, where $\alpha, \beta, \gamma, \delta \in \{-1, 1\}$.

By making a similar argument as above, we then derive (27).

Furthermore, by McCarthy inequality (25), we have for $\|x\| = 1$

$$\begin{aligned} \left| \left\langle \left(\alpha|B|^{2\lambda} + i\beta|C|^{2\lambda} \right) x, x \right\rangle \right|^2 &= \left| \alpha \langle |B|^{2\lambda} x, x\rangle + i\beta \langle |C|^{2\lambda} x, x\rangle \right|^2 \\ &= \langle |B|^{2\lambda} x, x\rangle^2 + \langle |C|^{2\lambda} x, x\rangle^2 \\ &\le \langle |B|^{4\lambda} x, x\rangle + \langle |C|^{4\lambda} x, x\rangle \\ &= \left\langle \left(|B|^{4\lambda} + |C|^{4\lambda} \right) x, x \right\rangle. \end{aligned}$$

By taking the supremum over $\|x\| = 1$, we obtain

$$\begin{aligned} w\left(\alpha|B|^{2\lambda} + i\beta|C|^{2\lambda} \right) &= \sup_{\|x\|=1} \left| \left\langle \left(\alpha|B|^{2\lambda} + i\beta|C|^{2\lambda} \right) x, x \right\rangle \right| \\ &\le \sup_{\|x\|=1} \left\langle \left(|B|^{4\lambda} + |C|^{4\lambda} \right) x, x \right\rangle^{1/2} = \left\| |B|^{4\lambda} + |C|^{4\lambda} \right\|^{1/2}. \end{aligned}$$

Similarly,

$$w\left(\gamma|B^*|^{2(1-\lambda)} + i\delta|C^*|^{2(1-\lambda)} \right) \le \left\| |B^*|^{4(1-\lambda)} + |C^*|^{4(1-\lambda)} \right\|^{1/2},$$

which proves the last part of (26). □

Several consequences of Theorem 2 can be presented. We start with the following corollary.

Corollary 6. *For any $B, C \in \mathcal{B}(H)$ and $\alpha, \beta, \gamma, \delta \in \{-1, 1\}$, we have*

$$\begin{aligned} w_e^2(B,C) &\le w(\alpha|B| + i\beta|C|) w(\gamma|B^*| + i\delta|C^*|) \\ &\le \left\| |B|^2 + |C|^2 \right\|^{1/2} \left\| |B^*|^2 + |C^*|^2 \right\|^{1/2} \end{aligned}$$

and

$$\begin{aligned} w_e^2(B,C) &\le w(\alpha|B| + i\delta|C^*|)w(\gamma|B^*| + i\beta|C|) \\ &\le \left\| |B|^2 + |C^*|^2 \right\|^{1/2} \left\| |B^*|^2 + |C|^2 \right\|^{1/2}. \end{aligned}$$

Proof. The proof follows directly from Theorem 2 by setting $\lambda = \frac{1}{2}$. □

Remark 6. *In ([12], Theorem 2.2), the authors proved that for any $B, C \in \mathcal{B}(H)$, the following result holds:*

$$\begin{aligned} &w_e^2(B,C) \\ &\le \min\left\{ w(|B| + i|C|)w(|B^*| + i|C^*|), w(|B| + i|C^*|)w\left(|B^*|^2 + i|C|\right) \right\} \\ &\le \min\left\{ \left\| |B|^2 + |C|^2 \right\|^{1/2} \left\| |B^*|^2 + |C^*|^2 \right\|^{1/2}, \left\| |B|^2 + |C^*|^2 \right\|^{1/2} \left\| |B^*|^2 + |C|^2 \right\|^{1/2} \right\}. \end{aligned}$$

This result can be derived from Corollary 6 by setting $\alpha = \beta = \gamma = \delta = 1$. The interested reader can derive other similar bounds by choosing some of $\alpha, \beta, \gamma, \delta$ to be -1.

We also present the following corollary, which, while straightforward, is of significant interest.

Corollary 7. *For any B, C self-adjoint operators, $\lambda \in [0,1]$ and $\alpha, \beta, \gamma, \delta \in \{-1, 1\}$, we have*

$$\begin{aligned} w_e^2(B,C) &\le w\left(\alpha|B|^{2\lambda} + i\beta|C|^{2\lambda}\right) w\left(\gamma|B|^{2(1-\lambda)} + i\delta|C|^{2(1-\lambda)}\right) \\ &\le \left\| |B|^{4\lambda} + |C|^{4\lambda} \right\|^{1/2} \left\| |B|^{4(1-\lambda)} + |C|^{4(1-\lambda)} \right\|^{1/2} \end{aligned}$$

and

$$\begin{aligned} w_e^2(B,C) &\le w\left(\alpha|B|^{2\lambda} + i\delta|C|^{2(1-\lambda)}\right) w\left(\gamma|B|^{2(1-\lambda)} + i\beta|C|^{2\lambda}\right) \\ &\le \left\| |B|^{4\lambda} + |C|^{4(1-\lambda)} \right\|^{1/2} \left\| |B|^{4(1-\lambda)} + |C|^{4\lambda} \right\|^{1/2}. \end{aligned}$$

Remark 7. *For $\lambda = 1/2$ in the above corollary, we obtain*

$$w_e^2(B,C) \le w(\alpha|B| + i\beta|C|)w(\gamma|B| + i\delta|C|) \le \left\| |B|^2 + |C|^2 \right\|$$

for $\alpha, \beta, \gamma, \delta \in \{-1, 1\}$.

Another important consequence of Theorem 2 is derived in the next corollary.

Corollary 8. *For $A \in \mathcal{B}(H)$, $\varphi, \psi \in \mathbb{C}$, $\lambda \in [0,1]$ and $\alpha, \beta, \gamma, \delta \in \{-1, 1\}$, we have the numerical radius inequalities*

$$\begin{aligned} &\left(|\varphi|^2 + |\psi|^2\right) w^2(A) \\ &\le w\left(\alpha|\varphi|^{2\lambda}|A|^{2\lambda} + i\beta|\psi|^{2\lambda}|A^*|^{2\lambda}\right) w\left(\gamma|\varphi|^{2(1-\lambda)}|A^*|^{2(1-\lambda)} + i\delta|\psi|^{2(1-\lambda)}|A|^{2(1-\lambda)}\right) \\ &\le \left\| |\varphi|^{4\lambda}|A|^{4\lambda} + |\psi|^{4\lambda}|A^*|^{4\lambda} \right\|^{1/2} \left\| |\varphi|^{4(1-\lambda)}|A^*|^{4(1-\lambda)} + |\psi|^{4(1-\lambda)}|A|^{4(1-\lambda)} \right\|^{1/2} \end{aligned}$$

and

$$\begin{aligned}&\left(|\varphi|^2+|\psi|^2\right)w^2(A)\\&\le w\left(\alpha|\varphi|^{2\lambda}|A|^{2\lambda}+i\delta|\psi|^{2(1-\lambda)}|A|^{2(1-\lambda)}\right)w\left(\gamma|\varphi|^{2(1-\lambda)}|A^*|^{2(1-\lambda)}+i\beta|\psi|^{2\lambda}|A^*|^{2\lambda}\right)\\&\le\left\||\varphi|^{4\lambda}|A|^{4\lambda}+|\psi|^{4(1-\lambda)}|A|^{4(1-\lambda)}\right\|^{1/2}\left\||\varphi|^{4(1-\lambda)}|A^*|^{4(1-\lambda)}+|\psi|^{4\lambda}|A^*|^{4\lambda}\right\|^{1/2}.\end{aligned}$$

Proof. We take $B=\varphi A$ and $C=\psi A^*$ in Theorem 2 to obtain the desired results. □

Remark 8. *We notice that for $\varphi=\psi=1$, we obtain the simpler inequalities*

$$\begin{aligned}w^2(A)&\le\frac{1}{2}w\left(\alpha|A|^{2\lambda}+i\beta|A^*|^{2\lambda}\right)w\left(\gamma|A^*|^{2(1-\lambda)}+i\delta|A|^{2(1-\lambda)}\right)\\&\le\left\||A|^{4\lambda}+|A^*|^{4\lambda}\right\|^{1/2}\left\||A^*|^{4(1-\lambda)}+|A|^{4(1-\lambda)}\right\|^{1/2}\end{aligned}$$

and

$$\begin{aligned}w^2(A)&\le\frac{1}{2}w\left(\alpha|A|^{2\lambda}+i\delta|A|^{2(1-\lambda)}\right)w\left(\gamma|A^*|^{2(1-\lambda)}+i\beta|A^*|^{2\lambda}\right)\\&\le\frac{1}{2}\left\||A|^{4\lambda}+|A|^{4(1-\lambda)}\right\|^{1/2}\left\||A^*|^{4(1-\lambda)}+|A^*|^{4\lambda}\right\|^{1/2}.\end{aligned}$$

We also have in the particular case of $\lambda=1/2$, that

Remark 9. *For $\lambda=1/2$ in Corollary 8, we obtain*

$$\begin{aligned}\left(|\varphi|^2+|\psi|^2\right)w^2(A)&\le w(\alpha|\varphi||A|+i\beta|\psi||A^*|)w(\gamma|\varphi||A^*|+i\delta|\psi||A|)\\&\le\left\||\varphi|^2|A|^2+|\psi|^2|A^*|^2\right\|^{1/2}\left\||\varphi|^2|A^*|^2+|\psi|^2|A|^2\right\|^{1/2}.\end{aligned}\tag{31}$$

If $|\varphi|^2+|\psi|^2=1$ in (31), then we obtain

$$\begin{aligned}w^2(A)&\le w(\alpha|\varphi||A|+i\beta|\psi||A^*|)w(\gamma|\varphi||A^*|+i\delta|\psi||A|)\\&\le\left\||\varphi|^2|A|^2+|\psi|^2|A^*|^2\right\|^{1/2}\left\||\varphi|^2|A^*|^2+|\psi|^2|A|^2\right\|^{1/2}.\end{aligned}\tag{32}$$

If we take $\varphi=\sin\theta$, $\psi=\cos\theta$, $\theta\in\mathbb{R}$ in (32), then we obtain

$$\begin{aligned}w^2(A)&\le w(\alpha|\sin\theta||A|+i\beta|\cos\theta||A^*|)w(\gamma|\sin\theta||A^*|+i\delta|\cos\theta||A|)\\&\le\left\|\sin^2\theta|A|^2+\cos^2\theta|A^*|^2\right\|^{1/2}\left\|\sin^2\theta|A^*|^2+\cos^2|A|^2\right\|^{1/2}.\end{aligned}$$

Also, if we take $\varphi=\psi=1$, then we obtain from (31) that

$$w^2(A)\le\frac{1}{2}w(\alpha|A|+i\beta|A^*|)w(\gamma|A^*|+i\delta|A|)\le\frac{1}{2}\left\||A|^2+|A^*|^2\right\|.$$

This is an improvement of the second Kittaneh's inequality in (2).

Another significant consequence of Theorem 2 is presented in the following corollary.

Corollary 9. *For any $A\in\mathcal{B}(H)$ and $\lambda\in[0,1]$, we have*

$$w^2(A) \le w\left(\alpha\left(\frac{A+A^*}{2}\right)^{2\lambda} + i\beta\left(\frac{A^*-A}{2}\right)^{2\lambda}\right) w\left(\gamma\left(\frac{A+A^*}{2}\right)^{2(1-\lambda)} + i\delta\left(\frac{A^*-A}{2}\right)^{2(1-\lambda)}\right) \tag{33}$$
$$\le \left\|\left(\frac{A+A^*}{2}\right)^{4\lambda} + \left(\frac{A^*-A}{2}\right)^{4\lambda}\right\|^{1/2} \left\|\left(\frac{A+A^*}{2}\right)^{4(1-\lambda)} + \left(\frac{A^*-A}{2}\right)^{4(1-\lambda)}\right\|^{1/2}$$

and

$$w^2(A) \le w\left(\alpha\left(\frac{A+A^*}{2}\right)^{2\lambda} + i\delta\left(\frac{A^*-A}{2}\right)^{2(1-\lambda)}\right) w\left(\gamma\left(\frac{A+A^*}{2}\right)^{2(1-\lambda)} + i\beta\left(\frac{A^*-A}{2}\right)^{2\lambda}\right) \tag{34}$$
$$\le \left\|\left(\frac{A+A^*}{2}\right)^{4\lambda} + \left(\frac{A^*-A}{2}\right)^{4(1-\lambda)}\right\|^{1/2} \left\|\left(\frac{A+A^*}{2}\right)^{4(1-\lambda)} + \left(\frac{A^*-A}{2}\right)^{4\lambda}\right\|^{1/2}.$$

Proof. Let $A = B + iC$ be the Cartesian decomposition of A, where

$$B = \frac{A+A^*}{2} = \mathfrak{R}(A) \text{ and } C = \frac{i}{2}(A^* - A) = \mathfrak{I}(A).$$

By applying Theorem 2 for this choice of B and C, we obtain (33) and (34). □

Remark 10. *For $\lambda = 1/2$ in Corollary 9, we obtain*

$$w^2(A) \le w\left(\alpha\left(\frac{A+A^*}{2}\right) + i\beta\left(\frac{A^*-A}{2}\right)\right) w\left(\gamma\left(\frac{A+A^*}{2}\right) + i\delta\left(\frac{A^*-A}{2}\right)\right) \le \frac{1}{2}\left\|A^2 + (A^*)^2\right\|$$

for $\alpha, \beta, \gamma, \delta \in \{-1, 1\}$.

This is another improvement of the second Kittaneh's inequality in (2).

By taking $(B, C) = \left(T, |T|^2\right)$ in Theorem 2, we obtain

Corollary 10. *For any $T \in \mathcal{B}(H)$, $\lambda \in [0, 1]$ and $\alpha, \beta, \gamma, \delta \in \{-1, 1\}$, we have*

$$dw^2(T) \le w\left(\alpha|T|^{2\lambda} + i\beta|T|^{4\lambda}\right) w\left(\gamma|T^*|^{2(1-\lambda)} + i\delta|T|^{4(1-\lambda)}\right)$$
$$\le \left\||T|^{4\lambda} + |T|^{8\lambda}\right\|^{1/2} \left\||T^*|^{4(1-\lambda)} + |T|^{8(1-\lambda)}\right\|^{1/2}$$

and

$$dw^2(T) \le w\left(\alpha|T|^{2\lambda} + i\delta|T|^{4(1-\lambda)}\right) w\left(\gamma|T^*|^{2(1-\lambda)} + i\beta|T|^{4\lambda}\right)$$
$$\le \left\||T|^{4\lambda} + |T|^{8(1-\lambda)}\right\|^{1/2} \left\||T^*|^{4(1-\lambda)} + |T|^{8\lambda}\right\|^{1/2}.$$

In particular for $\lambda = 1/2$, we have

$$dw^2(T) \le w\left(\alpha|T| + i\beta|T|^2\right) w\left(\gamma|T^*| + i\delta|T|^2\right) \tag{35}$$
$$\le \left\||T|^2 + |T|^4\right\|^{1/2} \left\||T^*|^2 + |T|^4\right\|^{1/2}$$

for all $T \in \mathcal{B}(H)$.

We observe that, if $|T^*|^2 \le |T|^2$, namely T is *hyponormal operator*, then

$$\left\||T^*|^2 + |T|^4\right\| \le \left\||T|^2 + |T|^4\right\|$$

and by (35), we obtain

$$dw^2(T) \le w\left(\alpha|T| + i\beta|T|^2\right)w\left(\gamma|T^*| + i\delta|T|^2\right)$$
$$\le \left\||T|^2 + |T|^4\right\|^{1/2}\left\||T^*|^2 + |T|^4\right\|^{1/2} \le \left\||T|^2 + |T|^4\right\|,$$

which is an improvement of inequality (5).

To obtain our next result, we make also use of the well-known McCarthy inequality (25).

Theorem 3. *For any $B, C \in \mathcal{B}(H)$, $\lambda \in [0,1]$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$w_e^2(B,C) \le \left\||B|^{2\lambda p} + |C|^{2\lambda p}\right\|^{1/p}\left\||B^*|^{2(1-\lambda)q} + |C^*|^{2(1-\lambda)q}\right\|^{1/q} \tag{36}$$

and

$$w_e^2(B,C) \le \left\||B|^{2\lambda p} + |C^*|^{2(1-\lambda)p}\right\|^{1/p}\left\||B^*|^{2(1-\lambda)q} + |C|^{2\lambda q}\right\|^{1/q}. \tag{37}$$

Proof. By Hölder's inequality, we have for $\lambda \in [0,1]$ that

$$\begin{aligned}&\left\langle |B|^{2\lambda}x,x\right\rangle\left\langle |B^*|^{2(1-\lambda)}x,x\right\rangle + \left\langle |C|^{2\lambda}x,x\right\rangle\left\langle |C^*|^{2(1-\lambda)}x,x\right\rangle \\ &\le \left(\left\langle |B|^{2\lambda}x,x\right\rangle^p + \left\langle |C|^{2\lambda}x,x\right\rangle^p\right)^{1/p}\left(\left\langle |B^*|^{2(1-\lambda)}x,x\right\rangle^q + \left\langle |C^*|^{2(1-\lambda)}x,x\right\rangle^q\right)^{1/q}\end{aligned} \tag{38}$$

for $x \in H$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

By utilizing McCarthy's inequality (25), we obtain

$$\begin{aligned}\left\langle |B|^{2\lambda}x,x\right\rangle^p + \left\langle |C|^{2\lambda}x,x\right\rangle^p &\le \left\langle |B|^{2\lambda p}x,x\right\rangle + \left\langle |C|^{2\lambda p}x,x\right\rangle \\ &= \left\langle \left(|B|^{2\lambda p} + |C|^{2\lambda p}\right)x,x\right\rangle\end{aligned}$$

and

$$\begin{aligned}\left\langle |B^*|^{2(1-\lambda)}x,x\right\rangle^q + \left\langle |C^*|^{2(1-\lambda)}x,x\right\rangle^q &\le \left\langle |B^*|^{2(1-\lambda)q}x,x\right\rangle + \left\langle |C^*|^{2(1-\lambda)q}x,x\right\rangle \\ &= \left\langle \left(|B^*|^{2(1-\lambda)q} + |C^*|^{2(1-\lambda)q}\right)x,x\right\rangle\end{aligned}$$

for $x \in H$, $\|x\| = 1$. So, by (38), we obtain

$$\begin{aligned}&\left\langle |B|^{2\lambda}x,x\right\rangle\left\langle |B^*|^{2(1-\lambda)}x,x\right\rangle + \left\langle |C|^{2\lambda}x,x\right\rangle\left\langle |C^*|^{2(1-\lambda)}x,x\right\rangle \\ &\le \left\langle \left(|B|^{2\lambda p} + |C|^{2\lambda p}\right)x,x\right\rangle^{1/p}\left\langle \left(|B^*|^{2(1-\lambda)q} + |C^*|^{2(1-\lambda)q}\right)x,x\right\rangle^{1/q}\end{aligned}$$

for $x \in H$, $\|x\| = 1$.

By making use of (28), we get

$$\begin{aligned}&\left|\left\langle Bx,x\right\rangle\right|^2 + \left|\left\langle Cx,x\right\rangle\right|^2 \\ &\le \left\langle \left(|B|^{2\lambda p} + |C|^{2\lambda p}\right)x,x\right\rangle^{1/p}\left\langle \left(|B^*|^{2(1-\lambda)q} + |C^*|^{2(1-\lambda)q}\right)x,x\right\rangle^{1/q}\end{aligned}$$

for $x \in H$, $\|x\| = 1$.

By taking the supremum over $\|x\| = 1$, we obtain (36).

We also have

$$\langle |B|^{2\lambda}x,x\rangle\langle |B^*|^{2(1-\lambda)}x,x\rangle+\langle |C|^{2\lambda}x,x\rangle\langle |C^*|^{2(1-\lambda)}x,x\rangle$$
$$\le \left(\langle |B|^{2\lambda}x,x\rangle^p+\langle |C^*|^{2(1-\lambda)}x,x\rangle^p\right)^{1/p}\left(\langle |B^*|^{2(1-\lambda)}x,x\rangle^q+\langle |C|^{2\lambda}x,x\rangle^q\right)^{1/q}$$

for $x\in H$ and $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, which gives in a similar way (37). □

Remark 11. *For $p=q=2$ in Theorem 3, we obtain the last upper bounds in Theorem 2, where $\lambda\in[0,1]$.*

For $\lambda=1/2$ in Theorem 3, we obtain

$$w_e^2(B,C)\le \left\||B|^p+|C|^p\right\|^{1/p}\left\||B^*|^q+|C^*|^q\right\|^{1/q}$$

and

$$w_e^2(B,C)\le \left\||B|^p+|C^*|^p\right\|^{1/p}\left\||B^*|^q+|C|^q\right\|^{1/q}$$

for $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Several consequences of Theorem 3 can be derived. As a first step, we present the following corollary.

Corollary 11. *For any $A\in\mathcal{B}(H)$, $\lambda\in[0,1]$ and $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, we have*

$$w^2(A)\le\frac{1}{2}\left\||A|^{2\lambda p}+|A^*|^{2\lambda p}\right\|^{1/p}\left\||A|^{2(1-\lambda)q}+|A^*|^{2(1-\lambda)q}\right\|^{1/q}$$

and

$$w^2(A)\le\frac{1}{2}\left\||A|^{2\lambda p}+|A|^{2(1-\lambda)p}\right\|^{1/p}\left\||A^*|^{2\lambda q}+|A^*|^{2(1-\lambda)q}\right\|^{1/q}.$$

In particular,

$$w^2(A)\le\frac{1}{2}\left\||A|^p+|A^*|^p\right\|^{1/p}\left\||A|^q+|A^*|^q\right\|^{1/q}.$$

Proof. It follows by taking in Theorem 3 $B=A$ and $C=A^*$. □

Our next corollary follows by taking $B=\mathfrak{R}(A)$ and $C=\mathfrak{I}(A)$ in Theorem 3.

Corollary 12. *For any $A\in\mathcal{B}(H)$, $\lambda\in[0,1]$ and $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, we have*

$$w^2(A)\le\left\||\mathfrak{R}(A)|^{2\lambda p}+|\mathfrak{I}(A)|^{2\lambda p}\right\|^{1/p}\left\||\mathfrak{R}(A)|^{2(1-\lambda)q}+|\mathfrak{I}(A)|^{2(1-\lambda)q}\right\|^{1/q}$$

and

$$w_e^2(B,C)\le\left\||\mathfrak{R}(A)|^{2\lambda p}+|\mathfrak{I}(A)|^{2(1-\lambda)p}\right\|^{1/p}\left\||\mathfrak{R}(A)|^{2(1-\lambda)q}+|\mathfrak{I}(A)|^{2\lambda q}\right\|^{1/q}.$$

In particular,

$$w^2(A)\le\left\||\mathfrak{R}(A)|^p+|\mathfrak{I}(A)|^p\right\|^{1/p}\left\||\mathfrak{R}(A)|^q+|\mathfrak{I}(A)|^q\right\|^{1/q}.$$

We derive also the next corollary.

Corollary 13. *For any $T\in\mathcal{B}(H)$, $\lambda\in[0,1]$ and $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, we have*

$$dw^2(T)\le\left\||T|^{2\lambda p}+|T|^{4\lambda p}\right\|^{1/p}\left\||T^*|^{2(1-\lambda)q}+|T|^{4(1-\lambda)q}\right\|^{1/q}$$

and

$$dw^2(T) \le \left\| |T|^{2\lambda p} + |T|^{4(1-\lambda)p} \right\|^{1/p} \left\| |T^*|^{2(1-\lambda)q} + |T|^{4\lambda q} \right\|^{1/q}.$$

In particular,

$$dw^2(T) \le \left\| |T|^{p} + |T|^{2p} \right\|^{1/p} \left\| |T^*|^{q} + |T|^{2q} \right\|^{1/q}.$$

Proof. The proof follows by taking $(B,C) = \left(T, |T|^2\right)$ in Theorem 3. □

We also have the following theorem.

Theorem 4. *For any $B, C \in \mathcal{B}(H)$ and $\lambda \in [0,1]$, we have*

$$w_e^2(B,C) \le \frac{1}{2}\left[\left\| |B|^{2\lambda} + |C|^{2\lambda} \right\| + \left\| \left| |B|^{2\lambda} - |C|^{2\lambda} \right| \right\|\right] \left\| |B^*|^{2(1-\lambda)} + |C^*|^{2(1-\lambda)} \right\| \tag{39}$$

and

$$w_e^2(B,C) \le \frac{1}{2}\left[\left\| |B|^{2\lambda} + |C^*|^{2(1-\lambda)} \right\| + \left\| \left| |B|^{2\lambda} - |C^*|^{2(1-\lambda)} \right| \right\|\right] \left\| |B^*|^{2(1-\lambda)} + |C|^{2\lambda} \right\|. \tag{40}$$

Proof. One can see that

$$\begin{aligned}
&\langle |B|^{2\lambda}x, x\rangle \langle |B^*|^{2(1-\lambda)}x, x\rangle + \langle |C|^{2\lambda}x, x\rangle \langle |C^*|^{2(1-\lambda)}x, x\rangle \\
&\le \max\left\{\langle |B|^{2\lambda}x, x\rangle, \langle |C|^{2\lambda}x, x\rangle\right\}\left[\langle |B^*|^{2(1-\lambda)}x, x\rangle + \langle |C^*|^{2(1-\lambda)}x, x\rangle\right] \\
&= \frac{1}{2}\left[\langle |B|^{2\lambda}x, x\rangle + \langle |C|^{2\lambda}x, x\rangle + \left|\langle |B|^{2\lambda}x, x\rangle - \langle |C|^{2\lambda}x, x\rangle\right|\right]\left[\langle |B^*|^{2(1-\lambda)}x, x\rangle + \langle |C^*|^{2(1-\lambda)}x, x\rangle\right] \\
&= \frac{1}{2}\left[\left\langle \left(|B|^{2\lambda} + |C|^{2\lambda}\right)x, x\right\rangle + \left|\left\langle \left(|B|^{2\lambda} - |C|^{2\lambda}\right)x, x\right\rangle\right|\right]\left\langle \left(|B^*|^{2(1-\lambda)} + |C^*|^{2(1-\lambda)}\right)x, x\right\rangle
\end{aligned}$$

for $x \in H$.

From (28), we obtain

$$\begin{aligned}
&|\langle Bx, x\rangle|^2 + |\langle Cx, x\rangle|^2 \\
&\le \frac{1}{2}\left[\left\langle \left(|B|^{2\lambda} + |C|^{2\lambda}\right)x, x\right\rangle + \left|\left\langle \left(|B|^{2\lambda} - |C|^{2\lambda}\right)x, x\right\rangle\right|\right]\left\langle \left(|B^*|^{2(1-\lambda)} + |C^*|^{2(1-\lambda)}\right)x, x\right\rangle
\end{aligned}$$

for $x \in H$.

By taking the supremum over $x \in H$, $\|x\| = 1$, we obtain the desired result (39).

Inequality (40) follows by the fact that

$$\begin{aligned}
&\langle |B|^{2\lambda}x, x\rangle \langle |B^*|^{2(1-\lambda)}x, x\rangle + \langle |C|^{2\lambda}x, x\rangle \langle |C^*|^{2(1-\lambda)}x, x\rangle \\
&\le \max\left\{\langle |B|^{2\lambda}x, x\rangle, \langle |C^*|^{2(1-\lambda)}x, x\rangle\right\}\left[\langle |B^*|^{2(1-\lambda)}x, x\rangle + \langle |C|^{2\lambda}x, x\rangle\right]
\end{aligned}$$

for $x \in H$. □

Remark 12. *If we take $\lambda = 1/2$ in Theorem 4, we obtain*

$$w_e^2(B,C) \le \frac{1}{2}[\||B| + |C|\| + \||B| - |C|\|]\||B^*| + |C^*|\|$$

and

$$w_e^2(B,C) \le \frac{1}{2}[\||B| + |C^*|\| + \||B| - |C^*|\|]\||B^*| + |C|\|.$$

The first consequence of Theorem 4 is presented in the following corollary.

Corollary 14. *For any $A \in \mathcal{B}(H)$ and $\lambda \in [0,1]$, we have*

$$w^2(A) \le \frac{1}{4}\left[\left\| |A|^{2\lambda} + |A^*|^{2\lambda} \right\| + \left\| |A|^{2\lambda} - |A^*|^{2\lambda} \right\|\right] \left\| |A^*|^{2(1-\lambda)} + |A|^{2(1-\lambda)} \right\|$$

and

$$w^2(A) \le \frac{1}{4}\left[\left\| |A|^{2\lambda} + |A|^{2(1-\lambda)} \right\| + \left\| |A|^{2\lambda} - |A|^{2(1-\lambda)} \right\|\right] \left\| |A^*|^{2(1-\lambda)} + |A^*|^{2\lambda} \right\|.$$

In particular,

$$w^2(A) \le \frac{1}{4}[\| |A| + |A^*| \| + \| |A| - |A^*| \|] \| |A^*| + |A| \|.$$

Proof. The proof follows by taking $B = A$ and $C = A^*$ in Theorem 4. □

The next corollary holds also.

Corollary 15. *For any $A \in \mathcal{B}(H)$, $\lambda \in [0,1]$, we have*

$$w^2(A) \le \frac{1}{2}\left[\left\| |\Re(A)|^{2\lambda} + |\Im(A)|^{2\lambda} \right\| + \left\| |\Re(A)|^{2\lambda} - |\Im(A)|^{2\lambda} \right\|\right] \left\| |\Re(A)|^{2(1-\lambda)} + |\Im(A)|^{2(1-\lambda)} \right\|$$

and

$$w^2(A) \le \frac{1}{2}\left[\left\| |\Re(A)|^{2\lambda} + |\Im(A)|^{2(1-\lambda)} \right\| + \left\| |\Re(A)|^{2\lambda} - |\Im(A)|^{2(1-\lambda)} \right\|\right] \left\| |\Re(A)|^{2(1-\lambda)} + |\Im(A)|^{2\lambda} \right\|.$$

In particular,

$$w^2(A) \le \frac{1}{2}[\| |\Re(A)| + |\Im(A)| \| + \| |\Re(A)| - |\Im(A)| \|] \| |\Re(A)| + |\Im(A)| \|.$$

Proof. The proof follows by taking $B = \Re(A)$ and $C = \Im(A)$ in Theorem 4. □

Finally, by selecting $B = T$ and $C = |T|^2$ in Theorem 4, we can also state the following corollary.

Corollary 16. *For any $T \in \mathcal{B}(H)$, $\lambda \in [0,1]$, we have*

$$dw^2(T) \le \frac{1}{2}\left[\left\| |T|^{2\lambda} + |T|^{4\lambda} \right\| + \left\| |T|^{2\lambda} - |T|^{4\lambda} \right\|\right] \left\| |T^*|^{2(1-\lambda)} + |T|^{4(1-\lambda)} \right\|$$

and

$$dw^2(T) \le \frac{1}{2}\left[\left\| |T|^{2\lambda} + |T|^{4(1-\lambda)} \right\| + \left\| |T|^{2\lambda} - |T|^{4(1-\lambda)} \right\|\right] \left\| |T^*|^{2(1-\lambda)} + |T|^{4\lambda} \right\|.$$

In particular

$$dw^2(T) \le \frac{1}{2}\left[\left\| |T| + |T|^2 \right\| + \left\| |T| - |T|^2 \right\|\right] \left\| |T^*| + |T|^2 \right\|.$$

4. Conclusions

In this paper, we have presented new lower and upper bounds for the Euclidean numerical radius of operator pairs in Hilbert spaces, improving upon recent results in the literature. We also derived new inequalities for the numerical radius and the Davis–Wielandt radius as natural consequences of our findings.

This work lays a strong foundation for future research in this area. In particular, extending these results to the setting of semi-Hilbert spaces would be a valuable direction to explore (see [22–24] for a solid background on operators in semi-Hilbert spaces). Another interesting direction for future work is to study the Berezin number of operator pairs in reproducing kernel Hilbert spaces (see [25,26] and the sources cited therein for background on reproducing kernel Hilbert spaces).

We hope this study will inspire further advancements in numerical radius inequalities and related concepts in operator theory.

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References

1. Kittaneh, F. Numerical radius inequalities for Hilbert space operators. *Studia Math.* **2005**, *168*, 73–80. [CrossRef]
2. Bag, S.; Bhunia, P.; Paul, K. Bounds of numerical radius of bounded linear operators using t-Aluthge transform. *Math. Inequal. Appl.* **2020**, *23*, 991–1004. [CrossRef]
3. Bhunia, P.; Paul, K. Furtherance of numerical radius inequalities of Hilbert space operators. *Arch. Math.* **2021**, *117*, 537–546. [CrossRef]
4. Bhunia, P.; Paul, K. Development of inequalities and characterization of equality conditions for the numerical radius. *Linear Algebra Appl.* **2021**, *630*, 306–315. [CrossRef]
5. Bhunia, P.; Paul, K. Proper improvement of well-known numerical radius inequalities and their applications. *Results Math.* **2021**, *76*, 177. [CrossRef]
6. Bhunia, P.; Paul, K. New upper bounds for the numerical radius of Hilbert space operators. *Bull. Sci. Math.* **2021**, *167*, 102959. [CrossRef]
7. Kittaneh, F. A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix. *Studia Math.* **2003**, *158*, 11–17. [CrossRef]
8. Yamazaki, T. On upper and lower bounds for the numerical radius and an equality condition. *Studia Math.* **2007**, *178*, 83–89. [CrossRef]
9. Popescu, G. Unitary invariants in multivariable operator theory. In *Memoirs of the American Mathematical Society*; American Mathematical Society: Providence, RI, USA, 2009; Volume 200, 91p.
10. Dragomir, S.S. Some inequalities for the Euclidean operator radius of two operators in Hilbert spaces. *Linear Algebra Its Appl.* **2006**, *419*, 256–264. [CrossRef]
11. Ammar, A.; Frakis, A.; Kittaneh, F. New bounds for the Euclidean operator radius of two Hilbert space operators with applications. *Bol. Soc. Mat. Mex.* **2024**, *30*, 45. [CrossRef]
12. Jana, S.; Bhunia, P.; Paul, K. Euclidean operator radius inequalities of a pair of bounded linear operators and their applications. *Bull. Braz. Math. Soc. New Ser.* **2023**, *54*, 1. [CrossRef]
13. Jana, S.; Bhunia, P.; Paul, K. Euclidean operator radius and numerical radius inequalities. *arXiv* **2023**, arXiv:2308.09252.
14. Davis, C. The shell of a Hilbert-space operator. *Acta Sci. Math.* **1968**, *29*, 69–86.
15. Wielandt, H. On eigenvalues of sums of normal matrices. *Pac. J. Math.* **1955**, *5*, 633–638. [CrossRef]
16. Bhunia, P.; Bhanja, A.; Bag, S.; Paul, K. Bounds for the Davis-Wielandt radius of bounded linear operators. *Ann. Funct. Anal.* **2021**, *12*, 18. [CrossRef]
17. Zamani, A.; Shebrawi, K. Some upper bounds for the Davis-Wielandt radius of Hilbert space operators. *Mediterr. J. Math.* **2020**, *17*, 25. [CrossRef]
18. Bhunia, P.; Paul, K.; Batik, S. Further refinemenmts of Davis-Wieland radius inequalities. *Oper. Matrices* **2023**, *17*, 767–778. [CrossRef]
19. Hirzallah, O.; Kittaneh, F.; Shebrawi, K. Numerical radius inequalities for certain 2×2 operator matrices. *Integr. Equ. Oper. Theory* **2011**, *71*, 129–147. [CrossRef]
20. Kato, T. Notes on some inequalities for linear operators. *Math. Ann.* **1952**, *125*, 208–212. [CrossRef]
21. Sattari, M.; Moslehian, M.S.; Yamazaki, T. Some generalized numerical radius inequalities for Hilbert space operators. *Linear Algebra Appl.* **2015**, *470*, 216–227. [CrossRef]

22. Arias, M.L.; Corach, G.; Gonzalez, M.C. Partial isometries in semi-Hilbertian spaces. *Linear Algebra Appl.* **2008**, *428*, 1460–1475. [CrossRef]
23. Arias, M.L.; Corach, G.; Gonzalez, M.C. Metric properties of projections in semi-Hilbertian spaces. *Integral Equ. Oper. Theory* **2008**, *62*, 11–28. [CrossRef]
24. Arias, M.L.; Corach, G.; Gonzalez, M.C. Lifting properties in operator ranges. *Acta Sci. Math.* **2009**, *75*, 635–653.
25. Altwaijry, N.; Feki, K.; Minculete, N. Some new estimates for the Berezin number of Hilbert space operators. *Axioms* **2022**, *11*, 683. [CrossRef]
26. Guesba, M. Some Euclidean Berezin number inequalities of a pair of operators and their applications. *Filomat* **2023**, *37*, 8777–8790. [CrossRef]