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Power Bounds for the Numerical Radius of the Off-Diagonal 2×2 Operator Matrix

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Abstract: In this paper, we employ a generalization of the Boas–Bellman inequality for inner products, as developed by Mitrinović–Pečarić–Fink, to derive several upper bounds for the $2p$-th power with $p \geq 1$ of the numerical radius of the off-diagonal operator matrix $\begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}$ for any bounded linear operators A and B on a complex Hilbert space $\mathcal{H}$. While the general matrix is not symmetric, a special case arises when $B = A^*$, where the matrix becomes symmetric. This symmetry plays a crucial role in the derivation of our bounds, illustrating the importance of symmetric structures in operator theory.

Keywords: power inequalities; operator norm; numerical radius; off-diagonal 2×2-operator matrix; Hilbert space

MSC: 46C05; 47A63; 47A99; 15A39; 26D15

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Citation: Altwaijry, N.; Dragomir, S.S; Feki, K. Power Bounds for the Numerical Radius of the Off-Diagonal 2×2 Operator Matrix. *Symmetry* **2024**, *16*, 1199. https://doi.org/10.3390/sym16091199

Academic Editor: Sergei D. Odintsov

Received: 17 August 2024
Revised: 4 September 2024
Accepted: 9 September 2024
Published: 12 September 2024

1. Introduction

Block matrix techniques play a pivotal role in matrix analysis and operator theory, offering theorems that characterize the properties of a matrix A, such as its norm, in terms of the properties of a larger matrix, in which A is embedded as a block. In areas like the theory of the Schur complement and the study of positive linear maps, particularly relevant in quantum information, 2×2 block matrices are fundamental. Key theorems in these areas are framed in terms of such block matrices (see, e.g., [1] and the survey article [2] in the book [3]).

This present article aims to illustrate this technique through a specific problem concerning the numerical radius. By utilizing a generalization of the Boas–Bellman inequality for inner products, as developed by Mitrinović–Pečarić–Fink, we derive several power bounds for the numerical radius of the off-diagonal 2×2 operator matrix.

Before presenting our results, we introduce relevant notions and notations. Throughout this paper, we consider $\mathcal{H}$ as a complex Hilbert space equipped with an inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$. Specifically, $\|x\| = \sqrt{\langle x, x \rangle}$ for all $x \in \mathcal{H}$. We denote the C^*-algebra of bounded linear operators on $\mathcal{H}$ as $\mathcal{B}(\mathcal{H})$. Let $A \in \mathcal{B}(\mathcal{H})$. The adjoint of A is denoted by A^*, and the positive square root of A^*A is expressed as $|A| = (A^*A)^{\frac{1}{2}}$. The real part and imaginary part of $A \in \mathcal{B}(\mathcal{H})$ are defined as $\mathrm{Re}(A) = \frac{1}{2}(A + A^*)$ and $\mathrm{Im}(A) = \frac{1}{2\mathrm{i}}(A - A^*)$, respectively. Furthermore, if $A \in \mathcal{B}(\mathcal{H})$, then the *norm* of A is defined as

$$\|A\| = \sup_{\|x\|=1} \|Ax\| = \sup_{\|x\|=\|y\|=1} |\langle Ax, y \rangle|.$$

while the numerical radius of A is given by

$$w(A) = \sup_{\|x\|=1} |\langle Ax, x \rangle|.$$

The numerical radius is also a norm on $\mathcal{B}(\mathcal{H})$. Comparing the numerical radius and the operator norm, we have the inequalities:

$$w(A) \leq \|A\| \leq 2w(A), \tag{1}$$

which hold for every $A \in \mathcal{B}(\mathcal{H})$. For further details, see, e.g., [4–10].

Kittaneh [11] established an improved version of these inequalities, demonstrating the following:

$$\frac{1}{2}\sqrt{\|A^*A + AA^*\|} \leq w(A) \leq \frac{\sqrt{2}}{2}\sqrt{\|A^*A + AA^*\|}. \tag{2}$$

To explore further progress regarding (1) and (2), interested individuals may consult [12–18].

Let us examine the off-diagonal part, denoted as $\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}$, of a 2×2 operator matrix $\begin{bmatrix} Z & X \\ Y & W \end{bmatrix}$ defined on the direct sum space $\mathcal{H} \oplus \mathcal{H}$.

For every $X, Y \in \mathcal{B}(\mathcal{H})$, it is well established that

$$w\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) = w\left(\begin{bmatrix} 0 & Y \\ X & 0 \end{bmatrix}\right),$$

$$w\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) = w\left(\begin{bmatrix} 0 & X \\ e^{i\alpha}Y & 0 \end{bmatrix}\right) \text{ for } \alpha \in \mathbb{R},$$

$$w\left(\begin{bmatrix} 0 & Y \\ Y & 0 \end{bmatrix}\right) = w(Y) \text{ and } w\left(\begin{bmatrix} 0 & Y \\ Y^* & 0 \end{bmatrix}\right) = \|Y\|.$$

In 2011, Hirzallah, Kittaneh, and Shebrawi [19] established several results, including the following double inequality:

$$\begin{aligned} \frac{1}{2}\max\{w(X+Y), w(X-Y)\} &\leq w\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) \\ &\leq \frac{1}{2}[w(X+Y) + w(X-Y)]. \end{aligned}$$

Additionally, they demonstrated that

$$w\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) \leq \min\{w(X), w(Y)\} + \frac{1}{2}\min\{\|X+Y\|, \|X-Y\|\}.$$

Furthermore, they derived several other notable inequalities of this kind.

In a significant publication by Kittaneh, Moslehian, and Yamazaki [20], they presented a refined version of the triangle inequality, which can be expressed as follows:

$$\left\|\frac{X+Y}{2}\right\| \leq w\left(\begin{bmatrix} 0 & X \\ Y^* & 0 \end{bmatrix}\right) \leq \frac{\|X\| + \|Y\|}{2}$$

for all $X, Y \in \mathcal{B}(\mathcal{H})$. Several upper and lower bounds for the numerical radius of the operator matrix

$$\begin{bmatrix} 0 & AX - XB \\ A^*X - XB^* & 0 \end{bmatrix}$$

were provided, considering different assumptions for the operators involved.

From a different perspective, in 2022, Bhunia and Paul obtained the following result, as stated in ([21], Lemma 2.4):

$$\omega^2\left(\begin{bmatrix} 0 & X \\ Y^* & 0 \end{bmatrix}\right) \leq \frac{1}{2}\left\|\frac{|X|^2+|Y|^2}{2}\right\| + \frac{1}{2}\omega(Y^*X).$$

Motivated by the above results, in this paper, by utilizing a generalization of the Boas–Bellman inequality for inner products due to Mitrinović–Pečarić–Fink ([22], p. 392), we obtain several upper bounds for the power $2p$ of the numerical radius of the off-diagonal operator matrix $\begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}$ for any operators $A, B \in \mathcal{B}(\mathcal{H})$ and $p \geq 1$. Applications in the case when $B = A^*$ are also provided. Although the matrix $\begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}$ is typically non-symmetric, it becomes symmetric in the special case where $B = A^*$. This symmetry is essential in deriving our bounds, highlighting the significance of symmetric structures in operator theory.

A key aspect of our results lies in providing general power inequalities that offer more comprehensive and diverse upper bounds for the numerical radius of off-diagonal 2×2 operator matrices compared to those presented in recent papers [19–21]. Since our bounds for the specific case $p = 1$ have distinct analytic expressions from those in [19,21], they are not directly comparable.

The novelty of our approach is in the application of a Mitrinović–Pečarić–Fink generalization of the well-known Boas–Bellman inequality for two vectors, a technique not previously employed by other authors to derive such bounds.

As highlighted in [5], upper bounds for the numerical radius can be useful in approximating polynomial solutions or estimating the numerical radius of the Frobenius companion matrix, as discussed in [17]. However, these applications require extensive research beyond the scope of this work and may be explored in future papers.

2. Some Preliminary Facts

In this section, we recall and establish some preliminary results that will be used in this work. We begin with the following generalization of Bessel's inequality, which states that if $x, y_1, \ldots, y_n$ are elements of $\mathcal{H}$, then the following inequality holds:

$$\sum_{i=1}^{n} |\langle x, y_i\rangle|^2 \leq \|x\|^2 \left[\max_{1\leq i\leq n} \|y_i\|^2 + \left(\sum_{1\leq i\neq j\leq n} |\langle y_i, y_j\rangle|^2\right)^{\frac{1}{2}}\right]. \tag{3}$$

The inequality (3) is known as the Boas–Bellman inequality (see [23,24]).

A generalization of the Boas–Bellman inequality was provided by Mitrinović–Pečarić–Fink ([22], p. 392). They proved that, if $x, y_1, \ldots, y_n$ are vectors in $\mathcal{H}$ and $c_1, \ldots, c_n \in \mathbb{C}$, then the following inequality holds:

$$\left|\sum_{i=1}^{n} c_i\langle x, y_i\rangle\right|^2 \leq \|x\|^2 \sum_{i=1}^{n} |c_i|^2 \left[\max_{1\leq i\leq n} \|y_i\|^2 + \left(\sum_{1\leq i\neq j\leq n} |\langle y_i, y_j\rangle|^2\right)^{\frac{1}{2}}\right]. \tag{4}$$

For $n = 2$ in (4), we have

$$\begin{aligned} &|c_1\langle x, y_1\rangle + c_2\langle x, y_2\rangle|^2 \\ &\leq \|x\|^2\left(|c_1|^2+|c_2|^2\right)\left[\max\left\{\|y_1\|^2, \|y_2\|^2\right\} + \sqrt{2}|\langle y_1, y_2\rangle|\right] \end{aligned} \tag{5}$$

for complex numbers c_1, c_2 and vectors $x, y_1, y_2 \in \mathcal{H}$.

By using this result, we can establish the following operator norm inequalities:

Lemma 1. *Let $\gamma, \delta \in \mathbb{C}$, $U, V \in \mathcal{B}(\mathcal{H})$ and $p \geq 1$. Then, we have*

$$\|\gamma U + \delta V\|^{2p} \leq 2^{p-1}\left(|\gamma|^2 + |\delta|^2\right)^p \left[2^{p/2} w^p(V^*U) + \begin{cases} \max\left\{\|U\|^{2p}, \|V\|^{2p}\right\}, \\ \frac{\left\||U|^2+|V|^2\right\|^p + \left\||U|^2-|V|^2\right\|^p}{2}. \end{cases}\right] \quad (6)$$

Proof. If we take in (5) $c_1 = \bar{\gamma}$, $\delta = \bar{\delta}$, $y_1 = Uy$ and $y_2 = Vy$, then we get

$$\begin{aligned} &|\langle x, (\gamma U + \delta V)y\rangle|^2 \\ &\leq \|x\|^2\left(|\gamma|^2 + |\delta|^2\right)\left[\max\left\{\|Uy\|^2, \|Vy\|^2\right\} + \sqrt{2}|\langle V^*Uy, y\rangle|\right] \\ &= \|x\|^2\left(|\gamma|^2 + |\delta|^2\right) \\ &\times \left[\frac{1}{2}\left[\left\langle\left(|U|^2 + |V|^2\right)y, y\right\rangle + \left|\left\langle\left(|U|^2 - |V|^2\right)y, y\right\rangle\right|\right] + \sqrt{2}|\langle V^*Uy, y\rangle|\right] \end{aligned}$$

since

$$\begin{aligned} \max\left\{\|Uy\|^2, \|Vy\|^2\right\} &= \max\left\{\left\langle|U|^2 y, y\right\rangle, \left\langle|V|^2 y, y\right\rangle\right\} \\ &= \frac{1}{2}\left[\left\langle|U|^2 y, y\right\rangle + \left\langle|V|^2 y, y\right\rangle\right] + \frac{1}{2}\left|\left\langle|U|^2 y, y\right\rangle - \left\langle|V|^2 y, y\right\rangle\right| \\ &= \frac{1}{2}\left\langle\left(|U|^2 + |V|^2\right)y, y\right\rangle + \frac{1}{2}\left|\left\langle\left(|U|^2 - |V|^2\right)y, y\right\rangle\right| \end{aligned}$$

for all $x, y \in \mathcal{H}$.

If we take the power $p \geq 1$, then we get

$$\begin{aligned} &|\langle x, (\gamma U + \delta V)y\rangle|^{2p} \\ &\leq \|x\|^{2p}\left(|\gamma|^2 + |\delta|^2\right)^p\left[\max\left\{\|Uy\|^2, \|Vy\|^2\right\} + \sqrt{2}|\langle V^*Uy, y\rangle|\right]^p \\ &= \|x\|^{2p}\left(|\gamma|^2 + |\delta|^2\right)^p \\ &\times \left[\frac{1}{2}\left[\left\langle\left(|U|^2 + |V|^2\right)y, y\right\rangle + \left|\left\langle\left(|U|^2 - |V|^2\right)y, y\right\rangle\right|\right] + \sqrt{2}|\langle V^*Uy, y\rangle|\right]^p \end{aligned} \quad (7)$$

for all $x, y \in \mathcal{H}$.

Using the elementary inequality that follows from the convexity of the power function,

$$(m+n)^p \leq 2^{p-1}(m^p + n^p), \quad m, n \geq 0, \quad p \geq 1,$$

we get

$$\begin{aligned} &\left[\max\left\{\|Uy\|^2, \|Vy\|^2\right\} + \sqrt{2}|\langle V^*Uy, y\rangle|\right]^p \\ &\leq 2^{p-1}\left[\max\left\{\|Uy\|^{2p}, \|Vy\|^{2p}\right\} + 2^{p/2}|\langle V^*Uy, y\rangle|^p\right] \end{aligned}$$

and

$$\begin{aligned} &\left[\frac{1}{2}\left[\left\langle\left(|U|^2 + |V|^2\right)y, y\right\rangle + \left|\left\langle\left(|U|^2 - |V|^2\right)y, y\right\rangle\right|\right] + \sqrt{2}|\langle V^*Uy, y\rangle|\right]^p \\ &\leq 2^{p-1}\left\{\left[\frac{\left\langle\left(|U|^2 + |V|^2\right)y, y\right\rangle + \left|\left\langle\left(|U|^2 - |V|^2\right)y, y\right\rangle\right|}{2}\right]^p + 2^{p/2}|\langle V^*Uy, y\rangle|^p\right\} \\ &\leq 2^{p-1}\left\{\frac{\left\langle\left(|U|^2 + |V|^2\right)y, y\right\rangle^p + \left|\left\langle\left(|U|^2 - |V|^2\right)y, y\right\rangle\right|^p}{2} + 2^{p/2}|\langle V^*Uy, y\rangle|^p\right\} \end{aligned}$$

for all $x, y \in \mathcal{H}$.

By utilizing (7) we derive

$$|\langle x, (\gamma U + \delta V)y\rangle|^{2p} \leq 2^{3p/2-1}\|x\|^{2p}\left(|\gamma|^2 + |\delta|^2\right)^p |\langle V^*Uy, y\rangle|^p + 2^{p-1}\|x\|^{2p}\left(|\gamma|^2 + |\delta|^2\right)^p \times \begin{cases} \max\left\{\|Uy\|^{2p}, \|Vy\|^{2p}\right\}, \\ \frac{\langle(|U|^2+|V|^2)y,y\rangle^p + |\langle(|U|^2-|V|^2)y,y\rangle|^p}{2} \end{cases}$$

for all $x, y \in \mathcal{H}$.

Therefore, by taking the supremum over $x, y \in \mathcal{H}$ with $\|x\| = \|y\| = 1$, we derive

$$\begin{aligned} &\|(\gamma U + \delta V)\|^{2p} \\ &= \sup_{\|x\|=\|y\|=1} |\langle x, (\gamma U + \delta V)y\rangle|^{2p} \\ &\leq 2^{3p/2-1}\left(|\gamma|^2 + |\delta|^2\right)^p \sup_{\|y\|=1} |\langle V^*Uy, y\rangle|^p \\ &+ 2^{p-1}\left(|\gamma|^2 + |\delta|^2\right)^p \begin{cases} \sup_{\|y\|=1} \max\left\{\|Uy\|^{2p}, \|Vy\|^{2p}\right\}, \\ \sup_{\|y\|=1}\left[\frac{\langle(|U|^2+|V|^2)y,y\rangle^p + |\langle(|U|^2-|V|^2)y,y\rangle|^p}{2}\right] \end{cases} \\ &\leq 2^{3p/2-1}\left(|\gamma|^2 + |\delta|^2\right)^p w^p(V^*U) + 2^{p-1}\left(|\gamma|^2 + |\delta|^2\right)^p \times \begin{cases} \max\left\{\|U\|^{2p}, \|V\|^{2p}\right\}, \\ \frac{\||U|^2+|V|^2\|^p + \||U|^2-|V|^2\|^p}{2}, \end{cases} \end{aligned}$$

which proves (6). □

Corollary 1. *With the assumptions of Lemma 1, we have*

$$\|\gamma U + \delta V\|^2 \leq \left(|\gamma|^2 + |\delta|^2\right)\left[\sqrt{2}w(V^*U) + \begin{cases} \max\left\{\|U\|^2, \|V\|^2\right\}, \\ \frac{\||U|^2+|V|^2\| + \||U|^2-|V|^2\|}{2} \end{cases}\right]$$

and

$$\|\gamma U + \delta V\|^4 \leq 2\left(|\gamma|^2 + |\delta|^2\right)^2\left[2w^2(V^*U) + \begin{cases} \max\left\{\|U\|^4, \|V\|^4\right\}, \\ \frac{\||U|^2+|V|^2\|^2 + \||U|^2-|V|^2\|^2}{2}. \end{cases}\right]$$

3. Inequalities for Off-Diagonal Operator 2 × 2 Matrix

In this section, we aim to establish some inequalities for the off-diagonal operator 2×2 matrix. Our first investigation focuses on stating a result concerning upper bounds for the power of the numerical radius of the operator matrix. Before stating it, we need to recall from [20] the following lemma.

Lemma 2. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

$$w\left(\begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}\right) = \frac{1}{2}\sup_{\alpha\in\mathbb{R}}\left\|e^{i\alpha}A + e^{-i\alpha}B\right\|. \tag{8}$$

Now, we can prove the following result.

Theorem 1. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and $p \geq 1$, then*

$$w^{2p}\left(\begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}\right) \leq \frac{1}{2}\left[2^{p/2}w^p(B^*A) + \begin{cases} \max\left\{\|A\|^{2p}, \|B\|^{2p}\right\}, \\ \frac{\left\||A|^2+|B|^2\right\|^p + \left\||A|^2-|B|^2\right\|^p}{2} \end{cases}\right] \tag{9}$$

and

$$w^{2p}\left(\begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}\right) \tag{10}$$
$$\leq \frac{1}{2}\left[\frac{1}{2^{p/2}}w^p\left((A-B)^*(A+B)\right) + \begin{cases} \frac{1}{2^p}\max\left\{\|A+B\|^{2p}, \|A-B\|^{2p}\right\}\right], \\ \frac{1}{2}\left(\left\||A|^2+|B|^2\right\|^p + 2^p\|Re(B^*A)\|^p\right)\right]. \end{cases}$$

Proof. In (6) choose $\gamma = e^{i\alpha}$, $\delta = e^{-i\alpha}$, $U = A$ and $V = B$, then

$$\left\|e^{i\alpha}A + e^{-i\alpha}B\right\|^{2p} \leq 2^{5p/2-1}w^p(B^*A) + 2^{2p-1} \times \begin{cases} \max\left\{\|A\|^{2p}, \|B\|^{2p}\right\}, \\ \frac{\left\||A|^2+|B|^2\right\|^p + \left\||A|^2-|B|^2\right\|^p}{2} \end{cases}$$

and dividing by 2^{2p}, we obtain

$$\frac{1}{2^{2p}}\left\|e^{i\alpha}A + e^{-i\alpha}B\right\|^{2p} \leq 2^{p/2-1}w^p(B^*A) + \frac{1}{2} \times \begin{cases} \max\left\{\|A\|^{2p}, \|B\|^{2p}\right\}, \\ \frac{\left\||A|^2+|B|^2\right\|^p + \left\||A|^2-|B|^2\right\|^p}{2} \end{cases}$$

Therefore, by (8), we deduce (9).

Further, observe that

$$\begin{aligned} e^{i\alpha}A + e^{-i\alpha}B &= (\cos\alpha + i\sin\alpha)A + (\cos\alpha - i\sin\alpha)B \\ &= \cos\alpha(A+B) + i\sin\alpha(A-B) \end{aligned} \tag{11}$$

for $\alpha \in \mathbb{R}$.

Now, if we take in (6) $\gamma = \cos\alpha$, $\delta = i\sin\alpha$, $U = A+B$ and $V = A-B$, then we get

$$\left\|e^{i\alpha}A + e^{-i\alpha}B\right\|^{2p} \tag{12}$$
$$\leq 2^{3p/2-1}w^p\left((A-B)^*(A+B)\right) + 2^{p-1} \times \begin{cases} \max\left\{\|A+B\|^{2p}, \|A-B\|^{2p}\right\}, \\ \frac{\left\||A+B|^2+|A-B|^2\right\|^p + \left\||A+B|^2-|A-B|^2\right\|^p}{2}. \end{cases}$$

Observe that

$$|A+B|^2 + |A-B|^2 = 2\left(|A|^2 + |B|^2\right)$$

and

$$|A+B|^2 - |A-B|^2 = 4\mathrm{Re}(B^*A)$$

and by (12), we obtain

$$\begin{aligned}&\left\|e^{i\alpha}A+e^{-i\alpha}B\right\|^{2p}\\&\le 2^{3p/2-1}w^{p}\left((A-B)^{*}(A+B)\right)+2^{p-1}\times\begin{cases}\max\left\{\|A+B\|^{2p},\|A-B\|^{2p}\right\},\\ \frac{2^{p}\left\||A|^{2}+|B|^{2}\right\|^{p}+2^{2p}\|\mathrm{Re}(B^{*}A)\|^{p}}{2}\end{cases}\end{aligned}$$

then dividing by 2^{2p} we get the desired inequality (10). □

We observe that for $p=1$ in (10), we obtain

$$\begin{aligned}&w^{2}\left(\begin{bmatrix}0&A\\B^{*}&0\end{bmatrix}\right)\\&\le\frac{\sqrt{2}}{4}w\left((A-B)^{*}(A+B)\right)+\frac{1}{2}\times\begin{cases}\frac{1}{2}\max\left\{\|A+B\|^{2},\|A-B\|^{2}\right\},\\ \frac{\left\||A|^{2}+|B|^{2}\right\|+2\|\mathrm{Re}(B^{*}A)\|}{2},\end{cases}\end{aligned}$$

while for $p=2$, we get

$$\begin{aligned}&w^{4}\left(\begin{bmatrix}0&A\\B^{*}&0\end{bmatrix}\right)\\&\le\frac{1}{4}w^{2}\left((A-B)^{*}(A+B)\right)+\frac{1}{2}\times\begin{cases}\frac{1}{4}\max\left\{\|A+B\|^{4},\|A-B\|^{4}\right\},\\ \frac{\left\||A|^{2}+|B|^{2}\right\|^{2}+4\|\mathrm{Re}(B^{*}A)\|^{2}}{2}.\end{cases}\end{aligned}$$

Another result is as follows:

Theorem 2. *Let $A,B\in\mathcal{B}(\mathcal{H})$ and $p\ge 1$, then*

$$w^{2p}\left(\begin{bmatrix}0&A\\B^{*}&0\end{bmatrix}\right)\le\frac{1}{2}\left\|\frac{|A|^{2}+|B|^{2}}{2}\right\|^{p}+\frac{\sqrt{2}}{4}\sqrt{\ell_{1}(A,B,p)+\ell_{2}(A,B,p)},\tag{13}$$

where

$$\ell_{1}(A,B,p):=2^{p/2}w^{p}\left((B^{*}A)^{2}\right)$$

and

$$\ell_{2}(A,B,p):=\begin{cases}\|B^{*}A\|^{2p},\\ \frac{\left\||B^{*}A|^{2}+|A^{*}B|^{2}\right\|^{p}+\left\||B^{*}A|^{2}-|A^{*}B|^{2}\right\|^{p}}{2}.\end{cases}$$

Proof. Consider

$$H_{\alpha,A,B}:=\frac{e^{i\alpha}A+e^{-i\alpha}B}{2},\text{ for }\alpha\in\mathbb{R}.$$

Observe that

$$\begin{aligned}4|H_{\alpha,A,B}|^{2}&=\left|e^{i\alpha}A+e^{-i\alpha}B\right|^{2}=\left(e^{i\alpha}A+e^{-i\alpha}B\right)^{*}\left(e^{i\alpha}A+e^{-i\alpha}B\right)\\&=\left(e^{-i\alpha}A^{*}+e^{i\alpha}B^{*}\right)\left(e^{i\alpha}A+e^{-i\alpha}B\right)\\&=A^{*}A+B^{*}B+e^{2i\alpha}B^{*}A+e^{-2i\alpha}A^{*}B\\&=|A|^{2}+|B|^{2}+e^{2i\alpha}B^{*}A+e^{-2i\alpha}A^{*}B\end{aligned}$$

which implies that

$$|H_{\alpha,A,B}|^2 = \frac{1}{2}\left(\frac{|A|^2+|B|^2}{2} + \frac{e^{2i\alpha}B^*A + e^{-2i\alpha}A^*B}{2}\right)$$

for $\alpha \in \mathbb{R}$.

By taking the norm, we have

$$\begin{aligned}\|H_{\alpha,A,B}\|^2 &= \left\|\frac{1}{2}\left(\frac{|A|^2+|B|^2}{2} + \frac{e^{2i\alpha}B^*A + e^{-2i\alpha}A^*B}{2}\right)\right\| \\ &\le \frac{1}{2}\left(\left\|\frac{|A|^2+|B|^2}{2}\right\| + \left\|\frac{e^{2i\alpha}B^*A + e^{-2i\alpha}A^*B}{2}\right\|\right)\end{aligned}$$

for $\alpha \in \mathbb{R}$.

By using the convexity of the power function, we have

$$\begin{aligned}\|H_{\alpha,A,B}\|^{2p} &\le \left[\frac{1}{2}\left(\left\|\frac{|A|^2+|B|^2}{2}\right\| + \left\|\frac{e^{2i\alpha}B^*A + e^{-2i\alpha}A^*B}{2}\right\|\right)\right]^p \\ &\le \frac{1}{2}\left(\left\|\frac{|A|^2+|B|^2}{2}\right\|^p + \left\|\frac{e^{2i\alpha}B^*A + e^{-2i\alpha}A^*B}{2}\right\|^p\right)\end{aligned} \tag{14}$$

for $\alpha \in \mathbb{R}$.

From (6), we get

$$\begin{aligned}&\left\|\frac{e^{2i\alpha}B^*A + e^{-2i\alpha}A^*B}{2}\right\|^{2p} \\ &\le \frac{1}{2}\left[2^{p/2}w^p\left((B^*A)^2\right) + \begin{cases}\|B^*A\|^{2p}, \\ \frac{\left\||B^*A|^2+|A^*B|^2\right\|^p + \left\||B^*A|^2-|A^*B|^2\right\|^p}{2},\end{cases}\right]\end{aligned}$$

which, by taking the square root, gives that

$$\begin{aligned}&\left\|\frac{e^{2i\alpha}B^*A + e^{-2i\alpha}A^*B}{2}\right\|^{p} \\ &\le \frac{\sqrt{2}}{2}\sqrt{2^{p/2}w^p\left((B^*A)^2\right) + \begin{cases}\|B^*A\|^{2p}, \\ \frac{\left\||B^*A|^2+|A^*B|^2\right\|^p + \left\||B^*A|^2-|A^*B|^2\right\|^p}{2}.\end{cases}}\end{aligned}$$

By making use of (14)

$$\begin{aligned}&\|H_{\alpha,A,B}\|^{2p} \\ &\le \frac{1}{2}\left\|\frac{|A|^2+|B|^2}{2}\right\|^p + \frac{\sqrt{2}}{4}\sqrt{2^{p/2}w^p\left((B^*A)^2\right) + \begin{cases}\|B^*A\|^{2p}, \\ \frac{\left\||B^*A|^2+|A^*B|^2\right\|^p + \left\||B^*A|^2-|A^*B|^2\right\|^p}{2}\end{cases}}\end{aligned}$$

for $\alpha \in \mathbb{R}$. By taking the supremum over $\alpha \in \mathbb{R}$ and utilizing the representation (8), we can deduce the desired result (13). □

The following remark provides a special case of the above theorem and is of interest.

Remark 1. *For $p = 1$ we get*

$$w^2\left(\begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}\right) \leq \frac{1}{2}\left\|\frac{|A|^2 + |B|^2}{2}\right\| + \frac{\sqrt{2}}{4}\sqrt{\ell_1(A,B,1) + \ell_2(A,B,1)}$$

where

$$\ell_1(A,B,1) := \sqrt{2}w\left((B^*A)^2\right)$$

and

$$\ell_2(A,B,1) := \begin{cases} \|B^*A\|^2, \\ \frac{\left\||B^*A|^2+|A^*B|^2\right\|+\left\||B^*A|^2-|A^*B|^2\right\|}{2}. \end{cases}$$

Our next result reads as follows.

Theorem 3. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and $p \geq 1$. Then*

$$w^{2p}\left(\begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}\right) \leq \frac{1}{2}\left\|\frac{|A|^2 + |B|^2}{2}\right\|^p + \sqrt{k_1(A,B,p) + k_2(A,B,p)}, \tag{15}$$

where

$$k_1(A,B,p) := 2^{p/2-1}w^p((A^*B - B^*A)(A^*B + B^*A))$$

and

$$k_2(A,B,p) := \frac{1}{2} \times \begin{cases} \max\left\{\|B^*A + A^*B\|^{2p}, \|B^*A - A^*B\|^{2p}\right\}, \\ \frac{2^p\left\||B^*A|^2+|A^*B|^2\right\|^p+4^p\left\|Re\left((B^*A)^2\right)\right\|^p}{2}. \end{cases}$$

Proof. Observe that

$$\begin{aligned} \frac{e^{2i\alpha}B^*A + e^{-2i\alpha}A^*B}{2} &= \frac{[\cos(2\alpha) + i\sin(2\alpha)]B^*A + [\cos(2\alpha) - i\sin(2\alpha)]A^*B}{2} \\ &= \frac{\cos(2\alpha)(B^*A + A^*B) + i\sin(2\alpha)(B^*A - A^*B)}{2}. \end{aligned}$$

From (6) we get

$$\begin{aligned} &\left\|\frac{\cos(2\alpha)(B^*A + A^*B) + i\sin(2\alpha)(B^*A - A^*B)}{2}\right\|^{2p} \\ &\leq 2^{p/2-1}w^p((A^*B - B^*A)(A^*B + B^*A)) \\ &+ \frac{1}{2} \times \begin{cases} \max\left\{\|B^*A + A^*B\|^{2p}, \|B^*A - A^*B\|^{2p}\right\}, \\ \frac{\left\||B^*A+A^*B|^2+|B^*A-A^*B|^2\right\|^p+\left\||B^*A+A^*B|^2-|B^*A-A^*B|^2\right\|^p}{2}. \end{cases} \end{aligned} \tag{16}$$

Observe that

$$|B^*A + A^*B|^2 + |B^*A - A^*B|^2 = 2\left(|B^*A|^2 + |A^*B|^2\right)$$

and

$$|B^*A + A^*B|^2 - |B^*A - A^*B|^2 = 4\mathrm{Re}\left((B^*A)^2\right)$$

and by (16) we get

$$\begin{aligned}
&\left\| \frac{\cos(2\alpha)(B^*A + A^*B) + i\sin(2\alpha)(B^*A - A^*B)}{2} \right\|^{2p} \\
&\le 2^{p/2-1} w^p((A^*B - B^*A)(A^*B + B^*A)) \\
&+ \frac{1}{2} \times \begin{cases} \max\left\{ \|B^*A + A^*B\|^{2p}, \|B^*A - A^*B\|^{2p} \right\}, \\ \frac{2^p \left\| \left(|B^*A|^2 + |A^*B|^2 \right) \right\|^p + 4^p \left\| \mathrm{Re}\left((B^*A)^2 \right) \right\|^p}{2}. \end{cases} \\
&= k_1(A,B,p) + k_2(A,B,p).
\end{aligned}$$

By utilizing (14) we then get the desired result (15). □

Remark 2. *The case $p = 1$ gives that*

$$w^2\left(\begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix} \right) \le \frac{1}{2} \left\| \frac{|A|^2 + |B|^2}{2} \right\| + \sqrt{k_1(A,B,1) + k_2(A,B,1)},$$

where

$$k_1(A,B,1) := \frac{\sqrt{2}}{2} w((A^*B - B^*A)(A^*B + B^*A))$$

and

$$k_2(A,B,1) := \frac{1}{2} \times \begin{cases} \max\left\{ \|B^*A + A^*B\|^2, \|B^*A - A^*B\|^2 \right\}, \\ \left\| |B^*A|^2 + |A^*B|^2 \right\| + 2\left\| Re\left((B^*A)^2 \right) \right\|. \end{cases}$$

Moreover, we can state the following result as well:

Theorem 4. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and $p \ge 1$. Then*

$$w^{2p}\left(\begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix} \right) \le \frac{1}{2^{p+1}} \|B^*A - A^*B\|^p + \frac{1}{2^{\frac{p+3}{2}}} \sqrt{m_1(A,B,p) + m_2(A,B,p)},$$

where

$$m_1(A,B,p) := 2^{p/2} w^p\left(|A-B|^2 |A+B|^2 \right)$$

and

$$m_2(A,B,p) := \begin{cases} \max\left\{ \|A+B\|^{4p}, \|A-B\|^{4p} \right\}, \\ \frac{\left\| |A+B|^4 + |A-B|^4 \right\|^p + \left\| |A+B|^4 - |A-B|^4 \right\|^p}{2}. \end{cases}$$

Proof. From (11) we have

$$\begin{aligned}
4|H_{\alpha,A,B}|^2 &= |\cos\alpha(A+B) + i\sin\alpha(A-B)|^2 \\
&= [\cos\alpha(A+B) + i\sin\alpha(A-B)]^*[\cos\alpha(A+B) + i\sin\alpha(A-B)] \\
&= \left[\cos\alpha(A+B)^* - i\sin\alpha(A-B)^* \right][\cos\alpha(A+B) + i\sin\alpha(A-B)] \\
&= \cos^2\alpha|A+B|^2 + \sin^2\alpha|A-B|^2 \\
&\quad + i\sin\alpha\cos\alpha\left[(A+B)^*(A-B) - (A-B)^*(A+B) \right]
\end{aligned} \tag{17}$$

for $\alpha \in \mathbb{R}$.

Observe that

$$\begin{aligned}&(A+B)^*(A-B)-(A-B)^*(A+B)\\&=(A^*+B^*)(A-B)-(A^*-B^*)(A+B)\\&=|A|^2+B^*A-A^*B-|B|^2-|A|^2+B^*A-A^*B+|B|^2\\&=2(B^*A-A^*B).\end{aligned}$$

So, by using (17), we get

$$4|H_{\alpha,A,B}|^2=\cos^2\alpha|A+B|^2+\sin^2\alpha|A-B|^2+2i\sin\alpha\cos\alpha(B^*A-A^*B)$$

namely

$$\begin{aligned}&|H_{\alpha,A,B}|^2\\&=\frac{1}{2}\left(\frac{\cos^2\alpha|A+B|^2+\sin^2\alpha|A-B|^2}{2}+i\sin\alpha\cos\alpha(B^*A-A^*B)\right)\end{aligned}\tag{18}$$

for $\alpha\in\mathbb{R}$.

If we take the norm in (18), we get

$$\begin{aligned}\|H_{\alpha,A,B}\|^2&=\left\|\frac{1}{2}\left(\frac{\cos^2\alpha|A+B|^2+\sin^2\alpha|A-B|^2}{2}+i\sin\alpha\cos\alpha(B^*A-A^*B)\right)\right\|\\&\le\frac{1}{2}\left(\left\|\frac{\cos^2\alpha|A+B|^2+\sin^2\alpha|A-B|^2}{2}\right\|+|\sin\alpha\cos\alpha|\|B^*A-A^*B\|\right)\end{aligned}$$

for $\alpha\in\mathbb{R}$.

Further, if we take the power $p\ge 1$ and use the convexity of the power function, we obtain

$$\begin{aligned}\|H_{\alpha,A,B}\|^{2p}&\le\frac{1}{2}\left\|\frac{\cos^2\alpha|A+B|^2+\sin^2\alpha|A-B|^2}{2}\right\|^p\\&+\frac{1}{2^{p+1}}|\sin(2\alpha)|^p\|B^*A-A^*B\|^p\\&\le\frac{1}{2}\left\|\frac{\cos^2\alpha|A+B|^2+\sin^2\alpha|A-B|^2}{2}\right\|^p\\&+\frac{1}{2^{p+1}}\|B^*A-A^*B\|^p\end{aligned}\tag{19}$$

for $\alpha\in\mathbb{R}$.

By (6) we get

$$\begin{aligned}&\left\|\frac{\cos^2\alpha|A+B|^2+\sin^2\alpha|A-B|^2}{2}\right\|^{2p}\\&\le 2^{p-1}\left(\frac{\cos^4\alpha+\sin^4\alpha}{2^2}\right)^p\\&\times\left[2^{p/2}w^p\left(|A-B|^2|A+B|^2\right)+\left\{\begin{array}{l}\max\left\{\left\||A+B|^2\right\|^{2p},\left\||A-B|^2\right\|^{2p}\right\},\\ \frac{\left\||A+B|^4+|A-B|^4\right\|^p+\left\||A+B|^4-|A-B|^4\right\|^p}{2}.\end{array}\right.\right]\\&\le\frac{1}{2^{p+1}}(m_1(A,B,p)+m_2(A,B,p))\end{aligned}$$

since

$$\cos^4\alpha + \sin^4\alpha \le \cos^2\alpha + \sin^2\alpha = 1,\ \alpha \in \mathbb{R},$$

which gives that

$$\left\| \frac{\cos^2\alpha|A+B|^2 + \sin^2\alpha|A-B|^2}{2} \right\|^p \le \frac{1}{2^{\frac{p+1}{2}}}\sqrt{m_1(A,B,p) + m_2(A,B,p)}.$$

By making use of (19)

$$\begin{aligned} &\|H_{\alpha,A,B}\|^{2p} \\ &\le \frac{1}{2}\left\| \frac{\cos^2\alpha|A+B|^2 + \sin^2\alpha|A-B|^2}{2} \right\|^p + \frac{1}{2^{p+1}}|\sin(2\alpha)|^p\|B^*A - A^*B\|^p \\ &\le \frac{1}{2^{p+1}}\|B^*A - A^*B\|^p + \frac{1}{2^{\frac{p+3}{2}}}\sqrt{m_1(A,B,p) + m_2(A,B,p)} \end{aligned}$$

for $\alpha \in \mathbb{R}$. By taking the supremum over $\alpha \in \mathbb{R}$ and utilizing the representation (8), we obtain the desired result. □

For $p = 1$, we obtain the simpler inequality

$$w^2\left(\begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}\right) \le \frac{1}{4}\|B^*A - A^*B\|^p + \frac{1}{4}\sqrt{m_1(A,B,1) + m_2(A,B,1)},$$

where

$$m_1(A,B,1) := \sqrt{2}w^p\left(|A-B|^2|A+B|^2\right)$$

and

$$m_2(A,B,1) := \begin{cases} \max\left\{\|A+B\|^4, \|A-B\|^4\right\}, \\ \frac{\left\||A+B|^4+|A-B|^4\right\| + \left\||A+B|^4-|A-B|^4\right\|}{2}. \end{cases}$$

We conclude this section with the following result.

Theorem 5. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and $p \ge 1$. Then for $\alpha \in \mathbb{R}$, we have*

$$\begin{aligned} w^{2p}\left(\begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}\right) &\le \frac{1}{2^{p+1}}w^p(K(\alpha,A,B)) \\ &\quad + 2^{p-1} \times \begin{cases} \frac{1}{2^{2p}}\max\left\{\|L(\alpha,A,B)\|^p, \|M(\alpha,A,B)\|^p\right\}, \\ \frac{\left\|\frac{|A|^2+|B|^2}{2} + \frac{1}{2}i\sin(2\alpha)(B^*A - A^*B)\right\|^p + |\cos(2\alpha)|^p\|Re(B^*A)\|^p}{2}, \end{cases} \end{aligned} \tag{20}$$

where

$$\begin{aligned} K(\alpha,A,B) &:= \sin(2\alpha)Re(A^*B) \\ &\quad + i\left[\cos^2\alpha(A^* - B^*)(A+B) + \sin^2\alpha(A^* + B^*)(A-B)\right], \end{aligned}$$

$$L(\alpha,A,B) := \cos^2\alpha|A+B|^2 + \sin^2\alpha|A-B|^2 + i\sin(2\alpha)(B^*A - A^*B)$$

and

$$M(\alpha,A,B) := \sin^2\alpha|A+B|^2 + \cos^2\alpha|A-B|^2 + i\sin(2\alpha)(B^*A - A^*B).$$

Proof. Observe that

$$H_{\alpha,A,B} := \frac{e^{i\alpha}A + e^{-i\alpha}B}{2} = \frac{1}{2}[\cos\alpha(A+B) + i\sin\alpha(A-B)]$$

for $\alpha \in \mathbb{R}$.

Observe that, for all $\alpha, \beta \in \mathbb{R}$, we have

$$\begin{aligned}
&\cos\beta H_{\alpha,A,B} + \sin\beta H_{\alpha+\frac{\pi}{2},A,B} \\
&= \frac{1}{2}\cos\beta[\cos\alpha(A+B) + i\sin\alpha(A-B)] \\
&+ \frac{1}{2}\sin\beta\Big[\cos\Big(\alpha+\frac{\pi}{2}\Big)(A+B) + i\sin\Big(\alpha+\frac{\pi}{2}\Big)(A-B)\Big] \\
&= \frac{1}{2}\cos\beta[\cos\alpha(A+B) + i\sin\alpha(A-B)] \\
&+ \frac{1}{2}\sin\beta[-\sin\alpha(A+B) + i\cos\alpha(A-B)] \\
&= \frac{1}{2}(\cos\beta\cos\alpha - \sin\beta\sin\alpha)(A+B) \\
&+ \frac{1}{2}i(\cos\beta\sin\alpha + \sin\alpha\cos\alpha)(A-B) \\
&= \frac{1}{2}[\cos(\alpha+\beta)(A+B) + i\sin(\alpha+\beta)(A-B)] = H_{\alpha+\beta,A,B}.
\end{aligned}$$

From (6), with $\gamma = \cos\beta$, $\delta = \sin\beta$, $U = H_{\alpha,A,B}$ and $V = H_{\alpha+\frac{\pi}{2},A,B}$, we obtain

$$\begin{aligned}
\left\|H_{\alpha+\beta,A,B}\right\|^{2p} \le 2^{3p/2-1}w^p\left(H^*_{\alpha+\frac{\pi}{2},A,B}H_{\alpha,A,B}\right) \\
+ 2^{p-1} \times \begin{cases} \max\left\{\|H_{\alpha,A,B}\|^{2p}, \left\|H_{\alpha+\frac{\pi}{2},A,B}\right\|^{2p}\right\}, \\ \dfrac{\left\||H_{\alpha,A,B}|^2 + \left|H_{\alpha+\frac{\pi}{2},A,B}\right|^2\right\|^p + \left\||H_{\alpha,A,B}|^2 - \left|H_{\alpha+\frac{\pi}{2},A,B}\right|^2\right\|^p}{2}. \end{cases}
\end{aligned} \tag{21}$$

Observe that

$$\begin{aligned}
&H^*_{\alpha+\frac{\pi}{2},A,B}H_{\alpha,A,B} \\
&= \frac{1}{4}[-\sin\alpha(A+B) + i\cos\alpha(A-B)]^*[\cos\alpha(A+B) + i\sin\alpha(A-B)] \\
&= \frac{1}{4}[-\sin\alpha(A^*+B^*) - i\cos\alpha(A^*-B^*)][\cos\alpha(A+B) + i\sin\alpha(A-B)] \\
&= \frac{1}{4}\Big[-\sin\alpha\cos\alpha|A+B|^2 - i\cos^2\alpha(A^*-B^*)(A+B) \\
&-i\sin^2\alpha(A^*+B^*)(A-B) + \sin\alpha\cos\alpha|A-B|^2\Big] \\
&= \frac{1}{4}\Big[\sin\alpha\cos\alpha\Big(|A-B|^2 - |A+B|^2\Big)\Big] \\
&- \frac{1}{4}i\Big[\cos^2\alpha(A^*-B^*)(A+B) + \sin^2\alpha(A^*+B^*)(A-B)\Big]
\end{aligned}$$

This implies that

$$\begin{aligned}
&H^*_{\alpha+\frac{\pi}{2},A,B}H_{\alpha,A,B}\\
&=\frac{1}{4}[-\sin\alpha\cos\alpha(B^*A+A^*B)]\\
&-\frac{1}{4}i\Big[\cos^2\alpha(A^*-B^*)(A+B)+\sin^2\alpha(A^*+B^*)(A-B)\Big]\\
&=-\frac{1}{4}[2\sin\alpha\cos\alpha\mathrm{Re}(A^*B)]\\
&-\frac{1}{4}i\Big[\cos^2\alpha(A^*-B^*)(A+B)+\sin^2\alpha(A^*+B^*)(A-B)\Big]\\
&=-\frac{1}{4}\{\sin(2\alpha)\mathrm{Re}(A^*B)\\
&+i\Big[\cos^2\alpha(A^*-B^*)(A+B)+\sin^2\alpha(A^*+B^*)(A-B)\Big]\Big\}.
\end{aligned}$$

So,

$$H^*_{\alpha+\frac{\pi}{2},A,B}H_{\alpha,A,B}=-\frac{1}{4}K(\alpha,A,B).$$

On the other hand, we see that

$$\begin{aligned}
&|H_{\alpha,A,B}|^2\\
&=\frac{1}{4}[\cos\alpha(A+B)+i\sin\alpha(A-B)]^*[\cos\alpha(A+B)+i\sin\alpha(A-B)]\\
&=\frac{1}{4}[\cos\alpha(A^*+B^*)-i\sin\alpha(A^*-B^*)][\cos\alpha(A+B)+i\sin\alpha(A-B)]\\
&=\frac{1}{4}\Big[\cos^2\alpha|A+B|^2+\sin^2\alpha|A-B|^2\Big]\\
&+\frac{1}{4}i\sin\alpha\cos\alpha[(A^*+B^*)(A-B)-(A^*-B^*)(A+B)]\\
&=\frac{1}{4}\Big[\cos^2\alpha|A+B|^2+\sin^2\alpha|A-B|^2\Big]+\frac{1}{4}i\sin(2\alpha)(B^*A-A^*B)\\
&=\frac{1}{4}\Big[\cos^2\alpha|A+B|^2+\sin^2\alpha|A-B|^2+i\sin(2\alpha)(B^*A-A^*B)\Big].
\end{aligned}$$

Thus,

$$|H_{\alpha,A,B}|^2=\frac{1}{4}L(\alpha,A,B).$$

In addition, we have

$$\begin{aligned}
&\Big|H_{\alpha+\frac{\pi}{2},A,B}\Big|^2\\
&=\frac{1}{4}[-\sin\alpha(A+B)+i\cos\alpha(A-B)]^*[-\sin\alpha(A+B)+i\cos\alpha(A-B)]\\
&=\frac{1}{4}[-\sin\alpha(A^*+B^*)-i\cos\alpha(A^*-B^*)][-\sin\alpha(A+B)+i\cos\alpha(A-B)]\\
&=\frac{1}{4}\Big[\sin^2\alpha|A+B|^2+\cos^2\alpha|A-B|^2\Big]\\
&+\frac{1}{8}i\sin(2\alpha)[(A^*-B^*)(A+B)-(A^*+B^*)(A-B)]\\
&=\frac{1}{4}\Big[\sin^2\alpha|A+B|^2+\cos^2\alpha|A-B|^2+i\sin(2\alpha)(B^*A-A^*B)\Big]\\
&=\frac{1}{4}M(\alpha,A,B).
\end{aligned}$$

This gives that

$$\begin{aligned}
&|H_{\alpha,A,B}|^2 + \left|H_{\alpha+\frac{\pi}{2},A,B}\right|^2 \\
&= \frac{1}{4}\left[\cos^2\alpha|A+B|^2 + \sin^2\alpha|A-B|^2 + i\sin(2\alpha)(B^*A - A^*B)\right] \\
&+ \frac{1}{4}\left[\sin^2\alpha|A+B|^2 + \cos^2\alpha|A-B|^2 + i\sin(2\alpha)(B^*A - A^*B)\right] \\
&= \frac{1}{4}\left[|A+B|^2 + |A-B|^2 + 2i\sin(2\alpha)(B^*A - A^*B)\right] \\
&= \frac{1}{4}\left[2\left(|A|^2 + |B|^2\right) + 2i\sin(2\alpha)(B^*A - A^*B)\right] \\
&= \frac{|A|^2 + |B|^2}{2} + \frac{1}{2}i\sin(2\alpha)(B^*A - A^*B)
\end{aligned}$$

and

$$\begin{aligned}
&|H_{\alpha,A,B}|^2 - \left|H_{\alpha+\frac{\pi}{2},A,B}\right|^2 \\
&= \frac{1}{4}\left[\cos^2\alpha|A+B|^2 + \sin^2\alpha|A-B|^2 + i\sin(2\alpha)(B^*A - A^*B)\right] \\
&- \frac{1}{4}\left[\sin^2\alpha|A+B|^2 + \cos^2\alpha|A-B|^2 + i\sin(2\alpha)(B^*A - A^*B)\right] \\
&= \frac{1}{4}\left[\cos(2\alpha)|A+B|^2 - \cos(2\alpha)|A-B|^2\right] \\
&= \frac{1}{4}\cos(2\alpha)\left[|A+B|^2 - |A-B|^2\right] = \cos(2\alpha)\mathrm{Re}(B^*A).
\end{aligned}$$

By using (21) we get

$$\begin{aligned}
\left\|H_{\alpha+\beta,A,B}\right\|^{2p} &\leq \frac{1}{2^{3p+1}}w^p(K(\alpha,A,B)) \\
&+ 2^{p-1}\begin{cases} \frac{1}{2^{2p}}\max\{\|L(\alpha,A,B)\|^p, \|M(\alpha,A,B)\|^p\}, \\ \frac{\left\|\frac{|A|^2+|B|^2}{2} + \frac{1}{2}i\sin(2\alpha)(B^*A - A^*B)\right\|^p + |\cos(2\alpha)|^p\|\mathrm{Re}(B^*A)\|^p}{2} \end{cases}
\end{aligned}$$

for $\alpha, \beta \in \mathbb{R}$. By taking the supremum over $\beta \in \mathbb{R}$, we get (20) as desired. □

Remark 3. *For $p = 1$, we get*

$$\begin{aligned}
&w^2\left(\begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}\right) \\
&\leq \frac{\sqrt{2}}{4}w(K(\alpha,A,B)) + \begin{cases} \frac{1}{4}\max\{\|L(\alpha,A,B)\|, \|M(\alpha,A,B)\|\}, \\ \frac{\left\|\frac{|A|^2+|B|^2}{2} + \frac{1}{2}i\sin(2\alpha)(B^*A - A^*B)\right\| + |\cos(2\alpha)|\|Re(B^*A)\|}{2} \end{cases}
\end{aligned}$$

For $\alpha = 0$ we have

$$K(0,A,B) := i(A^* - B^*)(A+B),$$
$$L(0,A,B) := |A+B|^2$$

and

$$M(0,A,B) := |A-B|^2$$

and by (20), we recapture (10).

If we take $\alpha = \pi/4$, then we have

$$K(\pi/4, A, B) = Re(A^*B) + \frac{1}{2}i[(A^* - B^*)(A+B) + (A^* + B^*)(A-B)]$$
$$= Re(A^*B) + i\left(|A|^2 - |B|^2\right)$$

$$L(\pi/4, A, B) = \frac{1}{2}\left[|A+B|^2 + |A-B|^2\right] + i(B^*A - A^*B)$$
$$= |A|^2 + |B|^2 + i(B^*A - A^*B) = |A - iB|^2$$

and

$$M(\pi/4, A, B) = L(\pi/4, A, B).$$

By (20), we get

$$w^{2p}\left(\begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}\right) \tag{22}$$
$$\leq \frac{1}{2^{p/2+1}} w^p\left(Re(A^*B) + i\left(|A|^2 - |B|^2\right)\right) + 2^{p-1} \times \begin{cases} \frac{1}{4^p}\|A - iB\|^{2p}, \\ \frac{\left\|\frac{|A|^2+|B|^2}{2} + \frac{1}{2}i(B^*A - A^*B)\right\|^p}{2}, \end{cases}$$

and for $p = 1$, we derive

$$w^2\left(\begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}\right) \leq \frac{\sqrt{2}}{4} w\left(Re(A^*B) + i\left(|A|^2 - |B|^2\right)\right) + \begin{cases} \frac{1}{4}\|A - iB\|^2, \\ \frac{\left\|\frac{|A|^2+|B|^2}{2} + \frac{1}{2}i(B^*A - A^*B)\right\|}{2} \end{cases}$$

4. Inequalities for One Operator

In this section, we present some applications of our results. By setting $B = A^*$ in Theorems 1 and 2, we obtain the following upper bounds for the power $2p$ of the numerical radius for $p \geq 1$:

$$w^{2p}(A) \leq \frac{1}{2}\left[2^{p/2} w^p\left(A^2\right) + \begin{cases} \|A\|^{2p}, \\ \frac{\left\||A|^2 + |A^*|^2\right\|^p + \left\||A|^2 - |A^*|^2\right\|^p}{2}, \end{cases}\right]$$

$$w^{2p}(A)$$
$$\leq \frac{1}{2}\left[\frac{1}{2^{p/2}} w^p((A^* - A)(A + A^*)) + \begin{cases} \frac{1}{2^p}\max\left\{\|A + A^*\|^{2p}, \|A - A^*\|^{2p}\right\}\right], \\ \frac{1}{2}\left(\left\||A|^2 + |A^*|^2\right\|^p + 2^p\|\mathrm{Re}(A^2)\|^p\right)\right] \end{cases}$$

and

$$w^{2p}(A) \leq \frac{1}{2}\left\|\frac{|A|^2 + |A^*|^2}{2}\right\|^p + \frac{\sqrt{2}}{4}\sqrt{2^{p/2} w^p(A^4) + \ell(A, Bp)}$$

where

$$\ell(A, p) := \begin{cases} \|A^2\|^{2p}, \\ \frac{\left\||A^2|^2 + |(A^*)^2|^2\right\|^p + \left\||A^2|^2 - |(A^*)^2|^2\right\|^p}{2}. \end{cases}$$

Now, if we utilize the inequalities from Theorems 3 and 5 for the same choice $B = A^*$, we have:

$$w^{2p}(A) \le \frac{1}{2}\left\|\frac{|A|^2 + |A^*|^2}{2}\right\|^p + \sqrt{k_1(A,p) + k_2(A,p)},$$

where

$$k_1(A,p) := 2^{p/2-1} w^p\left(\left((A^*)^2 - A^2\right)\left((A^*)^2 + A^2\right)\right)$$

and

$$k_2(A,p) := \frac{1}{2} \times \begin{cases} \max\left\{\left\|(A^*)^2 + A^2\right\|^{2p}, \left\|(A^*)^2 - A^2\right\|^{2p}\right\}, \\ \frac{2^p\left\|\left|A^2\right|^2 + \left|(A^*)^2\right|^2\right\|^p + 4^p\left\|\mathrm{Re}\left((A)^4\right)\right\|^p}{2}, \end{cases}$$

and

$$w^{2p}(A) \le \frac{1}{2^{p+1}}\left\|A^2 - (A^*)^2\right\|^p + \frac{1}{2^{\frac{p+3}{2}}}\sqrt{m_1(A,B,p) + m_2(A,B,p)},$$

where

$$m_1(A,p) := 2^{p/2} w^p\left(|A - A^*|^2 |A + A^*|^2\right)$$

and

$$m_2(A,B,p) := \begin{cases} \max\left\{\|A + A^*\|^{4p}, \|A - A^*\|^{4p}\right\}, \\ \frac{\left\||A+A^*|^4 + |A-A^*|^4\right\|^p + \left\||A+A^*|^4 - |A-A^*|^4\right\|^p}{2}. \end{cases}$$

Moreover, we have

$$\begin{aligned} &w^{2p}(A) \\ &\le \frac{1}{2^{p+1}} w^p(K(\alpha,A)) + 2^{p-1} \times \begin{cases} \frac{1}{2^{2p}} \max\{\|L(\alpha,A)\|^p, \|M(\alpha,A)\|^p\}, \\ \frac{\left\|\frac{|A|^2+|A^*|^2}{2} + \frac{1}{2} i \sin(2\alpha)\left(A^2 - (A^*)^2\right)\right\|^p + |\cos(2\alpha)|^p \left\|\mathrm{Re}\left(A^2\right)\right\|^p}{2}, \end{cases} \end{aligned}$$

where

$$\begin{aligned} K(\alpha,A) &:= \sin(2\alpha)\mathrm{Re}\left((A^*)^2\right) \\ &\quad + i\left[\cos^2\alpha(A^* - A)(A + A^*) + \sin^2\alpha(A^* + A)(A - A^*)\right], \end{aligned}$$

$$L(\alpha,A) := \cos^2\alpha|A + A^*|^2 + \sin^2\alpha|A - A^*|^2 + i\sin(2\alpha)\left(A^2 - (A^*)^2\right)$$

and

$$M(\alpha,A) := \sin^2\alpha|A + A^*|^2 + \cos^2\alpha|A - A^*|^2 + i\sin(2\alpha)\left(A^2 - (A^*)^2\right)$$

for $\alpha \in \mathbb{R}$.

Finally, from (22) we get for $B = A^*$ that

$$\begin{aligned} &w^{2p}(A) \\ &\le \frac{1}{2^{p/2+1}} w^p\left(\mathrm{Re}\left((A^*)^2\right) + i\left(|A|^2 - |A^*|^2\right)\right) + 2^{p-1} \times \begin{cases} \frac{1}{4^p}\|A - iA^*\|^{2p}, \\ \frac{\left\|\frac{|A|^2+|A^*|^2}{2} + \frac{1}{2} i\left(A^2 - (A^*)^2\right)\right\|^p}{2}. \end{cases} \end{aligned}$$

5. Conclusions

In this paper, we have used a generalization of the Boas–Bellman inequality, known as the Mitrinović–Pečarić–Fink inequality, to find new upper bounds for the $2p$-th power of the numerical radius of off-diagonal 2×2 operator matrices. Our results provide more varied and comprehensive bounds compared to those found in recent papers like [19–21]. Specifically, for the case $p = 1$, our bounds have different forms, making them unique and not directly comparable to those in the other papers.

The key innovation in our work is the use of the Mitrinović–Pečarić–Fink generalization, which has not been applied before in this context. This approach has allowed us to discover new inequalities, especially when the matrix becomes symmetric.

While our results add to the understanding of numerical radius inequalities, they also open up possibilities for future research. As noted in [5], upper bounds for the numerical radius can be useful in areas like approximating polynomial solutions or estimating the numerical radius of the Frobenius companion matrix, as discussed in [17]. These potential applications go beyond the scope of this paper and could be explored in future studies.

In summary, this work serves as a starting point for further research. We hope that the methods and results presented here will inspire more studies that continue to explore power inequalities and their uses in operator theory.

Author Contributions: The authors confirm that they have made equal contributions to this article. Each author played a significant role in its creation and provided valuable contributions. All authors have reviewed and approved the final version of the manuscript for publication.

Funding: Researchers Supporting Project number (RSP2024R187), King Saud University, Riyadh, Saudi Arabia.

Data Availability Statement: Data are contained within the article.

Acknowledgments: The authors sincerely thank the reviewers for their valuable comments and suggestions, which have significantly enhanced this paper. Furthermore, the first author wishes to express her deep appreciation for the support received from the Distinguished Scientist Fellowship Program under the Researchers Supporting Project number (RSP2024R187), King Saud University, Riyadh, Saudi Arabia.

Conflicts of Interest: The authors declare no conflicts of interest.

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