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Research Article

On the Hermite-Hadamard Inequality and Other Integral Inequalities Involving Two Functions

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We establish some new Hermite-Hadamard-type inequalities involving product of two functions. Other integral inequalities for two functions are obtained as well. The analysis used in the proofs is fairly elementary and based on the use of the Minkowski, Hölder, and Young inequalities.

1. Introduction

Integral inequalities have played an important role in the development of all branches of Mathematics.

In [1, 2], Pachpatte established some Hermite-Hadamard-type inequalities involving two convex and log-convex functions, respectively. In [3], Bakula et al. improved Hermite-Hadamard type inequalities for products of two m-convex and (α, m)-convex functions. In [4], analogous results for s-convex functions were proved by Kirmaci et al.. General companion inequalities related to Jensen's inequality for the classes of m-convex and (α, m)-convex functions were presented by Bakula et al. (see [5]).

For several recent results concerning these types of inequalities, see [6–12] where further references are listed.

The aim of this paper is to establish several new integral inequalities for nonnegative and integrable functions that are related to the Hermite-Hadamard result. Other integral inequalities for two functions are also established.

In order to prove some inequalities related to the products of two functions we need the following inequalities. One of inequalities of this type is the following one.

Barnes-Gudunova-Levin Inequality (see [13–15] and references therein)

Let f, g be nonnegative concave functions on $[a,b]$. Then, for $p,q>1$ we have

$$\left(\int_a^b f(x)^p dx\right)^{1/p}\left(\int_a^b g(x)^q dx\right)^{1/q} \le B(p,q)\int_a^b f(x)g(x)dx, \tag{1.1}$$

where

$$B(p,q) = \frac{6(b-a)^{(1/p)+(1/q)-1}}{(p+1)^{1/p}(q+1)^{1/q}}. \tag{1.2}$$

In the special case $q=p$ we have

$$\left(\int_a^b f(x)^p dx\right)^{1/p}\left(\int_a^b g(x)^p dx\right)^{1/p} \le B(p,p)\int_a^b f(x)g(x)dx \tag{1.3}$$

with

$$B(p,p) = \frac{6(b-a)^{(2/p)-1}}{(p+1)^{2/p}}. \tag{1.4}$$

To prove our main results we recall some concepts and definitions.

Let $x=(x_1,x_2,\ldots,x_n)$ and $p=(p_1,p_2,\ldots,p_n)$ be two positive n-tuples, and let $r\in\mathbb{R}\cup\{+\infty,-\infty\}$. Then, on putting $P_n=\sum_{k=1}^n p_k$, the rth power mean of x with weights p is defined [16] by

$$M_n^{[r]} = \begin{cases} \left(\dfrac{1}{P_n}\displaystyle\sum_{k=1}^n p_k x_k^r\right)^{1/r}, & r\neq +\infty,0,-\infty, \\ \left(\displaystyle\prod_{k=1}^n x_k^{p_k}\right)^{1/P_n}, & r=0, \\ \min(x_1,x_2,\ldots,x_n), & r=-\infty, \\ \max(x_1,x_2,\ldots,x_n), & r=\infty. \end{cases} \tag{1.5}$$

Note that if $-\infty \leq r < s \leq \infty$, then

$$M_n^{[r]} \leq M_n^{[s]} \tag{1.6}$$

(see, e.g., [10, page 15]).

Let $f : [a, b] \to \mathbb{R}$ and $p \geq 1$. The p-norm of the function f on $[a, b]$ is defined by

$$\|f\|_p = \begin{cases} \left(\int_a^b |f(x)|^p dx\right)^{1/p}, & 1 \leq p < \infty, \\ \sup|f(x)|, & p = \infty, \end{cases} \tag{1.7}$$

and $L^p([a, b])$ is the set of all functions $f : [a, b] \to R$ such that $\|f\|_p < \infty$.

One can rewrite the inequality (1.1) as follows:

$$\|f\|_p \|g\|_q \leq B(p, q) \int_a^b f(x) g(x) dx. \tag{1.8}$$

For several recent results concerning p-norms we refer the interested reader to [17].

Also, we need some important inequalities.

Minkowski Integral Inequality (see page 1 in [18])

Let $p \geq 1$, $0 < \int_a^b f(x)^p dx < \infty$, and $0 < \int_a^b g(x)^p dx < \infty$. Then

$$\left(\int_a^b (f(x) + g(x))^p dx\right)^{1/p} \leq \left(\int_a^b f(x)^p dx\right)^{1/p} + \left(\int_a^b g(x)^p dx\right)^{1/p}. \tag{1.9}$$

Hermite-Hadamard's Inequality (see page 10 in [10])

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on interval I of real numbers and $a, b \in I$ with $a < b$. Then the following Hermite-Hadamard inequality for convex functions holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}. \tag{1.10}$$

If the function f is concave, the inequality (1.10) can be written as follows:

$$\frac{f(a) + f(b)}{2} \leq \frac{1}{b-a} \int_a^b f(x) dx \leq f\left(\frac{a+b}{2}\right). \tag{1.11}$$

For recent results, refinements, counterparts, generalizations, and new Hermite-Hadamard-type inequalities, see [19–21].

A Reversed Minkowski Integral Inequality (see page 2 in [18])

Let f and g be positive functions satisfying

$$0 < m \leq \frac{f(x)}{g(\mathrm{x})} \leq M, \quad (x \in [a,b]). \tag{1.12}$$

Then, putting $c = (M(m+1) + (M+1))/((m+1)(M+1))$, we have

$$\left(\int_a^b f(x)^p dx\right)^{1/p} + \left(\int_a^b g(x)^p dx\right)^{1/p} \leq c\left(\int_a^b (f(x) + g(x))^p dx\right)^{1/p}. \tag{1.13}$$

One of the most important inequalities of analysis is Hölder's integral inequality which is stated as follows (for its variant see [10, page 106]).

Hölder Integral Inequality

Let $p > 1$ and $1/p + 1/q = 1$. If f and g are real functions defined on $[a,b]$ and if $|f|^p$ and $|g|^q$ are integrable functions on $[a,b]$, then

$$\int_a^b \left|f(x)g(x)\right| dx \leq \left(\int_a^b \left|f(x)\right|^p dx\right)^{1/p} \left(\int_a^b \left|g(x)\right|^q dx\right)^{1/q}, \tag{1.14}$$

with equality holding if and only if $A|f(x)|^p = B|g(x)|^q$ almost everywhere, where A and B are constants.

Remark 1.1. Observe that whenever, f^p is concave on $[a,b]$, the nonnegative function f is also concave on $[a,b]$. Namely,

$$\left(f(ta + (1-t)b)\right)^p \geq tf(a)^p + (1-t)f(b)^p, \tag{1.15}$$

that is,

$$f(ta + (1-t)b) \geq \left(tf(a)^p + (1-t)f(b)^p\right)^{1/p} \tag{1.16}$$

and $p > 1$; using the power-mean inequality (1.6), we obtain

$$f(ta + (1-t)b) \geq tf(a) + (1-t)f(b). \tag{1.17}$$

For $q > 1$, similarly if g^q is concave on $[a,b]$, the nonnegative function g is concave on $[a,b]$.

2. The Results

Theorem 2.1. *Let $p, q > 1$ and let $f, g : [a,b] \to \mathbb{R}$, $a < b$, be nonnegative functions such that f^p and g^q are concave on $[a,b]$. Then*

$$\frac{f(a)+f(b)}{2} \times \frac{g(a)+g(b)}{2} \leq \frac{1}{(b-a)^{1/p+1/q}} B(p,q) \int_a^b f(x)g(x)dx, \tag{2.1}$$

and if $1/p + 1/q = 1$, then one has

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right). \tag{2.2}$$

Here $B(\cdot,\cdot)$ is the Barnes-Gudunova-Levin constant given by (1.1).

Proof. Since f^p, g^q are concave functions on $[a,b]$, then from (1.11) and Remark 1.1 we get

$$\begin{gathered}\left(\frac{f(a)^p+f(b)^p}{2}\right)^{1/p} \leq \frac{1}{(b-a)^{1/p}}\left(\int_a^b f(x)^p dx\right)^{1/p} \leq f\left(\frac{a+b}{2}\right), \\ \left(\frac{g(a)^q+g(b)^q}{2}\right)^{1/q} \leq \frac{1}{(b-a)^{1/q}}\left(\int_a^b g(x)^q dx\right)^{1/q} \leq g\left(\frac{a+b}{2}\right).\end{gathered} \tag{2.3}$$

By multiplying the above inequalities, we obtain (2.4) and (2.5)

$$\begin{aligned}\left(\frac{f(a)^p+f(b)^p}{2}\right)^{1/p}\left(\frac{g(a)^q+g(b)^q}{2}\right)^{1/q} &\leq \frac{1}{(b-a)^{1/p+1/q}}\left(\int_a^b f(x)^p dx\right)^{1/p} \\ &\quad \times \left(\int_a^b g(x)^q dx\right)^{1/q},\end{aligned} \tag{2.4}$$

$$\frac{1}{(b-a)^{1/p+1/q}}\left(\int_a^b f(x)^p dx\right)^{1/p}\left(\int_a^b g(x)^q dx\right)^{1/q} \leq f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right). \tag{2.5}$$

If $p, q > 1$, then it easy to show that

$$\begin{gathered}\left(\frac{f(a)^p+f(b)^p}{2}\right)^{1/p} \geq \frac{f(a)+f(b)}{2}, \\ \left(\frac{g(a)^q+g(b)^q}{2}\right)^{1/q} \geq \frac{g(a)+g(b)}{2}.\end{gathered} \tag{2.6}$$

Thus, by applying Barnes-Gudunova-Levin inequality to the right-hand side of (2.4) with (2.6), we get (2.1).

Applying the Hölder inequality to the left-hand side of (2.5) with $1/p + 1/q = 1$, we get (2.2). □

Theorem 2.2. *Let* $p \geq 1$, $0 < \int_a^b f(x)^p dx < \infty$, *and* $0 < \int_a^b g(x)^p dx < \infty$, *and let* $f, g : [a,b] \to \mathbb{R}$ *be positive functions with*

$$0 < m \leq \frac{f}{g} \leq M, \quad \forall x \in [a,b], \ a < b. \tag{2.7}$$

Then

$$\frac{\|f\|_p^2 + \|g\|_p^2}{\|f\|_p \|g\|_p} \geq \frac{1}{s} - 2, \tag{2.8}$$

where $s = M/((M+1)(m+1))$.

Proof. Since f, g are positive, as in the proof of the inequality (1.13) (see [18, page 2]), we have that

$$\begin{aligned} \left(\int_a^b f(x)^p dx\right)^{1/p} &\leq \frac{M}{M+1}\left(\int_a^b (f(x)+g(x))^p dx\right)^{1/p}, \\ \left(\int_a^b g(x)^p dx\right)^{1/p} &\leq \frac{1}{m+1}\left(\int_a^b (f(x)+g(x))^p dx\right)^{1/p}. \end{aligned} \tag{2.9}$$

By multiplying the above inequalities, we get

$$\left(\int_a^b f(x)^p dx\right)^{1/p}\left(\int_a^b g(x)^p dx\right)^{1/p} \leq s\left(\int_a^b (f(x)+g(x))^p dx\right)^{2/p}. \tag{2.10}$$

Since $(\int_a^b f(x)^p dx)^{1/p} = \|f\|_p$ and $(\int_a^b g(x)^p dx)^{1/p} = \|g\|_p$, by applying the Minkowski integral inequality to the right hand side of (2.10), we obtain inequality (2.8). □

Theorem 2.3. *Let* f^p *and* g^q *be as in Theorem 2.1. Then the following inequality holds:*

$$\frac{(f(a)+f(b))^p (g(a)+g(b))^q}{2^{(p+q)}} \leq \frac{1}{(b-a)^2}\|f\|_p^p \|g\|_q^q. \tag{2.11}$$

Proof. If f^p, g^q are concave on $[a,b]$, then from (1.11) we get

$$\begin{aligned} \frac{f(a)^p + f(b)^p}{2} &\leq \frac{1}{(b-a)}\int_a^b f(x)^p dx, \\ \frac{g(a)^q + g(b)^q}{2} &\leq \frac{1}{(b-a)}\int_a^b g(x)^q dx, \end{aligned} \tag{2.12}$$

which imply that

$$\frac{[f(a)^p + f(b)^p][g(a)^q + g(b)^q]}{4} \le \frac{1}{(b-a)^2}\left(\int_a^b f(x)^p dx\right)\left(\int_a^b g(x)^q dx\right). \tag{2.13}$$

On the other hand, if $p, q \ge 1$, from (1.6) we get

$$\left(\frac{f(a)^p + f(b)^p}{2}\right)^{1/p} \ge 2^{-1}[f(a) + f(b)], \qquad \left(\frac{g(a)^q + g(b)^q}{2}\right)^{1/q} \ge 2^{-1}[g(a) + g(b)], \tag{2.14}$$

or

$$\frac{f(a)^p + f(b)^p}{2} \ge 2^{-p}[f(a) + f(b)]^p, \qquad \frac{g(a)^q + g(b)^q}{2} \ge 2^{-q}[g(a) + g(b)]^q, \tag{2.15}$$

which imply that

$$\frac{[f(a)^p + f(b)^p][g(a)^q + g(b)^q]}{4} \ge (f(a) + f(b))^p (g(a) + g(b))^q 2^{-(p+q)}. \tag{2.16}$$

Combining (2.13) and (2.16), we obtain the desired inequality as

$$(f(a) + f(b))^p (g(a) + g(b))^q 2^{-(p+q)} \le \frac{1}{(b-a)^2}\|f\|_p^p \|g\|_q^q, \tag{2.17}$$

that is,

$$(f(a) + f(b))^p (g(a) + g(b))^q \le \frac{2^{(p+q)}}{(b-a)^2}\|f\|_p^p \|g\|_q^q. \tag{2.18}$$

□

To prove the following theorem we need the following Young-type inequality (see [7, page 117]):

$$xy \le \frac{1}{p}x^p + \frac{1}{q}y^q, \quad \text{for any } x, y \ge 0,\ p > 1,\ \ \frac{1}{p} + \frac{1}{q} = 1. \tag{2.19}$$

Theorem 2.4. *Let $f, g : [a, b] \to \mathbb{R}^+$ be functions such that f^p, g^p, and fg are in $L_1[a, b]$, and*

$$0 < m \le \frac{f(x)}{g(x)} \le M, \quad \forall x \in [a, b],\ a, b \in [0, \infty). \tag{2.20}$$

Then

$$\int_a^b f(x)g(x)dx \le c_1\left(\frac{\|f\|_p^p + \|g\|_p^p}{2}\right) + c_2\left(\frac{\|f\|_q^q + \|g\|_q^q}{2}\right), \tag{2.21}$$

where

$$c_1 = \frac{2^p}{p}\left(\frac{M}{M+1}\right)^p, \qquad c_2 = \frac{2^q}{q}\left(\frac{1}{m+1}\right)^q \tag{2.22}$$

and $1/p + 1/q = 1$ *with* $p > 1$.

Proof. From $0 < m \le f(x)/g(x) \le M,$ for all $x \in [a,b]$, we have

$$\begin{aligned} f(x) &\le \frac{M}{M+1}(f(x)+g(x)), \\ g(x) &\le \frac{1}{m+1}(f(x)+g(x)). \end{aligned} \tag{2.23}$$

From (2.19) with (2.23) we obtain

$$\begin{aligned} \int_a^b f(x)g(x)dx &\le \frac{1}{p}\int_a^b f(x)^p dx + \frac{1}{q}\int_a^b g(x)^q dx \\ &\le \frac{1}{p}\left(\frac{M}{M+1}\right)^p \int_a^b (f(x)+g(x))^p dx + \frac{1}{q}\left(\frac{1}{m+1}\right)^q \int_a^b (f(x)+g(x))^q dx. \end{aligned} \tag{2.24}$$

Using the elementary inequality $(c+d)^p \le 2^{p-1}(c^p + d^p)$, ($p > 1$ and $c, d \in \mathbb{R}_+$) in (2.24), we get

$$\begin{aligned} \int_a^b f(x)g(x)dx &\le \frac{1}{p}\left(\frac{M}{M+1}\right)^p 2^{p-1}\int_a^b [f(x)^p + g(x)^p]dx \\ &\quad + \frac{1}{q}\left(\frac{1}{m+1}\right)^q 2^{q-1}\int_a^b [f(x)^q + g(x)^q]dx \\ &= \frac{2^p}{p}\left(\frac{M}{M+1}\right)^p\left(\frac{\|f\|_p^p + \|g\|_p^p}{2}\right) + \frac{2^q}{q}\left(\frac{1}{m+1}\right)^q\left(\frac{\|f\|_q^q + \|g\|_q^q}{2}\right). \end{aligned} \tag{2.25}$$

This completes the proof of the inequality in (2.21). □

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References

[1] B. G. Pachpatte, "On some inequalities for convex functions," *RGMIA Research Report Collection E*, vol. 6, 2003.
[2] B. G. Pachpatte, "A note on integral inequalities involving two log-convex functions," *Mathematical Inequalities & Applications*, vol. 7, no. 4, pp. 511–515, 2004.
[3] M. K. Bakula, M. E. Özdemir, and J. Pečarić, "Hadamard type inequalities for m-convex and (α, m)-convex functions," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 9, no. 4, article 96, 2008.
[4] U. S. Kirmaci, M. K. Bakula, M. E. Özdemir, and J. Pečarić, "Hadamard-type inequalities for s-convex functions," *Applied Mathematics and Computation*, vol. 193, no. 1, pp. 26–35, 2007.
[5] M. K. Bakula, J. Pečarić, and M. Ribičić, "Companion inequalities to Jensen's inequality for m-convex and (α, m)-convex functions," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 7, no. 5, article 194, 2006.
[6] M. K. Bakula and J. Pečarić, "Note on some Hadamard-type inequalities," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 5, no. 3, article 74, 2004.
[7] S. S. Dragomir, R. P. Agarwal, and N. S. Barnett, "Inequalities for beta and gamma functions via some classical and new integral inequalities," *Journal of Inequalities and Applications*, vol. 5, no. 2, pp. 103–165, 2000.
[8] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge Mathematical Library, Cambridge , UK, 1998.
[9] U. S. Kirmaci and M. E. Özdemir, "Some inequalities for mappings whose derivatives are bounded and applications to special means of real numbers," *Applied Mathematics Letters*, vol. 17, no. 6, pp. 641–645, 2004.
[10] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Classical and New Inequalities in Analysis*, vol. 61 of *Mathematics and Its Applications (East European Series)*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1993.
[11] M. E. Özdemir and U. S. Kırmacı, "Two new theorem on mappings uniformly continuous and convex with applications to quadrature rules and means," *Applied Mathematics and Computation*, vol. 143, no. 2-3, pp. 269–274, 2003.
[12] B. G. Pachpatte, *Inequalities for Differentiable and Integral Equations*, Academic Press, Boston, Mass, USA, 1997.
[13] J. Pečarić and T. Pejković, "On an integral inequality," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 5, no. 2, article 47, 2004.
[14] J. E. Pečarić, F. Proschan, and Y. L. Tong, *Convex Functions, Partial Orderings, and Statistical Applications*, vol. 187 of *Mathematics in Science and Engineering*, Academic Press, Boston, Mass, USA, 1992.
[15] T. K. Pogány, "On an open problem of F. Qi," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 3, no. 4, article 54, 2002.
[16] P. S. Bullen, D. S. Mitrinović, and P. M. Vasić, *Means and Their Inequalities*, vol. 31 of *Mathematics and Its Applications (East European Series)*, D. Reidel, Dordrecht, The Netherlands, 1988.
[17] U. S. Kirmaci, M. Klaričić, M. E. Özdemir, and J. Pečarić, "On some inequalities for p-norms," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 9, no. 1, article 27, 2008.
[18] L. Bougoffa, "On Minkowski and Hardy integral inequalities," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 7, no. 2, article 60, 2006.
[19] M. Alomari and M. Darus, "On the Hadamard's inequality for log-convex functions on the coordinates," *Journal of Inequalities and Applications*, vol. 2009, Article ID 283147, 13 pages, 2009.
[20] C. Dinu, "Hermite-Hadamard inequality on time scales," *Journal of Inequalities and Applications*, vol. 2008, Article ID 287947, 24 pages, 2008.
[21] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, Melbourne, Australia, 2000.