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Research Article

On Hadamard-Type Inequalities Involving Several Kinds of Convexity

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We do not only give the extensions of the results given by Gill et al. (1997) for log-convex functions but also obtain some new Hadamard-type inequalities for log-convex m-convex, and (α, m)-convex functions.

1. Introduction

The following inequality is well known in the literature as Hadamard's inequality:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2},\tag{1.1}$$

where $f : I \to R$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. This inequality is one of the most useful inequalities in mathematical analysis. For new proofs, note worthy extension, generalizations, and numerous applications on this inequality; see ([1–6]) where further references are given.

Let I be on interval in R. Then $f : I \rightarrow R$ is said to be convex if, for all $x, y \in I$ and $\lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \tag{1.2}$$

(see [5], Page 1). Geometrically, this means that if K, L, and M are three distinct points on the graph of f with L between K and M, then L is on or below chord KM.

Recall that a function $f : I \rightarrow (0, \infty)$ is said to be log-convex function if, for all $x, y \in I$ and $t \in [0, 1]$, one has the inequality (see [5], Page 3)

$$f(tx + (1 - t)y) \leq [f(x)]^{t}[f(y)]^{(1-t)}. \tag{1.3}$$

It is said to be log-concave if the inequality in (1.3) is reversed.

In [7], Toader defined m-convexity as follows.

Definition 1.1. The function $f : [0, b] \rightarrow R, b > 0$ is said to be m-convex, where $m \in [0, 1]$, if one has

$$f(tx + m(1 - t)y) \leq tf(x) + m(1 - t)f(y) \tag{1.4}$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$. We say that f is m-concave if $-f$ is m-convex.

Denote by $K_m(b)$ the class of all m-convex functions on $[0, b]$ such that $f(0) \leq 0$ (if $m < 1$). Obviously, if we choose $m = 1$, Definition 1.1 recaptures the concept of standard convex functions on $[0, b]$.

In [8], Miheşan defined (α, m)-convexity as in the following:

Definition 1.2. The function $f : [0, b] \rightarrow \mathbb{R}, b > 0$, is said to be (α, m)-convex, where $(\alpha, m) \in [0, 1]^2$, if one has

$$f(tx + m(1 - t)y) \leq t^{\alpha} f(x) + m(1 - t^{\alpha})f(y) \tag{1.5}$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

Denote by $K_m^{\alpha}(b)$ the class of all (α, m)-convex functions on $[0, b]$ for which $f(0) \leq 0$. It can be easily seen that for $(\alpha, m) = (1, m)$, (α, m)-convexity reduces to m-convexity and for $(\alpha, m) = (1, 1)$, (α, m)-convexity reduces to the concept of usual convexity defined on $[0, b]$, $b > 0$.

For recent results and generalizations concerning m-convex and (α, m)-convex functions, see ([9–12]).

In the literature, the logarithmic mean of the positive real numbers p, q is defined as the following:

$$L(p, q) = \frac{p - q}{\ln p - \ln q} \quad (p \neq q) \tag{1.6}$$

(for $p = q$, we put $L(p, p) = p$).

In [13], Gill et al. established the following results.

Theorem 1.3. *Let f be a positive,* log-*convex function on $[a,b]$. Then*

$$\frac{1}{b-a}\int_a^b f(t)dt \leq L(f(a), f(b)),$$

(1.7)

where $L(\cdot,\cdot)$ is a logarithmic mean of the positive real numbers as in (1.6).
    For f a positive log-*concave function, the inequality is reversed.*

Corollary 1.4. *Let f be positive* log-*convex functions on $[a,b]$. Then*

$$\frac{1}{b-a}\int_a^b f(t)dt \leq \min_{x\in[a,b]} \frac{(x-a)L(f(a),f(x)) + (b-x)L(f(x),f(b))}{b-a}.$$

(1.8)

If f is a positive log-*concave function, then*

$$\frac{1}{b-a}\int_a^b f(x)dx \geq \max_{x\in[a,b]} \frac{(x-a)L(f(a),f(x)) + (b-x)L(f(x),f(b))}{b-a}.$$

(1.9)

For some recent results related to the Hadamard's inequalities involving two log-convex functions, see [14] and the references cited therein. The main purpose of this paper is to establish the general version of inequalities (1.7) and new Hadamard-type inequalities involving two log-convex functions, two m-convex functions, or two (α, m)-convex functions using elementary analysis.

2. Main Results

We start with the following theorem.

Theorem 2.1. *Let $f_i : I \subset R \to (0, \infty)$ $(i = 1, 2, \ldots, n)$ be* log-*convex functions on I and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$\frac{1}{b-a}\int_a^b \prod_{i=1}^n f_i(x)dx \leq L\left(\prod_{i=1}^n f_i(a), \prod_{i=1}^n f_i(b)\right),$$

(2.1)

where L is a logarithmic mean of positive real numbers.
    For f a positive log-*concave function, the inequality is reversed.*

Proof. Since $f_i(i = 1, 2, \ldots, n)$ are log-convex functions on I, we have

$$f_i(ta + (1-t)b) \leq [f_i(a)]^t [f_i(b)]^{(1-t)} \tag{2.2}$$

for all $a, b \in I$ and $t \in [0,1]$. Writing (2.2) for $i = 1, 2, \ldots, n$ and multiplying the resulting inequalities, it is easy to observe that

$$\prod_{i=1}^{n} f_i(ta + (1-t)b) \leq \left[\prod_{i=1}^{n} f_i(a) \right]^t \left[\prod_{i=1}^{n} f_i(b) \right]^{(1-t)}$$

$$= \prod_{i=1}^{n} f_i(b) \left[\prod_{i=1}^{n} \frac{f_i(a)}{f_i(b)} \right]^t \tag{2.3}$$

for all $a, b \in I$ and $t \in [0,1]$.

Integrating inequality (2.3) on $[0,1]$ over t, we get

$$\int_0^1 \prod_{i=1}^{n} f_i(ta + (1-t)b)dt \leq \prod_{i=1}^{n} f_i(b) \int_0^1 \left[\prod_{i=1}^{n} \frac{f_i(a)}{f_i(b)} \right]^t dt. \tag{2.4}$$

As

$$\int_0^1 \prod_{i=1}^{n} f_i(ta + (1-t)b)dt = \frac{1}{b-a} \int_a^b \prod_{i=1}^{n} f_i(x)dx, \tag{2.5}$$

$$\int_0^1 \left[\prod_{i=1}^{n} \frac{f_i(a)}{f_i(b)} \right]^t dt = \frac{1}{\prod_{i=1}^{n} f_i(b)} L\left(\prod_{i=1}^{n} f_i(a), \prod_{i=1}^{n} f_i(b) \right), \tag{2.6}$$

the theorem is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 2.2. By taking $i = 1$ and $f_1 = f$ in Theorem 2.1, we obtain (1.7).

Corollary 2.3. *Let $f_i : I \subset R \to (0, \infty)$ $(i = 1, 2, \ldots, n)$ be log-convex functions on I and $a, b \in I$ with $a < b$. Then*

$$\frac{1}{b-a} \int_a^b \prod_{i=1}^{n} f_i(x)dx$$

$$\leq \min_{x \in [a,b]} \frac{(x-a)L\left(\prod_{i=1}^{n} f_i(a), \prod_{i=1}^{n} f_i(x) \right) + (b-x)L\left(\prod_{i=1}^{n} f_i(x), \prod_{i=1}^{n} f_i(b) \right)}{b-a}. \tag{2.7}$$

If $f_i (i = 1, 2, \ldots, n)$ are positive log-*concave functions, then*

$$\frac{1}{b - a} \int_a^b \prod_{i=1}^n f_i(x) dx$$

$$\geq \max_{x \in [a,b]} \frac{(x - a)L\left(\prod_{i=1}^n f_i(a), \prod_{i=1}^n f_i(x)\right) + (b - x)L\left(\prod_{i=1}^n f_i(x), \prod_{i=1}^n f_i(b)\right)}{b - a}. \tag{2.8}$$

Proof. Let $f_i (i = 1, 2, \ldots, n)$ be positive log-convex functions. Then by Theorem 2.1 we have that

$$\int_a^b \prod_{i=1}^n f_i(t) dt = \int_a^x \prod_{i=1}^n f_i(t) dt + \int_x^b \prod_{i=1}^n f_i(t) dt$$

$$\leq (x - a)L\left(\prod_{i=1}^n f_i(a), \prod_{i=1}^n f_i(x)\right) + (b - x)L\left(\prod_{i=1}^n f_i(x), \prod_{i=1}^n f_i(b)\right), \tag{2.9}$$

for all $x \in [a, b]$, whence (2.7). Similarly we can prove (2.8). □

Remark 2.4. By taking $i = 1$ and $f_1 = f$ in (2.7) and (2.8), we obtain the inequalities of Corollary 1.4.

We will now point out some new results of the Hadamard type for log-convex, m-convex, and (α, m)-convex functions, respectively.

Theorem 2.5. *Let $f, g : I \to (0, \infty)$ be log-convex functions on I and $a, b \in I$ with $a < b$. Then the following inequalities hold:*

$$f\left(\frac{a + b}{2}\right) g\left(\frac{a + b}{2}\right) \leq \frac{1}{2}\left\{\frac{1}{b - a} \int_a^b [f(x)f(a + b - x) + g(x)g(a + b - x)] dx\right\}$$

$$\leq \frac{f(a)f(b) + g(a)g(b)}{2}. \tag{2.10}$$

Proof. We can write

$$\frac{a + b}{2} = \frac{ta + (1 - t)b}{2} + \frac{(1 - t)a + tb}{2}. \tag{2.11}$$

Using the elementary inequality $cd \leq 1/2[c^2 + d^2]$ ($c, d \geq 0$ reals) and equality (2.11), we have

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)$$

$$\leq \frac{1}{2}\left[f^2\left(\frac{a+b}{2}\right) + g^2\left(\frac{a+b}{2}\right)\right]$$

$$= \frac{1}{2}\left[f^2\left(\frac{ta+(1-t)b}{2} + \frac{(1-t)a+tb}{2}\right) + g^2\left(\frac{ta+(1-t)b}{2} + \frac{(1-t)a+tb}{2}\right)\right]$$

$$\leq \frac{1}{2}\left\{\left[(f(ta+(1-t)b))^{1/2}\right]^2\left[(f((1-t)a+tb))^{1/2}\right]^2\right.$$

$$\left. + \left[(g(ta+(1-t)b))^{1/2}\right]^2\left[(g((1-t)a+tb))^{1/2}\right]^2\right\}$$

$$= \frac{1}{2}\left[f(ta+(1-t)b)f((1-t)a+tb) + g(ta+(1-t)b)g((1-t)a+tb)\right]. \tag{2.12}$$

Since f, g are log-convex functions, we obtain

$$\frac{1}{2}\left[f(ta+(1-t)b)f((1-t)a+tb) + g(ta+(1-t)b)g((1-t)a+tb)\right]$$

$$\leq \left\{\frac{1}{2}\left[f(a)\right]^t\left[f(b)\right]^{(1-t)}\left[f(a)\right]^{(1-t)}\left[f(b)\right]^t + \left[g(a)\right]^t\left[g(b)\right]^{(1-t)}\left[g(a)\right]^{(1-t)}\left[g(b)\right]^t\right\} \tag{2.13}$$

$$= \frac{f(a)f(b) + g(a)g(b)}{2}$$

for all $a, b \in I$ and $t \in [0, 1]$.

Rewriting (2.12) and (2.13), we have

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\left[f(ta+(1-t)b)f((1-t)a+tb) + g(ta+(1-t)b)g((1-t)a+tb)\right], \tag{2.14}$$

$$\frac{1}{2}\left[f(ta+(1-t)b)f((1-t)a+tb) + g(ta+(1-t)b)g((1-t)a+tb)\right] \leq \frac{f(a)f(b) + g(a)g(b)}{2}. \tag{2.15}$$

Integrating both sides of (2.14) and (2.15) on $[0, 1]$ over t, respectively, we obtain

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b [f(x)f(a+b-x) + g(x)g(a+b-x)]dx\right],$$

$$\frac{1}{2}\left[\frac{1}{b-a}\int_a^b [f(x)f(a+b-x) + g(x)g(a+b-x)]dx\right] \leq \frac{f(a)f(b) + g(a)g(b)}{2}. \tag{2.16}$$

Combining (2.16), we get the desired inequalities (2.10). The proof is complete.   □

Theorem 2.6. *Let $f, g : I \rightarrow (0, \infty)$ be* log-*convex functions on I and $a, b \in I$ with $a < b$. Then the following inequalities hold:*

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b \left[f^2(x) + g^2(x)\right]dx$$

$$\le \frac{f(a)+f(b)}{2}L(f(a), f(b)) + \frac{g(a)+g(b)}{2}L(g(a), g(b)),$$

(2.17)

where $L(\cdot, \cdot)$ is a logarithmic mean of positive real numbers.

Proof. From inequality (2.14), we have

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)$$

$$\le \frac{1}{2}\left[f(ta + (1-t)b)f((1-t)a + tb) + g(ta + (1-t)b)g((1-t)a + tb)\right].$$

(2.18)

for all $a, b \in I$ and $t \in [0, 1]$.

Using the elementary inequality $cd \le 1/2[c^2 + d^2]$ $(c, d \ge 0$ reals$)$ on the right side of the above inequality, we have

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)$$

$$\le \frac{1}{4}\left[f^2(ta + (1-t)b) + f^2((1-t)a + tb) + g^2(ta + (1-t)b) + g^2((1-t)a + tb)\right].$$

(2.19)

Since f, g are log-convex functions, then we get

$$\left[f^2(ta + (1-t)b) + f^2((1-t)a + tb) + g^2(ta + (1-t)b) + g^2((1-t)a + tb)\right]$$

$$\le \left\{ [f(a)]^{2t}[f(b)]^{(2-2t)} + [f(a)]^{(2-2t)}[f(b)]^{2t} + [g(a)]^{2t}[g(b)]^{(2-2t)} + [g(a)]^{(2-2t)}[g(b)]^{2t} \right\}$$

$$= \left[f^2(b)\left[\frac{f(a)}{f(b)}\right]^{2t} + f^2(a)\left[\frac{f(b)}{f(a)}\right]^{2t} + g^2(b)\left[\frac{g(a)}{g(b)}\right]^{2t} + g^2(a)\left[\frac{g(b)}{g(a)}\right]^{2t} \right].$$

(2.20)

Integrating both sides of (2.19) and (2.20) on $[0, 1]$ over t, respectively, we obtain

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b\left[f^2(x) + g^2(x)\right]dx,$$

$$\frac{1}{b-a}\int_a^b\left[f^2(x) + g^2(x)\right]dx$$

$$\leq \frac{1}{2}\left(f^2(b)\int_0^1\left[\frac{f(a)}{f(b)}\right]^{2t}dt + f^2(a)\int_0^1\left[\frac{f(b)}{f(a)}\right]^{2t}dt\right.$$

$$+g^2(b)\int_0^1\left[\frac{g(a)}{g(b)}\right]^{2t}dt + g^2(a)\int_0^1\left[\frac{g(b)}{g(a)}\right]^{2t}dt\right)$$

$$= \frac{1}{2}\left(f^2(b)\left[\frac{[f(a)/f(b)]^{2t}}{2\log f(a)/f(b)}\right]_0^1 + f^2(a)\left[\frac{[f(b)/f(a)]^{2t}}{2\log f(b)/f(a)}\right]_0^1\right.$$

$$\left.+g^2(b)\left[\frac{[g(a)/g(b)]^{2t}}{2\log g(a)/g(b)}\right]_0^1 + g^2(a)\left[\frac{[g(b)/g(a)]^{2t}}{2\log g(b)/g(a)}\right]_0^1\right) \qquad (2.21)$$

$$= \frac{1}{2}\left(\frac{f^2(a) - f^2(b)}{2(\log f(a) - \log f(b))} + \frac{f^2(b) - f^2(a)}{2(\log f(b) - \log f(a))}\right.$$

$$\left.+\frac{g^2(a) - g^2(b)}{2(\log g(a) - \log g(b))} + \frac{g^2(b) - g^2(a)}{2(\log g(b) - \log g(a))}\right)$$

$$= \frac{1}{2}\left(\frac{f(a) + f(b)}{2}L(f(a), f(b)) + \frac{f(a) + f(b)}{2}L(f(b), f(a))\right.$$

$$\left.+\frac{g(a) + g(b)}{2}L(g(a), g(b)) + \frac{g(a) + g(b)}{2}L(g(b), g(a))\right)$$

$$= \left\{\frac{f(a) + f(b)}{2}L(f(a), f(b)) + \frac{g(a) + g(b)}{2}L(g(a), g(b))\right\}.$$

Combining (2.21), we get the required inequalities (2.17). The proof is complete.   □

Theorem 2.7. *Let $f, g : [0, \infty) \rightarrow [0, \infty)$ be such that fg is in $L^1([a, b])$, where $0 \leq a < b < \infty$. If f is nonincreasing m_1-convex function and g is nonincreasing m_2-convex function on $[a, b]$ for some fixed $m_1, m_2 \in (0, 1]$, then the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \min\{S_1, S_2\}, \qquad (2.22)$$

where

$$S_1 = \frac{1}{6}\left[\left(f^2(a) + g^2(a)\right) + m_1 f(a)f\left(\frac{b}{m_1}\right) + m_2 g(a)g\left(\frac{b}{m_2}\right) + m_1^2 f^2\left(\frac{b}{m_1}\right) + m_2^2 g^2\left(\frac{b}{m_2}\right)\right],$$

(2.23)

$$S_2 = \frac{1}{6}\left[\left(f^2(b) + g^2(b)\right) + m_1 f(b)f\left(\frac{a}{m_1}\right) + m_2 g(b)g\left(\frac{a}{m_2}\right) + m_1^2 f^2\left(\frac{a}{m_1}\right) + m_2^2 g^2\left(\frac{a}{m_2}\right)\right]$$

(2.24)

Proof. Since f is m_1-convex function and g is m_2-convex function, we have

$$f(ta + (1-t)b) \leq tf(a) + m_1(1-t)f\left(\frac{b}{m_1}\right),$$

$$g(ta + (1-t)b) \leq tg(a) + m_2(1-t)g\left(\frac{b}{m_2}\right)$$

(2.25)

for all $t \in [0,1]$. It is easy to observe that

$$\int_a^b f(x)g(x)dx = (b-a)\int_0^1 f(ta + (1-t)b)g(ta + (1-t)b)dt.$$

(2.26)

Using the elementary inequality $cd \leq 1/2(c^2 + d^2)$ $(c, d \geq 0$ reals), (2.25) on the right side of (2.26) and making the charge of variable and since f, g is nonincreasing, we have

$$\int_a^b f(x)g(x)dx$$

$$\leq \frac{1}{2}(b-a)\int_0^1\left[\{f(ta + (1-t)b)\}^2 + \{g(ta + (1-t)b)\}^2\right]dt$$

$$\leq \frac{1}{2}(b-a)\int_0^1\left[\left(tf(a) + m_1(1-t)f\left(\frac{b}{m_1}\right)\right)^2 + \left(tg(a) + m_2(1-t)g\left(\frac{b}{m_2}\right)\right)^2\right]dt$$

$$= \frac{1}{2}(b-a)\left[\frac{1}{3}f^2(a) + \frac{1}{3}m_1^2 f^2\left(\frac{b}{m_1}\right) + \frac{1}{3}m_1 f(a)f\left(\frac{b}{m_1}\right) + \frac{1}{3}g^2(a) + \frac{1}{3}m_2^2 g^2\left(\frac{b}{m_2}\right)\right.$$

$$\left. + \frac{1}{3}m_2 g(a)g\left(\frac{b}{m_2}\right)\right]$$

(2.27)

$$= \frac{(b-a)}{6}\left[\left(f^2(a) + g^2(a)\right) + m_1 f(a)f\left(\frac{b}{m_1}\right) + m_2 g(a)g\left(\frac{b}{m_2}\right) + m_1^2 f^2\left(\frac{b}{m_1}\right)\right.$$

$$\left. + m_2^2 g^2\left(\frac{b}{m_2}\right)\right].$$

Analogously we obtain

$$\int_a^b f(x)g(x)dx$$

$$\leq \frac{(b-a)}{6}\left[\left(f^2(b)+g^2(b)\right)+m_1 f(b)f\left(\frac{a}{m_1}\right)+m_2 g(b)g\left(\frac{a}{m_2}\right)+m_1^2 f^2\left(\frac{a}{m_1}\right)+m_2^2 g^2\left(\frac{a}{m_2}\right)\right].$$

$$(2.28)$$

Rewriting (2.27) and (2.28), we get the required inequality in (2.22). The proof is complete. □

Theorem 2.8. *Let $f, g : [0, \infty) \to [0, \infty)$ be such that fg is in $L^1([a, b])$, where $0 \leq a < b < \infty$. If f is nonincreasing (α_1, m_1)-convex function and g is nonincreasing (α_2, m_2)-convex function on $[a, b]$ for some fixed $\alpha_1, m_1, \alpha_2, m_2 \in (0, 1]$. Then the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \min\{E_1, E_2\},$$

$$(2.29)$$

where

$$E_1 = \frac{1}{2}\left[\frac{1}{2\alpha_1+1}f^2(a) + \frac{2\alpha_1^2}{(\alpha_1+1)(2\alpha_1+1)}m_1^2 f^2\left(\frac{b}{m_1}\right)\right.$$

$$+ \frac{2\alpha_1}{(\alpha_1+1)(2\alpha_1+1)}m_1 f(a)f\left(\frac{b}{m_1}\right) + \frac{1}{2\alpha_2+1}g^2(a)$$

$$(2.30)$$

$$\left.+ \frac{2\alpha_2^2}{(\alpha_2+1)(2\alpha_2+1)}m_2^2 g^2\left(\frac{b}{m_2}\right) + \frac{2\alpha_2}{(\alpha_2+1)(2\alpha_2+1)}m_2 g(a)g\left(\frac{b}{m_2}\right)\right],$$

$$E_2 = \frac{1}{2}\left[\frac{1}{2\alpha_1+1}f^2(b) + \frac{2\alpha_1^2}{(\alpha_1+1)(2\alpha_1+1)}m_1^2 f^2\left(\frac{a}{m_1}\right)\right.$$

$$+ \frac{2\alpha_1}{(\alpha_1+1)(2\alpha_1+1)}m_1 f(b)f\left(\frac{a}{m_1}\right) + \frac{1}{2\alpha_2+1}g^2(b)$$

$$(2.31)$$

$$\left.+ \frac{2\alpha_2^2}{(\alpha_2+1)(2\alpha_2+1)}m_2^2 g^2\left(\frac{a}{m_2}\right) + \frac{2\alpha_2}{(\alpha_2+1)(2\alpha_2+1)}m_2 g(b)g\left(\frac{a}{m_2}\right)\right].$$

Proof. Since f is (α_1, m_1)-convex function and g is (α_2, m_2)-convex function, then we have

$$f(ta + (1-t)b) \leq t^{\alpha_1}f(a) + m_1(1-t^{\alpha_1})f\left(\frac{b}{m_1}\right),$$

$$g(ta + (1-t)b) \leq t^{\alpha_2}g(a) + m_2(1-t^{\alpha_2})g\left(\frac{b}{m_2}\right)$$

$$(2.32)$$

for all $t \in [0,1]$. It is easy to observe that

$$\int_a^b f(x)g(x)dx = (b-a)\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)dt. \tag{2.33}$$

Using the elementary inequality $cd \leq 1/2(c^2+d^2)$ $(c,d \geq 0$ reals$)$, (2.32) on the right side of (2.33) and making the charge of variable and since f,g is nonincreasing, we have

$$\int_a^b f(x)g(x)dx \leq \frac{1}{2}(b-a)\int_0^1 \left[\{f(ta+(1-t)b)\}^2 + \{g(ta+(1-t)b)\}^2\right]dt$$

$$\leq \frac{1}{2}(b-a)\int_0^1 \left[\left(t^{\alpha_1}f(a) + m_1(1-t^{\alpha_1})f\left(\frac{b}{m_1}\right)\right)^2 \right.$$

$$\left. + \left(t^{\alpha_2}g(a) + m_2(1-t^{\alpha_2})g\left(\frac{b}{m_2}\right)\right)^2\right]dt$$

$$= \frac{1}{2}(b-a)\left[\frac{1}{2\alpha_1+1}f^2(a) + \frac{2\alpha_1^2}{(\alpha_1+1)(2\alpha_1+1)}m_1^2 f^2\left(\frac{b}{m_1}\right)\right.$$

$$+ \frac{2\alpha_1}{(\alpha_1+1)(2\alpha_1+1)}m_1 f(a)f\left(\frac{b}{m_1}\right) + \frac{1}{2\alpha_2+1}g^2(a)$$

$$\left. + \frac{2\alpha_2^2}{(\alpha_2+1)(2\alpha_2+1)}m_2^2 g^2\left(\frac{b}{m_2}\right) + \frac{2\alpha_2}{(\alpha_2+1)(2\alpha_2+1)}m_2 g(a)g\left(\frac{b}{m_2}\right)\right]$$

$$\tag{2.34}$$

Analogously we obtain

$$\int_a^b f(x)g(x)dx$$

$$\leq \frac{1}{2}(b-a)\left[\frac{1}{2\alpha_1+1}f^2(b) + \frac{2\alpha_1^2}{(\alpha_1+1)(2\alpha_1+1)}m_1^2 f^2\left(\frac{a}{m_1}\right)\right.$$

$$+ \frac{2\alpha_1}{(\alpha_1+1)(2\alpha_1+1)}m_1 f(b)f\left(\frac{a}{m_1}\right) + \frac{1}{2\alpha_2+1}g^2(b) \tag{2.35}$$

$$\left. + \frac{2\alpha_2^2}{(\alpha_2+1)(2\alpha_2+1)}m_2^2 g^2\left(\frac{a}{m_2}\right) + \frac{2\alpha_2}{(\alpha_2+1)(2\alpha_2+1)}m_2 g(b)g\left(\frac{a}{m_2}\right)\right].$$

Rewriting (2.34) and (2.35), we get the required inequality in (2.29). The proof is complete.
□

Remark 2.9. In Theorem 2.8, if we choose $\alpha_1 = \alpha_2 = 1$, we obtain the inequality of Theorem 2.7.

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