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Research Article

Harmonic Numbers and Cubed Binomial Coefficients

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Euler related results on the sum of the ratio of harmonic numbers and cubed binomial coefficients are investigated in this paper. Integral and closed-form representation of sums are developed in terms of zeta and polygamma functions. The given representations are new.

1. Introduction

The well-known Riemann zeta function is defined as

$$\zeta(z) = \sum_{r=1}^{\infty} \frac{1}{r^z}, \quad \operatorname{Re}(z) > 1. \tag{1.1}$$

The generalized harmonic numbers of order α are given by

$$H_n^{(\alpha)} = \sum_{r=1}^{n} \frac{1}{r^\alpha} \quad \text{for } (\alpha, n) \in \mathbb{N} \times \mathbb{N}, \ \mathbb{N} := \{1, 2, 3, \ldots\}, \tag{1.2}$$

and for $\alpha = 1$,

$$H_n^{(1)} = H_n = \int_0^1 \frac{1 - t^n}{1 - t} dt = \sum_{r=1}^{n} \frac{1}{r} = \gamma + \psi(n+1), \tag{1.3}$$

where γ denotes the Euler-Mascheroni constant defined by

$$\gamma = \lim_{n\to\infty}\left(\sum_{r=1}^{n}\frac{1}{r} - \log(n)\right) = -\psi(1) \approx 0.5772156649\ldots, \tag{1.4}$$

and where $\psi(z)$ denotes the Psi, or digamma function defined by

$$\psi(z) = \frac{d}{dz}\log\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)} = \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+z}\right) - \gamma, \tag{1.5}$$

and the Gamma function $\Gamma(z) = \int_0^\infty u^{z-1}e^{-u}du$, for $\mathbb{R}(z) > 0$.

Variant Euler sums of the form

$$\sum_{n=1}^{\infty}\frac{\left(H_{2n}^{(1)} - (1/2)H_n^{(1)}\right)x^{2n}}{n^p} \quad \text{for } p = 1 \text{ and } p = 2 \tag{1.6}$$

have been considered by Chen [1], and recently Boyadzhiev [2] evaluated various binomial identities involving power sums with harmonic numbers $H_n^{(1)}$. Other remarkable harmonic number identities known to Euler are

$$\sum_{n=1}^{\infty}\frac{H_n^{(1)}}{(n+1)^2} = \zeta(3), \qquad \sum_{n=1}^{\infty}\frac{H_n^{(1)}}{n^3} = \frac{5}{4}\zeta(4), \tag{1.7}$$

there is also a recurrence formula

$$(2n+1)\zeta(2n) = 2\sum_{r=1}^{n-1}\zeta(2r)\zeta(2n-2r), \tag{1.8}$$

which shows that in particular, for $n = 2,\ 5\zeta(4) = 2(\zeta(2))^2$ and more generally that $\zeta(2n)$ is a rational multiple of $(\zeta(n))^2$. Another elegant recursion known to Euler [3] is

$$2\sum_{n=1}^{\infty}\frac{H_n^{(1)}}{n^q} = (q+2)\zeta(q+1) - \sum_{r=1}^{q-2}\zeta(r+1)\zeta(q-r). \tag{1.9}$$

Further work in the summation of harmonic numbers and binomial coefficients has also been done by Flajolet and Salvy [4] and Basu [5]. In this paper it is intended to add, in a small way, some results related to (1.7) and to extend the result of Cloitre, as reported in [6], $\sum_{n=1}^{\infty}(H_n^{(1)}/\binom{n+k}{k}) = (k/(k-1)^2)$ for $k > 1$. Specifically, we investigate integral representations and closed form representations for sums of harmonic numbers and cubed binomial coefficients. The works of [7–13] also investigate various representations of binomial sums and zeta functions in simpler form by the use of the Beta function and other techniques. Some of the material in this paper was inspired by the work of Mansour, [8], where he used, in part, the Beta function to obtain very general results for finite binomial sums.

2. Integral Representations and Identities

The following Lemma, given by Sofo [11], is stated without proof and deals with the derivative of a reciprocal binomial coefficient.

Lemma 2.1. *Let a be a positive real number, $z \geq 0$, n is a positive integer and let $Q(an,z) = \binom{an+z}{z}^{-1}$ be an analytic function of z. Then,*

$$Q'(an,z) = \frac{dQ}{dz} = \begin{cases} -Q(an,z)\left[\psi(z+1+an) - \psi(z+1)\right] & \text{for } z > 0, \\ -H_n^{(1)}, & \text{for } z = 0 \text{ and } a = 1. \end{cases} \tag{2.1}$$

Theorem 2.2. *Let $a, b, c, d \geq 0$ be real positive numbers, $|t| \leq 1$, $p \geq 0$ and let $j, k, l, m \geq 0$ be real positive numbers. Then*

$$\begin{aligned} &\sum_{n\geq 1} \frac{t^n \binom{n+p-1}{n-1} \left[\psi(j+1+an) - \psi(j+1)\right]}{n\binom{an+j}{j}\binom{bn+k}{k}\binom{cn+l}{l}\binom{dn+m}{m}} \\ &\quad = -aklmt \int\!\!\int\!\!\int\!\!\int_0^1 \frac{(1-x)^j (1-y)^{k-1}(1-z)^{l-1}(1-w)^{m-1}}{\left(1 - tx^a y^b z^c w^d\right)^{p+1}} \\ &\qquad \times \ln(1-x) \cdot x^{a-1} y^b z^c w^d \, dx\, dy\, dz\, dw. \end{aligned} \tag{2.2}$$

Proof. Expand

$$\begin{aligned} &\sum_{n\geq 1} \frac{t^n \binom{n+p-1}{n-1}}{n\binom{an+j}{j}\binom{bn+k}{k}\binom{cn+l}{l}\binom{dn+m}{m}} \\ &\quad = \sum_{n\geq 1}\left[t^n \binom{n+p-1}{n-1} \frac{an\Gamma(an)\Gamma(j+1)\Gamma(bn+1)k\cdot\Gamma(k)}{n\Gamma(an+j+1)\Gamma(bn+k+1)} \right. \\ &\qquad \left. \times \frac{\Gamma(cn+1)l\cdot\Gamma(l)\Gamma(dn+1)m\cdot\Gamma(m)}{\Gamma(cn+l+1)\Gamma(dn+m+1)} \right] \\ &\quad = aklm \int_0^1 \frac{(1-x)^j}{x} \sum_{n\geq 1} x^{an} t^n \binom{n+p-1}{n-1} \times B(bn+1,k)B(cn+1,l)B(dn+1,m)dx, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} B(\alpha,\beta) &= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 (1-y)^{\alpha-1} y^{\beta-1} dy \\ &= \int_0^1 (1-y)^{\beta-1} y^{\alpha-1} dy, \quad \text{for } \alpha > 0 \text{ and } \beta > 0 \end{aligned} \tag{2.4}$$

is the classical Beta function. Differentiating with respect to the parameter j, and utilizing Lemma 2.1 implies the resulting equation is as follows:

$$\begin{aligned}
&\sum_{n\geq 1}\frac{t^n\binom{n+p-1}{n-1}\left(\psi(j+1+an)-\psi(j+1)\right)}{n\binom{an+j}{j}\binom{bn+k}{k}\binom{cn+l}{l}\binom{dn+m}{m}}\\
&\quad= -aklm\int_0^1\frac{(1-x)^j}{x}\ln(1-x)\sum_{n\geq 1}x^{an}t^n\binom{n+p-1}{n-1}\\
&\qquad\times B(bn+1,k)B(cn+1,l)B(dn+1,m)dx,\\
&\quad= -aklm\int\iiint_0^1\frac{(1-x)^j\ln(1-x)}{x}(1-y)^{k-1}(1-z)^{l-1}(1-w)^{m-1}\\
&\qquad\times\sum_{n\geq 1}\binom{n+p-1}{n-1}\left(tx^ay^bz^cw^d\right)^n dx\,dy\,dz\,dw\\
&\quad= -aklmt\int\iiint_0^1\frac{(1-x)^j\ln(1-x)(1-y)^{k-1}(1-z)^{l-1}(1-w)^{m-1}}{\left(1-tx^ay^bz^cw^d\right)^{p+1}}\\
&\qquad\times x^{a-1}y^bz^cw^d dx\,dy\,dz\,dw
\end{aligned} \tag{2.5}$$

for $|tx^ay^bz^cw^d|<1$. □

In the following three corollaries we encounter harmonic numbers at possible rational values of the argument, of the form $H^{(\alpha)}_{(r/b)-1}$ where $r=1,2,3,\ldots,k$, $\alpha=1,2,3,\ldots$, and $k\in\mathbb{N}$. The polygamma function $\psi^{(\alpha)}(z)$ is defined as

$$\psi^{(\alpha)}(z)=\frac{d^{\alpha+1}}{dz^{\alpha+1}}\left[\log\Gamma(z)\right]=\frac{d^\alpha}{dz^\alpha}\left[\psi(z)\right],\quad z\neq\{0,-1,-2,-3,\ldots\}. \tag{2.6}$$

To evaluate $H^{(\alpha)}_{(r/b)-1}$ we have available a relation in terms of the polygamma function $\psi^{(\alpha)}(z)$, for rational arguments z,

$$H^{(\alpha+1)}_{(r/b)-1}=\zeta(\alpha+1)+\frac{(-1)^\alpha}{\alpha!}\psi^{(\alpha)}\left(\frac{r}{b}\right), \tag{2.7}$$

where $\zeta(z)$ is the Riemann zeta function. We also define

$$H^{(1)}_{(r/b)-1}=\gamma+\psi\left(\frac{r}{b}\right),\qquad H^{(\alpha)}_0=0. \tag{2.8}$$

The evaluation of the polygamma function $\psi^{(\alpha)}(r/b)$ at rational values of the argument can be explicitly done via a formula as given by Kölbig [14] (see also [15]) or Choi and Cvijović [16] in terms of the polylogarithmic or other special functions. Some specific values are given as

$$\psi^{(n)}\left(\frac{1}{2}\right) = (-1)^n n!\left(2^{n+1}-1\right)\zeta(n+1)$$
$$H^{(5)}_{(-2/3)} = \zeta(5) - \frac{1}{24}\psi^{(4)}\left(\frac{1}{3}\right) = -\frac{2\pi^5\sqrt{3}}{9} - 120\zeta(5), \qquad H^{(2)}_{1/4} = 16 - 8G - 5\zeta(2), \tag{2.9}$$

and can be confirmed on a mathematical computer package, such as Mathematica [17].

Corollary 2.3. *Let $a = 1,\ d = c = b > 0,\ t = 1,\ p = 0,\ j = 0$ and let $l = m = k \geq 1$ be a positive integer. Then*

$$\sum_{n\geq 1}\frac{H^{(1)}_n}{n\binom{bn+k}{k}^3}$$
$$= -k^3\int\!\!\!\int\!\!\!\int\!\!\!\int_0^1 \frac{\left[(1-y)(1-z)(1-w)\right]^{k-1}}{\left(1-x(yzw)^b\right)}\ln(1-x)\cdot(yzw)^b\,dx\,dy\,dz\,dw \tag{2.10}$$

$$= \sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}^3$$
$$\times\Bigg[-\frac{r^2}{4b^2}\zeta(4) - \left(\frac{r^2}{b^2}H^{(1)}_{(r/b)-1} + \frac{r}{b} + \frac{r^2}{b}XR(k)\right)\zeta(3)$$
$$+\left(1 - \frac{r}{b}H^{(1)}_{(r/b)-1} - \frac{r^2}{b^2}H^{(2)}_{(r/b)-1} + r\left\{1 - \frac{r}{b}H^{(1)}_{(r/b)-1}\right\}XR(k) + r^2YR(k)\right)\zeta(2)$$
$$+\frac{1}{2}\left\{\left(H^{(1)}_{(r/b)-1}\right)^2 + H^{(2)}_{(r/b)-1}\right\} + \frac{r}{b}\left\{H^{(1)}_{(r/b)-1}H^{(2)}_{(r/b)-1} + H^{(3)}_{(r/b)-1}\right\}$$
$$+\frac{r^2}{2b^2}\left\{\left(H^{(2)}_{(r/b)-1}\right)^2 + 2H^{(1)}_{(r/b)-1}H^{(3)}_{(r/b)-1} + 3H^{(4)}_{(r/b)-1}\right\}$$
$$+\left(\frac{r}{2}\left\{\left(H^{(1)}_{(r/b)-1}\right)^2 + H^{(2)}_{(r/b)-1}\right\} + \frac{r^2}{b}\left\{H^{(1)}_{(r/b)-1}H^{(2)}_{(r/b)-1} + H^{(3)}_{(r/b)-1}\right\}\right)XR(k)$$
$$+\frac{r^2}{2}\left\{\left(H^{(1)}_{(r/b)-1}\right)^2 + H^{(2)}_{(r/b)-1}\right\}YR(k)\Bigg], \tag{2.11}$$

where

$$XR(k) := -3\left(H_{k-r}^{(1)} - H_{r-1}^{(1)}\right), \tag{2.12}$$

$$YR(k) := \frac{3}{2}\left[\frac{XR^2(k)}{3} + H_{k-r}^{(2)} + H_{r-1}^{(2)}\right]. \tag{2.13}$$

Proof. Let

$$\begin{aligned}\sum_{n\geq 1}\frac{H_n^{(1)}}{n\binom{bn+k}{k}^3} &= \sum_{n\geq 1}\frac{H_n^{(1)}(k!)^3}{n\prod_{r=1}^{k}(bn+r)^3} \\ &= \sum_{n\geq 1}\frac{H_n^{(1)}(k!)^3}{n}\sum_{r=1}^{k}\left[\frac{A_r}{bn+r} + \frac{B_r}{(bn+r)^2} + \frac{C_r}{(bn+r)^3}\right],\end{aligned} \tag{2.14}$$

where

$$\begin{aligned}C_r &= \lim_{n\to(-r/b)}\left\{\frac{(bn+r)^3}{\prod_{r=1}^{k}(bn+r)^3}\right\} = (-1)^{r+1}\left(\frac{r}{k!}\binom{k}{r}\right)^3, \\ B_r &= \lim_{n\to(-r/b)}\frac{d}{dn}\left\{\frac{(bn+r)^3}{\prod_{r=1}^{k}(bn+r)^3}\right\} \\ &= 3(-1)^{r}\left(\frac{r}{k!}\binom{k}{r}\right)^3\left[H_{k-r}^{(1)} - H_{r-1}^{(1)}\right], \\ A_r &= \frac{1}{2}\lim_{n\to(-r/b)}\frac{d^2}{dn^2}\left\{\frac{(bn+r)^3}{\prod_{r=1}^{k}(bn+r)^3}\right\} \\ &= \frac{3}{2}(-1)^{r+1}\left(\frac{r}{k!}\binom{k}{r}\right)^3\left[3\left(H_{k-r}^{(1)} - H_{r-1}^{(1)}\right)^2 + H_{k-r}^{(2)} + H_{r-1}^{(2)}\right].\end{aligned} \tag{2.15}$$

Now, by interchanging sums, we have

$$\sum_{n\geq 1}\frac{H_n^{(1)}}{n\binom{bn+k}{k}^3} = \sum_{r=1}^{k}\left[A_r(k!)^3\sum_{n\geq 1}\frac{H_n^{(1)}}{n(bn+r)} + B_r(k!)^3\sum_{n\geq 1}\frac{H_n^{(1)}}{n(bn+r)^2} + C_r(k!)^3\sum_{n\geq 1}\frac{H_n^{(1)}}{n(bn+r)^3}\right]. \tag{2.16}$$

We can evaluate

$$\begin{aligned}\sum_{n\ge1}\frac{H_n^{(1)}}{n(bn+r)} &= \sum_{n\ge1}\left(\frac{\psi(n)+(1/n)}{n(bn+r)}+\frac{\gamma}{n(bn+r)}\right)=\sum_{n\ge1}\left(\frac{\psi(n)}{n(bn+r)}+\frac{1}{n^2(bn+r)}+\frac{\gamma}{n(bn+r)}\right)\\ &= \sum_{n\ge1}\left(\frac{\psi(n)}{n(bn+r)}+\frac{1}{rn^2}\right)\\ &\quad+\frac{(\gamma-(b/r))}{r}\sum_{n\ge0}\left(\frac{1}{n+1}-\frac{1}{n+(r/b)}+\left(\frac{1}{n+(r/b)}-\frac{1}{n+1+(r/b)}\right)\right)\\ &= -\frac{b\gamma}{r^2}-\frac{\gamma^2}{2r}+\frac{3\zeta(2)}{2r}+\frac{(\psi(r/b))^2}{2r}-\frac{\psi'(r/b)}{2r}+\frac{\gamma}{r}\left(\psi\left(\frac{r}{b}\right)+\gamma+b\right),\end{aligned} \tag{2.17}$$

here we have used the result from [18]

$$\sum_{n\ge1}\frac{\psi(n)}{n(bn+r)}=\frac{1}{2r}\left(\left(\psi\left(\frac{r}{b}+1\right)\right)^2-\gamma^2+\zeta(2)-\psi'\left(\frac{r}{b}+1\right)\right). \tag{2.18}$$

Now using (2.7) and (2.8), we may write

$$\sum_{n\ge1}\frac{H_n^{(1)}}{n(bn+r)}=\frac{\zeta(2)}{r}+\frac{1}{2r}\left\{\left(H_{(r/b)-1}^{(1)}\right)^2+H_{(r/b)-1}^{(2)}\right\}. \tag{2.19}$$

Similarly

$$\begin{aligned}\sum_{n\ge1}\frac{H_n^{(1)}}{n(bn+r)^2} &= \frac{1}{r}\sum_{n\ge1}\frac{H_n^{(1)}}{n(bn+r)}-\frac{b}{r}\sum_{n\ge1}\frac{H_n^{(1)}}{(bn+r)^2}\\ &= -\frac{\zeta(3)}{br}+\frac{\zeta(2)}{r^2}-\frac{\zeta(2)H_{(r/b)-1}^{(1)}}{br}+\frac{1}{2r^2}\left\{\left(H_{(r/b)-1}^{(1)}\right)^2+H_{(r/b)-1}^{(2)}\right\}\\ &\quad+\frac{1}{br}\left\{H_{(r/b)-1}^{(1)}H_{(r/b)-1}^{(2)}+H_{(r/b)-1}^{(3)}\right\},\end{aligned}$$

$$\sum_{n\geq 1}\frac{H_n^{(1)}}{n(bn+r)^3}=\sum_{n\geq 1}H_n^{(1)}\left[\frac{1}{r^2n(bn+r)}-\frac{b}{r^2(bn+r)^2}-\frac{b}{r(bn+r)^3}\right]$$
$$=-\frac{\zeta(4)}{4b^2r}-\frac{\zeta(3)}{br^2}-\frac{\zeta(3)H_{(r/b)-1}^{(1)}}{b^2r}+\frac{\zeta(2)}{r^3}-\frac{\zeta(2)H_{(r/b)-1}^{(1)}}{br^2}$$
$$-\frac{\zeta(2)H_{(r/b)-1}^{(2)}}{b^2r}+\frac{1}{2r^3}\left\{\left(H_{(r/b)-1}^{(1)}\right)^2+H_{(r/b)-1}^{(2)}\right\}$$
$$+\frac{1}{br^2}\left\{H_{(r/b)-1}^{(1)}H_{(r/b)-1}^{(2)}+H_{(r/b)-1}^{(3)}\right\}$$
$$+\frac{1}{2b^2r}\left\{\left(H_{(r/b)-1}^{(2)}\right)^2+2H_{(r/b)-1}^{(1)}H_{(r/b)-1}^{(3)}+3H_{(r/b)-1}^{(4)}\right\}. \tag{2.20}$$

Substituting (2.19), (2.20) into (2.16) where $XR(k)$ and $YR(k)$ are given by (2.12) and (2.13), respectively, on simplifying the identity (2.11) is realized. □

For $k=1$ and $b=1$ the following identity is valid:

$$\sum_{n\geq 1}\frac{H_n^{(1)}}{n(n+1)^3}=\zeta(2)-\zeta(3)-\frac{\zeta(4)}{4}. \tag{2.21}$$

Theorem 2.4.

$$\sum_{n\geq 1}\frac{t^n\binom{n+p-1}{n-1}\left(\psi(j+1+an)-\psi(j+1)\right)}{n^2\binom{an+j}{j}\binom{bn+k}{k}\binom{cn+l}{l}\binom{dn+m}{m}}$$
$$=-ablmt\int\iiint_0^1\frac{(1-x)^j(1-y)^k(1-z)^{l-1}(1-w)^{m-1}}{\left(1-tx^ay^bz^cw^d\right)^{p+1}} \tag{2.22}$$
$$\times\ln(1-x)\cdot x^{a-1}y^{b-1}z^cw^d\,dx\,dy\,dz\,dw.$$

Proof. The proof of this theorem is very similar to that of Theorem 2.2 and will not be given here. □

Corollary 2.5. *Let $a = 1,\ d = c = b > 0,\ t = 1,\ p = 0, j = 0$, and let $l = m = k \geq 1$ be a positive integer. Then*

$$\sum_{n\geq 1}\frac{H_n^{(1)}}{n^2\binom{bn+k}{k}^3}$$

$$= -bk^2\int\iiint_0^1 \frac{(1-y)^k[(1-z)(1-w)]^{k-1}}{\left(1-x(yzw)^b\right)}\ln(1-x)\cdot y^{b-1}(zw)^b dx\,dy\,dz\,dw \tag{2.23}$$

$$\begin{aligned}
&= \sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}^3 \\
&\times\Bigg[\frac{r}{4b}\zeta(4) + \left(4+\frac{r}{b}H_{(r/b)-1}^{(1)} + 3rXR(k) + 2r^2YR(k)\right)\zeta(3) \\
&\quad + \left(-\frac{3b}{r} + 2H_{(r/b)-1}^{(1)} + \frac{r}{b}H_{(r/b)-1}^{(2)} + \left\{rH_{(r/b)-1}^{(1)} - 2b\right\}XR(k) - brYR(k)\right)\zeta(2) \\
&\quad - \frac{3b}{2r}\left\{\left(H_{(r/b)-1}^{(1)}\right)^2 + H_{(r/b)-1}^{(2)}\right\} - 2\left\{H_{(r/b)-1}^{(1)}H_{(r/b)-1}^{(2)} + H_{(r/b)-1}^{(3)}\right\} \\
&\quad - \frac{r}{2b}\left\{\left(H_{(r/b)-1}^{(2)}\right)^2 + 2H_{(r/b)-1}^{(1)}H_{(r/b)-1}^{(3)} + 3H_{(r/b)-1}^{(4)}\right\} \\
&\quad - \left(b\left\{\left(H_{(r/b)-1}^{(1)}\right)^2 + H_{(r/b)-1}^{(2)}\right\} + r\left\{H_{(r/b)-1}^{(1)}H_{(r/b)-1}^{(2)} + H_{(r/b)-1}^{(3)}\right\}\right)XR(k) \\
&\quad - \frac{br}{2}\left\{\left(H_{(r/b)-1}^{(1)}\right)^2 + H_{(r/b)-1}^{(2)}\right\}YR(k)\Bigg],
\end{aligned} \tag{2.24}$$

where $XR(k)$ is given by (2.12) *and $YR(k)$ is given by* (2.13).

Proof. Following similar steps to Corollary 2.3, we may write

$$\sum_{n\geq 1}\frac{H_n^{(1)}}{n^2\binom{bn+k}{k}^3}$$

$$= \sum_{r=1}^{k}\left[A_r(k!)^3\sum_{n\geq 1}\frac{H_n^{(1)}}{n^2(bn+r)} + B_r(k!)^3\sum_{n\geq 1}\frac{H_n^{(1)}}{n^2(bn+r)^2} + C_r(k!)^3\sum_{n\geq 1}\frac{H_n^{(1)}}{n^2(bn+r)^3}\right], \tag{2.25}$$

and evaluate

$$\begin{aligned}
\sum_{n\geq 1}\frac{H_n^{(1)}}{n^2(bn+r)} &= \sum_{n\geq 1}H_n^{(1)}\left[\frac{1}{rn^2}-\frac{b}{rn(bn+r)}\right] \\
&= \frac{2\zeta(3)}{r}-\frac{b\zeta(2)}{r^2}-\frac{b}{2r^2}\left\{\left(H_{(r/b)-1}^{(1)}\right)^2+H_{(r/b)-1}^{(2)}\right\}, \\
\sum_{n\geq 1}\frac{H_n^{(1)}}{n^2(bn+r)^2} &= \frac{3\zeta(3)}{r^2}-\frac{2b\zeta(2)}{r^3}+\frac{\zeta(2)H_{(r/b)-1}^{(1)}}{r^2} \\
&\quad -\frac{b}{r^3}\left\{\left(H_{(r/b)-1}^{(1)}\right)^2+H_{(r/b)-1}^{(2)}\right\}-\frac{1}{r^2}\left\{H_{(r/b)-1}^{(1)}H_{(r/b)-1}^{(2)}+H_{(r/b)-1}^{(3)}\right\}, \\
\sum_{n\geq 1}\frac{H_n^{(1)}}{n^2(bn+r)^3} &= \frac{\zeta(4)}{4br^2}-\frac{3b\zeta(2)}{r^4}+\frac{2\zeta(2)H_{(r/b)-1}^{(1)}}{r^3}+\frac{\zeta(2)H_{(r/b)-1}^{(2)}}{br^2}+\frac{4\zeta(3)}{r^3}+\frac{\zeta(3)H_{(r/b)-1}^{(1)}}{br^2} \\
&\quad -\frac{3b}{2r^4}\left\{\left(H_{(r/b)-1}^{(1)}\right)^2+H_{(r/b)-1}^{(2)}\right\}-\frac{2}{r^3}\left\{H_{(r/b)-1}^{(1)}H_{(r/b)-1}^{(2)}+H_{(r/b)-1}^{(3)}\right\} \\
&\quad -\frac{1}{2br^2}\left\{\left(H_{(r/b)-1}^{(2)}\right)^2+2H_{(r/b)-1}^{(1)}H_{(r/b)-1}^{(3)}+3H_{(r/b)-1}^{(4)}\right\}.
\end{aligned} \tag{2.26}$$

By substituting (2.26) into (2.25) and collecting zeta functions, the identity (2.24) is obtained. □

For $k=1$ and $b=1$ the following identity is valid:

$$\sum_{n\geq 1}\frac{H_n^{(1)}}{n^2(n+1)^3}=\frac{\zeta(4)}{4}+4\zeta(3)-3\zeta(2). \tag{2.27}$$

Theorem 2.6.

$$\begin{aligned}
&\sum_{n\geq 1}\frac{t^n\binom{n+p-1}{n-1}(\psi(j+1+an)-\psi(j+1))}{n^3\binom{an+j}{j}\binom{bn+k}{k}\binom{cn+l}{l}\binom{dn+m}{m}} \\
&\quad = -abcmt\int\!\!\iiint_0^1\frac{(1-x)^j(1-y)^k(1-z)^l(1-w)^{m-1}}{(1-tx^ay^bz^cw^d)^{p+1}} \\
&\qquad \times \ln(1-x)\cdot x^{a-1}y^{b-1}z^{c-1}w^d\,dx\,dy\,dz\,dw.
\end{aligned} \tag{2.28}$$

Proof. The proof of this theorem is very similar to that of Theorem 2.2 and will not be given here. □

Corollary 2.7. *Let $a = 1,\ d = c = b > 0,\ t = 1,\ j = 0$, and let $l = m = k \geq 1$ be a positive integer. Then*

$$\sum_{n\geq 1}\frac{H_n^{(1)}}{n^3\binom{bn+k}{k}^3} = -b^2k\int\!\!\int\!\!\int\!\!\int_0^1 \frac{((1-y)(1-z))^k(1-w)^{k-1}}{\left(1-x(yzw)^b\right)}\ln(1-x)\cdot(yz)^{b-1}w^b\,dx\,dy\,dz\,dw \tag{2.29}$$

$$\begin{aligned}
&= \sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}^3\\
&\quad\times\Bigg[\left(1+\frac{5}{4}XR(k)+\frac{5}{4}r^2YR(k)\right)\zeta(4)\\
&\qquad-\left(\frac{9b}{r}+H^{(1)}_{(r/b)-1}+5bXR(k)+2brYR(k)\right)\zeta(3)\\
&\qquad+\left(\frac{6b^2}{r^2}-\frac{3bH^{(1)}_{(r/b)-1}}{r}-H^{(2)}_{(r/b)-1}-\left\{\frac{3b^2}{r}+bH^{(1)}_{(r/b)-1}\right\}XR(k)+b^2YR(k)\right)\zeta(2)\\
&\qquad+\frac{3b^2}{r^2}\left\{\left(H^{(1)}_{(r/b)-1}\right)^2+H^{(2)}_{(r/b)-1}\right\}+\frac{3b}{r}\left\{H^{(1)}_{(r/b)-1}H^{(2)}_{(r/b)-1}+H^{(3)}_{(r/b)-1}\right\}\\
&\qquad+\frac{1}{2}\left\{\left(H^{(2)}_{(r/b)-1}\right)^2+2H^{(1)}_{(r/b)-1}H^{(3)}_{(r/b)-1}+3H^{(4)}_{(r/b)-1}\right\}\\
&\qquad+\left(\frac{3b^2}{2r}\left\{\left(H^{(1)}_{(r/b)-1}\right)^2+H^{(2)}_{(r/b)-1}\right\}+b\left(H^{(1)}_{(r/b)-1}H^{(2)}_{(r/b)-1}+H^{(3)}_{(r/b)-1}\right)\right)XR(k)\\
&\qquad+\frac{b^2}{2}\left\{\left(H^{(1)}_{(r/b)-1}\right)^2+H^{(2)}_{(r/b)-1}\right\}YR(k)\Bigg],
\end{aligned} \tag{2.30}$$

where $XR(k)$ is given by (2.12) *and $YR(k)$ is given by* (2.13).

Proof. We follow similar steps as the previous corollary so that

$$\sum_{n\geq 1}\frac{H_n^{(1)}}{n^3\binom{bn+k}{k}^3} = \sum_{r=1}^{k}\left[A_r(k!)^3\sum_{n\geq 1}\frac{H_n^{(1)}}{n^3(bn+r)}+B_r(k!)^3\sum_{n\geq 1}\frac{H_n^{(1)}}{n^3(bn+r)^2}+C_r(k!)^3\sum_{n\geq 1}\frac{H_n^{(1)}}{n^3(bn+r)^3}\right]. \tag{2.31}$$

After much algebraic simplification, the following identity is obtained:

$$\begin{aligned}
&\sum_{n\geq 1}\frac{H_n^{(1)}}{n^3(bn+r)^3}\\
&\quad=\sum_{n\geq 1}H_n^{(1)}\left[\frac{1}{r^3n^3}-\frac{3b}{r^4n^2}-\frac{b^3}{r^3(bn+r)^3}-\frac{3b^3}{r^4(bn+r)^2}+\frac{6b^2}{r^4n(bn+r)}\right]\\
&\quad=\frac{\zeta(4)}{r^3}-\frac{9b\zeta(3)}{r^4}-\frac{\zeta(3)H_{(r/b)-1}^{(1)}}{r^3}+\frac{6b^2\zeta(2)}{r^5}-\frac{3b\zeta(2)H_{(r/b)-1}^{(1)}}{r^4}-\frac{\zeta(2)H_{(r/b)-1}^{(2)}}{r^3}\\
&\qquad+\frac{3b^2}{r^5}\left\{\left(H_{(r/b)-1}^{(1)}\right)^2+H_{(r/b)-1}^{(2)}\right\}+\frac{3b}{r^4}\left\{H_{(r/b)-1}^{(1)}H_{(r/b)-1}^{(2)}+H_{(r/b)-1}^{(3)}\right\}\\
&\qquad+\frac{1}{2r^3}\left\{\left(H_{(r/b)-1}^{(2)}\right)^2+2H_{(r/b)-1}^{(1)}H_{(r/b)-1}^{(3)}+3H_{(r/b)-1}^{(4)}\right\},\\
&\sum_{n\geq 1}\frac{H_n^{(1)}}{n^3(bn+r)^2}=\frac{5\zeta(4)}{4r^2}-\frac{5b\zeta(3)}{r^3}+\frac{3b^2\zeta(2)}{r^4}-\frac{b\zeta(2)H_{(r/b)-1}^{(1)}}{r^3}\\
&\qquad\qquad+\frac{3b^2}{2r^4}\left\{\left(H_{(r/b)-1}^{(1)}\right)^2+H_{(r/b)-1}^{(2)}\right\}+\frac{b}{r^3}\left\{H_{(r/b)-1}^{(1)}H_{(r/b)-1}^{(2)}+H_{(r/b)-1}^{(3)}\right\},\\
&\sum_{n\geq 1}\frac{H_n^{(1)}}{n^3(bn+r)}=\frac{5\zeta(4)}{4r}+\frac{b^2\zeta(2)}{r^3}-\frac{2b\zeta(3)}{r^2}+\frac{b^2}{2r^3}\left\{\left(H_{(r/b)-1}^{(1)}\right)^2+H_{(r/b)-1}^{(2)}\right\}.
\end{aligned}\tag{2.32}$$

Now we can substitute (2.32) into (2.31), collecting zeta functions and using (2.12) and (2.13) for $XR(k)$ and $YR(k)$, respectively, the identity (2.30) is obtained. □

Some specific examples of Corollary 2.7 are as follows.
For $k=1$ and $b=1$ the following identity is valid,

$$\begin{aligned}
&\sum_{n\geq 1}\frac{H_n^{(1)}}{n^3(n+1)^3}=\zeta(4)-9\zeta(3)+\frac{1}{6}\zeta(2),\\
&\sum_{n\geq 1}\frac{H_n^{(1)}}{n^3\binom{4n+8}{8}^3}=\frac{12729073000}{27}-\frac{2641017400832}{11025}\ln 2+\frac{440380018688}{3675}(\ln 2)^2\\
&\qquad-\frac{16604790784}{315}G+11272192G^2+\frac{1576747008}{35}G\ln 2\\
&\qquad-\frac{1390679293952\pi}{33075}-\frac{172233728\pi^3}{105}-\frac{203992775989}{2450}\zeta(2)\\
&\qquad-\frac{4069970159}{210}\zeta(3)+28974848\ln 2\zeta(3)-\frac{62246737}{2}\zeta(4),
\end{aligned}\tag{2.33}$$

where G is Catalan's constant, defined by

$$G = \frac{1}{2}\int_0^1 K(s)ds = \sum_{r=1}^{\infty}\frac{(-1)^r}{(2r+1)^2} \approx 0.915965\ldots, \tag{2.34}$$

and $K(s)$ is the complete elliptic integral of the first kind. The degenerate case $k = 0$, gives the well-known result

$$\sum_{n\geq 1}\frac{H_n^{(1)}}{n^3} = \frac{5}{4}\zeta(4). \tag{2.35}$$

Remark 2.8. Corollaries 2.3, 2.5, and 2.7 are important and can be evaluated as demonstrated independently of their integral representations. Similarly the proofs of Corollaries 2.3, 2.5, and 2.7 are not obvious therefore their explicit representations is desired.

Remark 2.9. Theoretically it should be possible to obtain an integral representation for the general sum

$$\sum_{n\geq 1}\frac{t^n\binom{n+p-1}{n-1}(\psi(j+1+an)-\psi(j+1))}{n^q\binom{an+j}{j}\binom{bn+k}{k}\binom{cn+l}{l}\binom{dn+m}{m}}, \quad \text{for } q = 1,2,3,\ldots, \tag{2.36}$$

with its associated corollaries. This work will be investigated in a forthcoming paper.

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